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On the Geroch group

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ABSTRACT. — The set of stationary, axially symmetric solutions of Einstein's vacuum field equations is acted on by some infinite dimensional group (Geroch). A precise definition of this group is given as the central extension of a group of holomorphic functions with values in SL(2). This group acts in the non-linear way known from σ-models on functions with values in SL(2) which are solutions of a system of linear differential equations and at the same time parametrize an infinite dimensional coset space. This implementation is shown to be directly related to the « Inverse Scattering Method » known for « Completely Integrable Systems ».

RÉSUMÉ. — L'ensemble des solutions stationnaires à symétrie axiale des équations d'Einstein du vide admet l'action d'un groupe de dimension infinie (Geroch). On donne une définition précise de ce groupe comme extension centrale d'un groupe de fonctions holomorphes à valeurs dans SL(2). Ce groupe agit de la façon non linéaire connue à partir des modèles σ sur des fonctions à valeurs dans SL(2) qui sont solutions d'un système d'équations différentielles linéaires et paramétrisent en même temps un espace quotient de dimension infinie. On montre que cette réalisation est directement reliée à la « méthode inverse de diffusion » connue pour les systèmes complètement intégrables.

1. INTRODUCTION

The subject of stationary, axially symmetric solutions of the Einstein vacuum field equations resp. the Einstein-Maxwell equations experienced a dramatic boost with the development of « solution generating methods » in the years 1978-1980. These methods are based on an observation of Geroch [*1*] that each given stationary, axially symmetric solution was accompanied by an infinite family of potentials, which in turn allowed for an infinite parameter set of infinitesimal transformations acting on the initial solution. What remained, however, unclear in Geroch's work was the precise Lie algebra structure of these infinitesimal transformations and even more so the structure of a corresponding group of finite transformations. Later the problem of constructing finite transformations, i. e. elements of the « Geroch group », found a number of seemingly different solutions in the form of the already mentioned solution generating methods (compare [*2*] for an exhaustive list of references). In 1980 Cosgrove [*3*] made the heroic effort to unravel the interrelationships between all these various methods. Unfortunately his work did not really shine much light on the underlying mathematical structure. It is the aim of the present work to provide a clear group-theoretical picture of all the solution generating methods. First attempts in this direction were already undertaken by B. Julia [*4*] and the present authors [*5*].

In contrast to the infinitesimal transformations, which show a rather obvious Lie algebra structure, the group structure behind the various explicitly known finite transformations is less evident due to their highly non-linear and non-local action on the solutions. For the analysis of the group-theoretical significance of various steps in the implementation of the infinite dimensional « Geroch group » it turned out important that similar structures prevail in so-called Kaluza-Klein theories [*6*] [*7*] [*8*] resp. extended supergravity theories [*4*] [*9*].

Quite naturally the field equations for stationary, axially symmetric solutions of the 4-dimensional theory can be reduced to those of a 2-dimensional theory. Remarkably in all the cases mentioned above the latter turns out to have the structure of a non-linear σ-model for a non-compact symmetric space G/H. It is this group theoretical structure that provides the clue to our analysis of the Geroch group resp. its implementation. In fact, we construct a natural extension $(G^{(\infty)}, H^{(\infty)}, \tau^{(\infty)})$ of the triple (G, H, τ) defining the symmetric space G/H (where τ is the involutive automorphism leaving H invariant). Here $G^{(\infty)}$ and $H^{(\infty)}$ are infinite dimensional groups of holomorphic functions with values in the complexification of G, corresponding to the algebra of infinitesimal transformations discovered by

Geroch. The elements of the coset space $G^{(\infty)}/H^{(\infty)}$ are solutions $\mathscr{P}(t, x)$ of a system of linear differential equations, whose integrability conditions are the field equations of the non-linear σ-model. The latter situation is typical for « Completely Integrable Systems ». In fact, we shall show that the action of the Geroch group is directly related to the « Inverse Scattering Method » developed for these systems. Implementing the group $G^{(\infty)}$ on the coset space $G^{(\infty)}/H^{(\infty)}$ turns out to be equivalent to the solution of a factorization problem for group-valued analytic functions, a so-called Riemann-Hilbert problem. The well-known Bäcklund transformations [10] correspond to meromorphic group elements.

Besides the fields parametrizing the coset space G/H there remains another field from the original 4-dimensional theory, a conformal factor λ describing the reduced 2-dimensional geometry. Following a suggestion of B. Julia [4] we show that the transformation of this conformal factor leads to a central extension of the group $G^{(\infty)}$ acting on the solutions of the non-linear σ-model. We construct the corresponding group 2-cocycle Ω and derive the beautiful explicit formula for the conformal factor $\ln \lambda = \frac{1}{2}\,\Omega(\tau^{(\infty)}\mathscr{P}^{-1}, \mathscr{P})$.

The Geroch group is thus given a precise meaning as the central extension $G_{ce}^{(\infty)}$ of a group of holomorphic functions $G^{(\infty)}$ (with values in $G = SL(2)$) acting in the usual non-linear way on the infinite dimensional coset space $G_{ce}^{(\infty)}/H^{(\infty)}$ of group-valued holomorphic functions.

In the present paper we shall restrict ourselves to the simplest case of pure Einstein gravity in four dimensions leading to the coset space SL(2)/SO(2). Using the so-called Kramer-Neugebauer transformation [11] we shall derive in a rather elementary way the infinite dimensional Lie algebra of Geroch. In a subsequent paper we shall dispose of the use the Kramer-Neugebauer transformation and derive the action of the group $G^{(\infty)}$ in a more abstract form exploiting the structure of the linear spectral problem adjoint to the non-linear σ-model. In this form the analysis applies immediately to σ-models for arbitrary symmetric spaces G/H.

2. DIMENSIONAL REDUCTION FROM 4 TO 2 DIMENSIONS

Stationary, axially symmetric space-times are characterized by the existence of two Killing vector fields, $\vec{K}$ describing asymptotically time translations and $\vec{M}$ describing axial rotations. In adapted coordinates t and φ the Killing vector fields are given by $\partial/\partial t$ and $\partial/\partial\varphi$ resp. and hence the 4-metric is independent of t and φ. This fact allows to « dimensionally reduce » the 4-dimensional theory to a 2-dimensional one. There are several possibilities to construct such a 2-dimensional field theory describing 4-dimensional space-times with the considered symmetry. They differ by dualizations of various potentials describing the 4-dimensional geometry. At the level of equations of motion these dualizations consist in an exchange of genuine field equations with Bianchi identities [*12*]. This requires that the potentials enter the equations of motion only through their field-strength (and not through covariant derivatives). At the level of the action such dualized potentials appear as Lagrange multipliers for the corresponding Bianchi identities.

We may use the invariance under local Lorentz transformations to bring the 4-bein into the following triangular form

$$E^{A}{}_{K} = \begin{pmatrix} \lambda \tilde{e}^{a}{}_{k} & 0 \\ \bar{e}^{\bar{a}}{}_{\bar{m}} B^{\bar{m}}{}_{k} & \bar{e}^{\bar{a}}{}_{\bar{k}} \end{pmatrix}, \tag{2.1}$$

adapted to the $2 + 2$ dimensional coordinates $X^{K} = (x^{k}, \bar{x}^{\bar{k}})$ with $\bar{x}^{\bar{k}} = (\varphi, t)$. The field $\bar{e}^{\bar{a}}{}_{\bar{k}}$ is a 2-bein for the isometry group, whereas $\tilde{e}^{a}{}_{k}$ is a 2-bein for the orbit space and λ is a suitable conformal factor chosen later. The fields $B^{\bar{m}}{}_{k}$ are a column of two vector fields. The vanishing of the field strengths $B^{\bar{m}}{}_{kl} = \partial_{k} B^{\bar{m}}{}_{l} - \partial_{l} B^{\bar{m}}{}_{k}$ is the condition for the hypersurface orthogonality of the Killing vector fields. From $\bar{e}^{\bar{a}}{}_{\bar{m}}$ we can build the metric m as usual $m = \bar{e}^{T}\bar{\eta}\bar{e}$ with $\bar{\eta} = (-\ +)$.

The 4-bein field $E^{A}{}_{K}$ transforms covariantly under diffeomorphisms and local Lorentz transformations, but the special triangular form (1) is preserved only by a subgroup consisting of [*13*]

i) diffeomorphisms and local Lorentz transformations in 2 dimensions acting on $\tilde{e}^{a}{}_{k}$,

ii) local Lorentz transformations in 2 dimensions depending only on x and not on $\bar{x} = (\varphi, t)$ acting on $\bar{e}^{\bar{a}}{}_{\bar{k}}$,

iii) global GL(2) transformations acting linearly on the coordinates $\bar{x}$,

iv) diffeomorphisms of the special form $\bar{x} \to \bar{x} + \bar{f}(x)$ acting as gauge transformations $B^{\bar{m}}{}_{k} \to B^{\bar{m}}{}_{k} + \partial_{k}\bar{f}^{\bar{m}}(x)$ on the vector fields.

The Lagrangean for the 4-dimensional gravitational theory can be expressed by the 2-dimensional fields of the parametrization (1) (ignoring surface terms)

$$\begin{aligned} \mathcal{L}^{(4,2)} = -\frac{1}{2}ER = \tilde{e}\rho\Big[& -\frac{1}{2}\tilde{R} - \frac{1}{8\lambda^{2}} B^{T}_{kl} m B^{kl} \\ & + \frac{\tilde{h}^{kl}}{8}(\mathrm{Tr}\,(m^{-1}\partial_{k} m m^{-1}\partial_{l} m) - 4\rho^{-2}\partial_{k}\rho\partial_{l}\rho - 8\lambda^{-1}\partial_{k}\lambda\rho^{-1}\partial_{l}\rho)\Big] \end{aligned} \tag{2.2}$$

where $\tilde{h}_{kl} = -\eta_{ab}\tilde{e}^a{}_k\tilde{e}^b{}_l$ and $\rho = \det(\bar{e})$. R resp. $\tilde{\mathrm{R}}$ are the scalar curvature of $\mathrm{E^A{}_K}$ resp. $\tilde{e}^a{}_k$.

The vector fields $\mathrm{B}^{\overline{m}}{}_k$ have no dynamical degree of freedom in 2 dimensions. From the field equation

$$\nabla_k(\rho\lambda^{-2}m\mathrm{B}^{kl}) = 0 \tag{2.3}$$

it follows that the dual field strength $*\mathrm{B} = \frac{1}{2\tilde{e}}\varepsilon^{kl}\rho\lambda^{-2}m\mathrm{B}_{kl}$ obeys the equation

$$\varepsilon^{kl}\partial_l{*\mathrm{B}} = 0 \tag{2.4}$$

and hence $*\mathrm{B} = \text{const}$. For asymptotically Minkowskian solutions $*\mathrm{B}$ vanishes at infinity and hence the constant vanishes. This means that $\mathrm{B}_{kl} \equiv 0$, i. e. the Killing vector fields are hypersurface orthogonal. This will be assumed in the following. Then $\mathscr{L}$ simplifies to

$$\mathscr{L}^{(4,2)} = \tilde{e}\rho\left[-\frac{1}{2}\tilde{\mathrm{R}} + \frac{1}{8}\mathrm{Tr}\,(m^{-1}\partial m m^{-1}\partial m) - \frac{1}{2}\rho^{-2}\partial\rho\partial\rho - \lambda^{-1}\partial\lambda\rho^{-1}\partial\rho\right]. \tag{2.5}$$

It turns out to be convenient to choose also $\bar{e}$ in a triangular gauge. We use $\Delta = \vec{\mathrm{K}}^2$ and $\psi = \dfrac{\vec{\mathrm{M}}\cdot\vec{\mathrm{K}}}{\Delta}$ in order to parametrize $\bar{e}$ and put

$$\mathrm{P} \equiv \frac{1}{\sqrt{\rho}}\bar{e} = \begin{pmatrix} \sqrt{\dfrac{\rho}{\Delta}} & 0 \\ \sqrt{\dfrac{\Delta}{\rho}}\,\psi & \sqrt{\dfrac{\Delta}{\rho}} \end{pmatrix} \tag{2.6}$$

which makes sense as long as $\Delta > 0$. We could as well have used $\Delta' = -\vec{\mathrm{M}}^2$ and $\psi' = \dfrac{\vec{\mathrm{M}}\cdot\vec{\mathrm{K}}}{\Delta'}$; although this latter choice has certain advantages for the description of rotating black holes, since $\Delta' > 0$ outside the horizon and off the rotation axis (whereas Δ vanishes on the ergosurface), we prefer to use Δ and ψ which tend to constants at infinity.

The metric $m = \bar{e}^{\mathrm{T}}\bar{\eta}\bar{e} = \rho\mathrm{M}$ with

$$\mathrm{M} = \mathrm{P}^{\mathrm{T}}\bar{\eta}\mathrm{P} = \frac{1}{\rho}\begin{pmatrix} \Delta\psi^2 - \dfrac{\rho^2}{\Delta} & \Delta\psi \\ \Delta\psi & \Delta \end{pmatrix} \tag{2.7}$$

corresponds to the so-called « Lewis » form [*14*].

Rescaling $\lambda \to \rho^{\frac{1}{4}}\lambda$ we obtain

$$\mathscr{L}^{(4,2)} = \tilde{e}\rho\left[-\frac{1}{2}\tilde{R} + \frac{1}{8}\operatorname{Tr}(M^{-1}\partial M M^{-1}\partial M) - \lambda^{-1}\partial\lambda\rho^{-1}\partial\rho\right]$$
$$= \tilde{e}\rho\left[-\frac{1}{2}\tilde{R} + \frac{(\partial\Delta)^2}{4\Delta^2} - \frac{\Delta^2(\partial\psi)^2}{4\rho^2} - \left(\frac{\partial\lambda}{\lambda} - \frac{\partial\rho}{4\rho} + \frac{\partial\Delta}{2\Delta}\right)\frac{\partial\rho}{\rho}\right] \quad (2.8)$$

The field equations derived from $\mathscr{L}^{(4,2)}$ are (we omit the field equation for λ, since it is of no relevance)

$$\tilde{R}_{kl} - \frac{1}{2}\tilde{h}_{kl}\tilde{R} = +\frac{1}{4}\operatorname{Tr}(M^{-1}\partial_k M M^{-1}\partial_l M) - 2\lambda^{-1}\partial_{(k}\lambda\rho^{-1}\partial_{l)}\rho$$
$$- \frac{1}{2}\tilde{h}_{kl}\left(\frac{1}{4}\operatorname{Tr}(M^{-1}\partial M M^{-1}\partial M) - 2\lambda^{-1}\partial\lambda\rho^{-1}\partial\rho\right), \quad (2.9)$$

$$\nabla(\rho M^{-1}\partial M) = 0, \quad (2.10)$$

$$\nabla\partial\rho = 0. \quad (2.11)$$

The 2-dimensional metric $\tilde{h}_{kl} = -(\tilde{e}^T\eta\tilde{e})_{kl}$ with $\eta = (-\ -)$ can be brought to the conformally flat form $\tilde{h}_{kl} = h\delta_{kl}$ by a suitable choice of coordinates. Finally h can be absorbed into the conformal factor λ leading to $\tilde{h}_{kl} = \delta_{kl}$. Since the field equations for M and ρ are conformally invariant they become those on flat space. In particular ρ is a harmonic function on $\mathbf{R}^2$. Together with its conjugate harmonic function z defined by $\partial z = -{*\partial\rho}$ (note that $**\partial = -\partial$ in a 2 dimensional space with definite metric) it provides a canonical coordinatization of the 2-dimensional reduced manifold as long as $\partial\rho \neq 0$ (Weyl's canonical coordinates). With this choice, which we shall always make from now on, the equation for M is that of a generalized (not translationally invariant) SL(2)/SO(1, 1) non-linear σ-model on flat space independent of λ.

Eq. (9) turns into an equation for λ, since the left hand side vanishes for $\tilde{h}_{kl} = \delta_{kl}$. One finds

$$\lambda^{-1}\partial_z\lambda = \frac{\rho}{4}\operatorname{Tr}(M^{-1}\partial_\rho M M^{-1}\partial_z M)$$

$$\lambda^{-1}\partial_\rho\lambda = \frac{\rho}{8}(\operatorname{Tr}(M^{-1}\partial_\rho M)^2 - \operatorname{Tr}(M^{-1}\partial_z M)^2). \quad (2.12)$$

From these equations λ can be computed by a simple integration once M is known. The integrability conditions are satisfied if eq. (10) is fulfilled.

An alternative description is obtained using the so-called twist potential $\tilde{\psi}$ instead of ψ, defined by

$$*\partial\tilde{\psi} = \frac{\Delta^2}{\rho}\partial\psi. \quad (2.13)$$

In order to switch from ψ to $\tilde{\psi}$ we add the Bianchi identity $*\partial\partial\psi=0$ with the Lagrange multiplier $\tilde{\psi}$ to $\mathscr{L}^{(4,2)}$, i. e. replace $\mathscr{L}^{(4,2)}$ by $\mathscr{L}^{(4,2)} + \frac{1}{2}\tilde{\psi}*\partial\partial\psi$. The resulting field equation (13) for ψ can be solved algebraically in terms of $\tilde{\psi}$. Substitution into $\mathscr{L}^{(4,2)}$ yields

$$\begin{aligned}\tilde{\mathscr{L}}^{(4,2)} &= \tilde{e}\rho\left[-\frac{1}{2}\tilde{\mathrm{R}} + \frac{(\partial\Delta)^2}{4\Delta^2} + \frac{(\partial\tilde{\psi})^2}{4\Delta^2} - \left(\frac{\partial\lambda}{\lambda} - \frac{\partial\rho}{4\rho} + \frac{\partial\Delta}{2\Delta}\right)\frac{\partial\rho}{\rho}\right] \\ &= \tilde{e}\rho\left[-\frac{1}{2}\tilde{\mathrm{R}} + \frac{1}{8}\mathrm{Tr}\,(\tilde{\mathrm{M}}^{-1}\partial\tilde{\mathrm{M}}\tilde{\mathrm{M}}^{-1}\partial\tilde{\mathrm{M}}) - \tilde{\lambda}^{-1}\partial\tilde{\lambda}\rho^{-1}\partial\rho\right] \end{aligned} \tag{2.14}$$

with

$$\tilde{\lambda} = \rho^{-\frac{1}{4}}\Delta^{\frac{1}{2}}\lambda \tag{2.15}$$

and

$$\tilde{\mathrm{M}} = \begin{pmatrix} \Delta + \dfrac{\tilde{\psi}^2}{\Delta} & \dfrac{\tilde{\psi}}{\Delta} \\ \dfrac{\tilde{\psi}}{\Delta} & \dfrac{1}{\Delta} \end{pmatrix}. \tag{2.16}$$

$\tilde{\mathrm{M}}$ can be factorized again in the form $\tilde{\mathrm{M}} = \tilde{\mathrm{P}}^{\mathrm{T}}\tilde{\mathrm{P}}$ with

$$\tilde{\mathrm{P}} = \begin{pmatrix} \sqrt{\Delta} & 0 \\ \dfrac{\tilde{\psi}}{\sqrt{\Delta}} & \dfrac{1}{\sqrt{\Delta}} \end{pmatrix} \tag{2.17}$$

parametrizing an SL(2)/SO(2) non-linear σ-model.

Since the group-theoretical structure of the symmetric space SL(2)/SO(2) plays an important role in our investigation we shall mention a few relevant features in appendix A.

3. THE LINEAR SYSTEM

In chapter 2 we saw how the replacement $\mathrm{K}:(\mathrm{P},\lambda) \to (\tilde{\mathrm{P}},\tilde{\lambda})$ leads from an SL(2)/SO(1, 1) σ-model structure to one for SL(2)/SO(2). The transformation K was first considered by Kramer and Neugebauer [11] and will be called the Kramer-Neugebauer transformation. In terms of the component fields it is

$$\mathrm{K}:\quad \begin{aligned} \frac{\rho}{\Delta} &\to \Delta \\ \psi &\to \tilde{\psi} \\ \lambda &\to \tilde{\lambda}. \end{aligned} \tag{3.1}$$

Obviously we have to distinguish the two SL(2) groups acting on P resp. $\tilde{\mathrm{P}}$. Let us call the first one, acting on P the « Matzner-Misner » group G and the second one, acting on $\tilde{\mathrm{P}}$ the « Ehlers » group $\tilde{\mathrm{G}}$. In the following we shall demonstrate how their interplay leads to the infinite-dimensional Geroch group. Our approach differs from that of Geroch, who exploited the interplay of the Ehlers group $\tilde{\mathrm{G}}$ (acting on Δ and $\tilde{\psi}$) with the group GL(2)—mentioned in chapter 2 as item *iii*)—mixing the two Killing vector fields $\vec{\mathrm{K}}$ and $\vec{\mathrm{M}}$. The result, however, is the same.

Let us study the action of the Ehlers group on $\tilde{\mathrm{P}}$ in some detail. There are three different generators $\delta\tilde{g}$ which yield the following infinitesimal transformations:

i) $\delta\tilde{g} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$, a shift of $\tilde{\psi}$: $\delta\Delta = 0$, $\delta\tilde{\psi} = 1$;

ii) $\delta\tilde{g} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, a scaling: $\delta\Delta = 2\Delta$, $\delta\tilde{\psi} = 2\tilde{\psi}$;

iii) $\delta\tilde{g} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, the « Ehlers » transformation: $\delta\Delta = 2\Delta\tilde{\psi}$, $\delta\tilde{\psi} = \tilde{\psi}^2 - \Delta^2$.

For the transformation of $\tilde{\mathrm{P}}$ we require the SO(2) element $\tilde{h}(\tilde{\mathrm{P}}, \tilde{g})$ restoring the triangular form of $\tilde{\mathrm{P}}$ (compare eq. (A.2)). Only *iii*) requires a non-vanishing $\delta\tilde{h}$ given by

$$\delta\tilde{h}(\tilde{\mathrm{P}}, \delta\tilde{g}) = \begin{pmatrix} 0 & \Delta \\ -\Delta & 0 \end{pmatrix} \in \mathrm{so}(2). \tag{3.2}$$

Clearly $\tilde{\lambda}$ is invariant under these transformations.

The interesting question now is, how these transformations act on (P, λ). First of all the invariance of $\tilde{\lambda}$ together with eq. (2.15) yields $\dfrac{\delta\lambda}{\lambda} = -\dfrac{\delta\Delta}{2\Delta}$. Obviously *i*) does not act on (P, λ) at all. Under *ii*) ψ scales oppositely to $\tilde{\psi}$ and

$$\delta\lambda = -\lambda. \tag{3.3 ii}$$

The « Ehlers » transformation *iii*) yields

$$\delta\lambda = -\tilde{\psi}\lambda, \tag{3.3 iii}$$

but $\delta\psi$ is more complicated, since $\tilde{\psi}$ and ψ are related through the differential equation (2.13):

$$*\partial\delta\psi = 2\rho(\Delta^{-1}\partial\Delta - \Delta^{-2}\tilde{\psi}\partial\tilde{\psi}). \tag{3.4}$$

The r. h. s. is one of the conserved currents in the matrix $\rho\tilde{\mathrm{M}}^{-1}\partial\tilde{\mathrm{M}}$ giving rise to a potential ϕ via dualization $*\partial\phi = \rho(\Delta^{-1}\partial\Delta - \Delta^{-2}\tilde{\psi}\partial\tilde{\psi})$, i. e.

$$\delta\psi = 2\phi. \tag{3.5}$$

Two more potentials (one of them is $\tilde{\psi}$) are required by covariance under the « Matzner-Misner » SL(2). This already shows that implementing the « Ehlers » SL(2) on (P, λ) requires new potentials not yet contained in P or $\tilde{\mathrm{P}}$. This phenomenon repeats itself if we try to apply the « Ehlers » SL(2) to these new potentials and in fact does not stop after finitely many steps. Hence, the attempt to implement both SL(2)'s on (P, λ) (or equivalently on ($\tilde{\mathrm{P}}$, $\tilde{\lambda}$)) requires infinitely many new potentials, which can be derived recursively. The integrability conditions for these new potentials are a consequence of the equation of motion (2.10) for $\tilde{\mathrm{M}}$. Instead of resolving the recursion we shall employ a simpler but equivalent method. Let us introduce a generating function $\mathscr{P}(s, x) = \mathrm{P}(x) + s\mathrm{P}_1(x) + \ldots$ incorporating all the required potentials. For reasons of covariance under the Matzner-Misner group the new currents constructed from these potentials are expected to be linear combinations of the original currents $\mathscr{J}$ and their duals. This suggests the following ansatz (compare eq. (A.5))

$$\begin{aligned}\partial\mathscr{P}\mathscr{P}^{-1} &= \partial\mathrm{PP}^{-1} + 2a\mathscr{J} + 2b{}^*\mathscr{J} \\ &= a\partial\mathrm{PP}^{-1} + (1-a)\tau\partial\mathrm{PP}^{-1} + b({}^*\partial\mathrm{PP}^{-1} - \tau{}^*\partial\mathrm{PP}^{-1})\end{aligned} \quad (3.6)$$

where a and b are functions of s. In order to determine the functions a and b we require that the integrability conditions for eq. (6) are satisfied if and only if P satisfies the field equation

$$\nabla(\rho\partial\mathrm{PP}^{-1}) - \nabla(\rho\tau(\partial\mathrm{PP}^{-1})) + \rho[\partial\mathrm{PP}^{-1}, \tau(\partial\mathrm{PP}^{-1})] = 0 \quad (3.7)$$

equivalent to eq. (2.10) and find that this is the case if the algebraic equation

$$a - a^2 - b^2 = 0 \quad (3.8)$$

and the differential equation

$${}^*\partial a + \rho\partial\frac{b}{\rho} = 0 \quad (3.9)$$

are satisfied. The algebraic equation (8) can be solved in terms of one unknown function $t\left(= -\frac{b}{a}\right)$ such that the linear system (6) takes the form

$$\partial\mathscr{P}\mathscr{P}^{-1} = \frac{1}{1+t^2}(\partial\mathrm{PP}^{-1} - t{}^*\partial\mathrm{PP}^{-1} + t\tau({}^*\partial\mathrm{PP}^{-1}) + t^2\tau(\partial\mathrm{PP}^{-1})). \quad (3.10)$$

The function t must satisfy

$$(1-t^2)\partial t + 2t{}^*\partial t = t(1+t^2)\frac{\partial\rho}{\rho} \quad (3.11)$$

or equivalently

$$\partial t = \frac{t}{(1+t^2)\rho}((1-t^2)\partial\rho - 2t{}^*\partial\rho) \quad (3.12)$$

leading to

$$\partial\left(\frac{\rho}{2}\left(\frac{1}{t} - t\right)\right) = *\partial\rho = -\,\partial z \qquad \text{i. e.} \qquad \frac{1}{t} - t = \frac{2}{\rho}(w - z) \quad (3.13)$$

with a constant of integration w. The quadratic equation for t has the solutions

$$t_{\pm}(w, x) = \frac{1}{\rho}\left((z - w) \pm \sqrt{(z - w)^2 + \rho^2}\right) = -\frac{1}{t_{\mp}}. \quad (3.14)$$

The pairs (w, t) solving eq. (13) define for each given $x = (z, \rho)$ with $\rho \neq 0$ a two-sheeted Riemann surface with the branch points $w = z \pm i\rho$. The replacement $t \to -\frac{1}{t}$ exchanges the two sheets. For z, ρ and w real we choose t_+, i. e. a positive value for the square root, in the first (physical) sheet. We can choose the branch cut along the line segment $(z = \Re w,\ \rho \leq |\Im w|)$. With this choice $t_{\pm}$ is purely imaginary on the branch cut and $\Re t \geq 0$ and $\Re t_- \leq 0$ for all (w, x). At the branch points $w = w_{\pm}(x) \equiv z \pm i\rho$ we find $t_+(w_{\pm}(x), x) = t_-(w_{\pm}(x), x) = \mp i$. For $z < \Re w$ resp. $z > \Re w$ the value of t_+ lies inside resp. outside and t_- lies outside resp. inside of the unit circle. Finally we can identify

$$w = \frac{1}{2s} \quad (3.15)$$

in order to obtain the correct behaviour for small s (in the physical sheet) and find

$$t_+ \underset{s \to 0+}{\longrightarrow} \rho s, \qquad t_- \underset{s \to 0+}{\longrightarrow} -\frac{1}{\rho s}. \quad (3.16)$$

For $\rho = 0$ the Riemann surface degenerates and splits into two disconnected planes $w = z$. The function $t(w)$ becomes singular and we find in particular

$$t_+ \underset{\rho \to 0}{\longrightarrow} \begin{cases} 0 \\ \infty \end{cases}, \quad t_- \underset{\rho \to 0}{\longrightarrow} \begin{cases} \infty & \text{for} \quad z < \Re w, \\ 0 & \text{for} \quad z > \Re w. \end{cases} \quad (3.17)$$

We will see later that the solution of eq. (10) has a branch point as well, i. e. we have in fact two solutions $\mathscr{P}_+(w, x)$ and $\mathscr{P}_-(w, x)$ which are analytic continuations of each other and which satisfy

$$\partial\mathscr{P}_{\pm}\mathscr{P}_{\pm}^{-1} = \frac{1}{1+t_{\pm}^2}\left(\partial \mathrm{PP}^{-1} - t_{\pm}{*\partial \mathrm{PP}^{-1}} + t_{\pm}\tau(*\partial \mathrm{PP}^{-1}) + t_{\pm}^2\tau(\partial \mathrm{PP}^{-1})\right). \quad (3.18)$$

The two branches $\mathscr{P}_{\pm}(w, x)$ correspond to one function $\mathscr{P}(t, x)$ defined on the Riemann surface with

$$\mathscr{P}_{\pm}(w, x) = \mathscr{P}(t_{\pm}(w, x), x) \quad (3.19)$$

which satisfies eq. (10). Note however that $\partial\mathscr{P}$ is to be interpreted as differentiation with w held fixed, i. e.

$$\begin{aligned}\partial\mathscr{P}(t, x) &= \partial\mathscr{P}(t, x)\,|_t + \partial t(w, x)\frac{\partial\mathscr{P}(t, x)}{\partial t}\\ &= \partial\mathscr{P}(t, x)\,|_t + \frac{t}{\rho(1 + t^2)}((1 - t^2)\partial\rho + 2t\partial z)\frac{\partial\mathscr{P}(t, x)}{\partial t}. \qquad (3.20)\end{aligned}$$

These differential operators commute with the substitution $t \to -\frac{1}{t}$. They are, apart from the replacement $t \to \rho^{-1}\lambda$, just the differential operators introduced in [*15*].

It is sometimes convenient to introduce slightly different linear systems for functions U and V constructed from $\mathscr{P}(t, x)$ and $\mathrm{P}(x) = \mathscr{P}(0, x)$. The function

$$\mathrm{U}(t, x) = \mathrm{P}^{-1}(x)\mathscr{P}(t, x) \qquad \text{with} \qquad \mathrm{U}(0, x) = \mathbf{1} \qquad (3.21)$$

or $\mathrm{U}_\pm(w, x) = \mathrm{P}^{-1}(x)\mathscr{P}_\pm(w, x)$ satisfies

$$\partial\mathrm{U}\mathrm{U}^{-1} = \frac{t}{1 + t^2}\mathrm{M}^{-1}(-\,{}^*\partial\mathrm{M} - t\partial\mathrm{M}) \qquad (3.22)$$

whereas

$$\mathrm{V}(t, x) = \mathrm{M}(x)\mathrm{U}(t, x) = \mathrm{P}^{\mathrm{T}}(x)\bar{\eta}\mathscr{P}(t, x) \qquad \text{with} \qquad \mathrm{V}(0, x) = \mathrm{M}(x) \quad (3.23)$$

or $\mathrm{V}_\pm(w, x)$ satisfies

$$\partial\mathrm{V}\mathrm{V}^{-1} = \frac{1}{1 + t^2}(\partial\mathrm{M} - t^*\partial\mathrm{M})\mathrm{M}^{-1}. \qquad (3.24)$$

They have been introduced previously [*16*] and $\mathrm{V}(t, x)$ is nothing but the function $\rho^{-1}\psi(\lambda)$ of Belinskiĭ and Sakharov [*15*].

The linear system eq. (22) is analogous to the one proposed by Pohlmeyer [*17*] for the standard O(3) non-linear σ-model, but with the essential difference of the explicit ρ-dependence of eqs. (2.10) resp. (7), leading in turn to the (ρ, z)-dependence of the spectral parameter t.

There exists a simple relation [*3*] between $\mathrm{U}_\pm(w, x)$, $\mathrm{V}_\pm(w, x)$ and a generating function $\mathrm{F}_\pm(w, x)$ introduced by Kinnersley and Chitre [*18*], namely

$$\mathrm{F}(t, x) = \frac{1}{1 + t^2}\sqrt{\frac{t}{\rho s}}(\mathrm{U}(t, x) - it\varepsilon\mathrm{V}(t, x)) \qquad \text{with} \qquad \varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (3.25)$$

The linear system for $\mathrm{F}_\pm(w, x)$ has been used extensively by Ernst and Hauser [*19*].

Clearly U, $\mathscr{P}$ and V are not determined uniquely by the differential equations. For each value of s there are constants of integration which have to be determined in such a way that $\mathscr{P}(t, x)$ has a Taylor series expansion

$$\mathscr{P}(t, x) = \sum_{k=0}^{\infty} t^k \mathrm{P}_k(x) \qquad \text{with} \qquad \mathrm{P}_0(x) = \mathrm{P}(x). \tag{3.26}$$

The remaining ambiguity corresponds to gauge transformations, such as $\tilde{\psi} \rightarrow \tilde{\psi} + \text{const.}$, and will be discussed later on.

Rewriting eq. (10) in the form

$$\partial\mathscr{P}\mathscr{P}^{-1} = \mathscr{A} + \frac{1-t^2}{1+t^2}\mathscr{J} - \frac{2t}{1+t^2}{}^*\mathscr{J} \tag{3.27}$$

with $\mathscr{A}$ and $\mathscr{J}$ as in eq. (A.5) one can easily verify that it is invariant under the transformation

$$\tau^{(\infty)}: \mathscr{P}(t, x) \rightarrow \tau\left(\mathscr{P}\left(-\frac{1}{t}, x\right)\right). \tag{3.28}$$

This transformation $\tau^{(\infty)}$ will play an important group-theoretical role later as a natural extension of the automorphism τ of SL(2) to the infinite dimensional Geroch group. The function $\mathscr{P}(t, x)$ considered as an element of that group is again « triangular » in the sense that it has a Taylor series expansion (i. e. only positive powers of t) and the t-independent term is triangular.

Guided by eq. (A.3) we can construct a new function

$$\begin{aligned} \mathscr{M}(t, x) &= \bar{\eta}\tau^{(\infty)}(\mathscr{P}^{-1}(t, x))\mathscr{P}(t, x) \equiv \mathscr{P}^{\mathrm{T}}\left(-\frac{1}{t}, x\right)\bar{\eta}\mathscr{P}(t, x) \\ &\equiv \mathrm{U}^{\mathrm{T}}\left(-\frac{1}{t}, x\right)\mathrm{V}(t, x) \equiv \mathrm{V}^{\mathrm{T}}\left(-\frac{1}{t}, x\right)\mathrm{U}(t, x). \end{aligned} \tag{3.29}$$

Since $\mathscr{P}(t, x)$ and $\tau\left(\mathscr{P}\left(-\frac{1}{t}, x\right)\right)$ are solutions of the same differential equation we find, as a consequence, that $\mathscr{M}(t, x)$ satisfies $\partial\mathscr{M}(t, x) = 0$ and the corresponding functions $\mathscr{M}_\pm(w)$ are x-independent and therefore cannot have branch points at $w = z \pm i\rho$, i. e. $\mathscr{M}_+(w) = \mathscr{M}_-(w)$ and hence $\mathscr{M}_\pm(w)$ is a symmetric matrix. It is important to observe that, although $\mathscr{P}(t, x)$ and $\tau\left(\mathscr{P}\left(-\frac{1}{t}, x\right)\right)$ are solutions of the same differential equation, the constants of integration are different and hence $\mathscr{M}_\pm$ is in general a non-trivial function of w.

The linear systems (10, 22, 24) with spectral parameter t resp. s have in common that their integrability conditions are equivalent to the non-linear field equation (7) for $\mathscr{P}$ resp. (2.10) for M. This situation is typical for

« completely integrable » dynamical systems, although it is difficult to find a generally applicable and accepted definition of this term. In any case, it allows to code the information about a solution of some non-linear PDE into the analyticity properties of special solutions (like $\mathscr{P}(t, x)$) of the linear system resp. their scattering or monodromy matrices (like $\mathscr{M}(w)$) considered as functions of the spectral parameter [*20*]. Along with that goes the possibility to produce solutions of the non-linear equation through the construction of solutions of the linear system with the required analyticity in the spectral parameter (« Inverse Scattering Method »). In fact, it is the transition from one such solution to another one with the required properties that incorporates the action of the infinite-dimensional Geroch group as will be demonstrated in chapter 6. Before we describe an adaption of the « Inverse Scattering Method » (ISM) in chapter 5 we shall study the analytic properties of $\mathscr{P}$ and $\mathscr{M}$.

4. PROPERTIES OF THE GENERATING FUNCTIONS $\mathscr{P}$, U, V AND $\mathscr{M}$

In this chapter we want to derive some properties of the generating functions $\mathscr{P}$, U, V and $\mathscr{M}$. We will first determine the behaviour of these functions on the z-axis ($\rho = 0$) and will then study their domains of analyticity.

We prefer to give the discussion for the generating functions $\tilde{\mathscr{P}}$ etc. referring to $\tilde{\mathrm{P}}$. The group SL(2) is, in this case, the « Ehlers » group and the vacuum solution (4-dimensional Minkowski space) is described by the constant matrix $\tilde{\mathrm{P}}(x) = \mathbf{1}$ (and thus $\tilde{\mathscr{P}}(t, x) = \mathbf{1}$). This parametrization avoids the explicit powers of ρ present in the « Lewis » form denoted by $\mathrm{P}(x)$ in chapter 2.

In a configuration with (one or several) black holes $\rho = 0$ corresponds to the horizons and to the pieces of the axis outside the horizons. In this case $\mathrm{M}(x)$ is an analytic function of z and ρ in the whole $x = (z, \rho)$ half plane ($\rho \geq 0$) with the exception of the points $(z_i, 0)$ where the axis meets a horizon (where coordinate singularities are present) and of possible naked singularities. $\mathrm{P}(x)$ and $\tilde{\mathrm{P}}(x)$ have a « coordinate singularity » at the ergosurface where Δ vanishes and correspondingly the factorizations $\mathrm{M} = \mathrm{P}^{\mathrm{T}}\bar{\eta}\mathrm{P}$ resp. $\tilde{\mathrm{M}} = \tilde{\mathrm{P}}^{\mathrm{T}}\tilde{\mathrm{P}}$ become singular. This problem can be avoided using Δ' (compare chapter 2).

Let us assume that $\tilde{\mathrm{P}}(x)$ is an analytic function of $x = (z, \rho)$ in a simply connected domain $\mathscr{X}$ of the (z, ρ) half plane and that the complement $\bar{\mathscr{X}} = \{ x = (z, \rho) \colon \rho \geq 0, x \notin \mathscr{X} \}$ of $\mathscr{X}$ is closed and contained in a (semi) disk

$$\mathscr{D}_{\mathrm{R}} = \{ (z, \rho) \colon \rho \geq 0, r \equiv \sqrt{z^2 + \rho^2} < \mathrm{R} \} \tag{4.1}$$

of radius R. This assumption allows not only black hole configurations but the exterior solution of localized matter distributions as well. The regularity of the 4-dimensional solution at the rotation axis $\rho = 0$ is guaranteed by the regularity of $\tilde{\mathrm{M}}$. Let us further assume that the configuration is asymptotically flat and sufficiently regular, i. e.

$$\tilde{\mathrm{P}}(x) = \mathbf{1} + \mathrm{O}\left(\frac{1}{r}\right), \qquad \partial\tilde{\mathrm{P}}(x) = \mathrm{O}\left(\frac{1}{r^2}\right). \tag{4.2}$$

We shall call such solutions « physically acceptable », although the required conditions may have to be supplemented by additional ones as e. g. positivity of the mass etc., for physically really meaningful solutions.

We have to choose the w-dependent constants of integration in $\tilde{\mathscr{P}}_\pm(w, x)$ such that in the limit $\rho \to 0$, $z \to -\infty$ (i. e. $t_+ \to 0$)

$$\tilde{\mathscr{P}}_+(w, x) \xrightarrow[\substack{\rho\to 0\\ z\to -\infty}]{} \mathbf{1}. \tag{4.3}$$

Due to the asymptotic behaviour (2) of $\tilde{\mathrm{P}}(x)$ we can integrate the differential equation (3.18) along a large circle and find

$$\mathscr{P}_+(w, x) \xrightarrow[r\to\infty]{} \mathbf{1} \tag{4.4}$$

in the whole asymptotic region.

4.1. Behaviour on the z-axis.

Let us keep w fixed $\Im w \neq 0$. If we integrate eq. (3.18) along the z-axis we find (for $|z| > \mathrm{R}$ and $z \neq \Re w$) taking into account eq. (3.17)

$$\begin{aligned}
\tilde{\mathscr{P}}_+(w, (z, 0)) &= \begin{cases} \tilde{\mathrm{P}}(z)\mathrm{C}_1(w) & \text{for} \quad z < \Re w, \\ (\tilde{\mathrm{P}}^{\mathrm{T}})^{-1}(z)\mathrm{C}_2(w) & \text{for} \quad z > \Re w, \end{cases} \\
\tilde{\mathscr{P}}_-(w, (z, 0)) &= \begin{cases} (\tilde{\mathrm{P}}^{\mathrm{T}})^{-1}(z)\mathrm{C}_2(w) & \text{for} \quad z < \Re w, \\ \tilde{\mathrm{P}}(z)\mathrm{C}_1(w) & \text{for} \quad z > \Re w, \end{cases}
\end{aligned} \tag{4.5}$$

where $\tilde{\mathrm{P}}(z)$ is a shorthand for $\tilde{\mathrm{P}}((z, 0))$. The constant matrices $\mathrm{C}_1(w)$ and $\mathrm{C}_2(w)$ will, in general, be different for $z < -\mathrm{R}$ and $z > \mathrm{R}$. The relations (5) remain obviously valid in the limit $\Im w \to 0$. Finally for $|\Re w| > \mathrm{R} + |\Im w|$ we can integrate eq. (3.18) around a circle of radius $|\Im w|$ centered at $x = (\Re w, |\Im w|)$ in order to relate $\tilde{\mathscr{P}}_+$ and $\tilde{\mathscr{P}}_-$. Due to the assumed analyticity of $\tilde{\mathrm{P}}(x)$ the rhs. of eq. (3.18) is uniformly bounded on all such circles and therefore

$$\lim_{\Im w\to 0} \tilde{\mathscr{P}}_+(w, (\Re w, 0)) = \lim_{\Im w\to 0} \tilde{\mathscr{P}}_-(w, (\Re w, 0)) \qquad \text{for} \qquad |\Re w| > \mathrm{R}. \tag{4.6}$$

With this information we can determine $\tilde{\mathscr{P}}_{\pm}(w,(z,0))$ for all real w with $|w| > \mathrm{R}$ and all z with $|z| > \mathrm{R}$. For $w < -\mathrm{R}$ we find

$$\tilde{\mathscr{P}}_{+}(w,(z,0)) = \begin{cases} \tilde{\mathrm{P}}(z) & \text{for} \quad z < w < -\mathrm{R}\,, \\ (\tilde{\mathrm{P}}^{\mathrm{T}})^{-1}(z)\tilde{\mathrm{M}}(w) & \text{for} \quad w < z < -\mathrm{R}\,, \\ (\tilde{\mathrm{P}}^{\mathrm{T}})^{-1}(z) & \text{for} \quad w < -\mathrm{R},\ \mathrm{R} < z\,, \end{cases}$$
$$\tilde{\mathscr{P}}_{-}(w,(z,0)) = \begin{cases} (\tilde{\mathrm{P}}^{\mathrm{T}})^{-1}(z)\tilde{\mathrm{M}}(w) & \text{for} \quad z < w < -R\,, \\ \tilde{\mathrm{P}}(z) & \text{for} \quad w < z < -\mathrm{R}\,, \\ \tilde{\mathrm{P}}(z)\tilde{\mathrm{M}}(w) & \text{for} \quad w < -\mathrm{R},\ \mathrm{R} < z\,, \end{cases} \tag{4.7}$$

whereas for $\mathrm{R} < w$

$$\tilde{\mathscr{P}}_{+}(w,(z,0)) = \begin{cases} \tilde{\mathrm{P}}(z) & \text{for} \quad z < -\mathrm{R},\ \mathrm{R} < w\,, \\ \tilde{\mathrm{P}}(z)\tilde{\mathrm{M}}^{-1}(w) & \text{for} \quad \mathrm{R} < z < w\,, \\ (\tilde{\mathrm{P}}^{\mathrm{T}})^{-1}(z) & \text{for} \quad \mathrm{R} < w < z \end{cases}$$
$$\tilde{\mathscr{P}}_{-}(w,(z,0)) = \begin{cases} (\tilde{\mathrm{P}}^{\mathrm{T}})^{-1}(z)\tilde{\mathrm{M}}^{-1}(w) & \text{for} \quad z < -\mathrm{R},\ \mathrm{R} < w\,, \\ (\tilde{\mathrm{P}}^{\mathrm{T}})^{-1}(z) & \text{for} \quad \mathrm{R} < z < w\,, \\ \tilde{\mathrm{P}}(z)\tilde{\mathrm{M}}^{-1}(w) & \text{for} \quad \mathrm{R} < w < z \end{cases} \tag{4.8}$$

and thus

$$\tilde{\mathscr{M}}_{\pm}(w) = \begin{cases} \tilde{\mathrm{M}}(w) & \text{for} \quad w < -\mathrm{R}\,, \\ \tilde{\mathrm{M}}^{-1}(w) & \text{for} \quad \mathrm{R} < w\,. \end{cases} \tag{4.9}$$

We see from eq. (9) that although $\tilde{\mathscr{M}}_{\pm}(w)$ is x-independent, it contains enough information to determine the behaviour of $\tilde{\mathrm{M}}(x)$, and thus $\tilde{\mathrm{P}}(x)$ on the axis ($|z| > \mathrm{R}$, $\rho = 0$). Together with the assumed analyticity properties this is sufficient to determine $\tilde{\mathrm{P}}(x)$ everywhere [*19*].

4.2. Domains of analyticity.

Starting from the asymptotic value $\tilde{\mathscr{P}}_{+}(w,x) \underset{r\to\infty}{\longrightarrow} \mathbf{1}$ we can, for a fixed value of w, determine $\tilde{\mathscr{P}}_{\pm}(w,x)$ by integration of eq. (3.18) along a suitable path in the (z,ρ) half plane. The r. h. s. of eq. (3.18) is analytic (in x and w) for $x \in \mathscr{X}$ except for the branch cut starting at $x = (\Re w, |\Im w|)$. The resulting $\tilde{\mathscr{P}}_{\pm}(w,x)$ will therefore be an analytic function of w and x as long as we can find a path of integration avoiding these singularities. For $\tilde{\mathscr{P}}_{+}$ (i. e. in the physical sheet) this is possible for all $x \in \mathscr{X}$ except those on the branch cut. In order to determine $\tilde{\mathscr{P}}_{-}$ we have to reach the second sheet, i. e. w must lie in the domain $\mathscr{W} = \{ w : (\Re w, |\Im w|) \in \mathscr{X} \}$ of the complex w plane. For the moment we need, as before, the additional assumption that $\Im w \neq 0$ but again the results remain true for $\Im w = 0$. Except for the branch cut the function $\tilde{\mathscr{P}}_{-}(w,x)$ will be analytic if $(w,x) \in \mathscr{W} \times \mathscr{X}$.

We can finally analyze, for $w \in \mathscr{W}$, the behaviour near the branch point and find

$$\tilde{\mathscr{P}}_{\pm}(w, x) = \tilde{\mathscr{P}}_1(w, x) + t_{\pm}(w, x)\tilde{\mathscr{P}}_2(w, x) \tag{4.10}$$

with matrix valued functions $\tilde{\mathscr{P}}_1$ and $\tilde{\mathscr{P}}_2$ analytic in a neighborhood of the branch point.

In order to characterize the domain of analyticity of $\tilde{\mathscr{P}}(t, x)$ we need some definitions. Let $\overline{\mathscr{W}} = \mathbf{C}\backslash\mathscr{W}$ be the complement of $\mathscr{W}$. For each point x in the (z, ρ) half plane we define the domains $\overline{\mathscr{C}}^x_{\pm}$, $\mathscr{X}^x_{\pm}$ and $\mathscr{C}^x$

$$\overline{\tau}^x_{\pm} = \{ t : t = t_{\mp}(w, x), w \in \overline{\mathscr{W}} \}, \qquad \mathscr{C}^x_{\pm} = \mathbf{C}\backslash\overline{\mathscr{C}}^x_{\pm}, \qquad \mathscr{C}^x = \mathscr{C}^x_{+} \cap \mathscr{C}^x_{-}. \tag{4.11}$$

The transformation $t \to -\frac{1}{t}$ will clearly map $\mathscr{C}^x_{\pm}$ onto $\mathscr{C}^x_{\mp}$ and will therefore leave their intersection $\mathscr{C}^x$ invariant. Furthermore let $\mathscr{D}_{\pm}$ and $\mathscr{D}$ be the domains

$$\mathscr{D}_{\pm} = \{ (t, x) : t \in \mathscr{C}^x_{\pm}, x \in \mathscr{X} \}, \qquad \mathscr{D} = \mathscr{D}_{+} \cap \mathscr{D}_{-}. \tag{4.12}$$

We can then use the analyticity of $\tilde{\mathscr{P}}_{\pm}(w, x)$ and the behaviour (10) near the branch point to deduce that $\tilde{\mathscr{P}}(t, x)$ is an analytic function in the domain $\mathscr{D}_{+}$ and that $\tilde{\mathscr{P}}\left(-\frac{1}{t}, x\right)$ is analytic in $\mathscr{D}_{-}$. $\tilde{\mathscr{M}}(t, x)$ will therefore be analytic in $\mathscr{D}$, but the domain of analyticity is in fact much larger. Since $\tilde{\mathscr{M}}_{\pm}(w)$ is x-independent it is analytic in $\mathscr{W}$ and $\tilde{\mathscr{M}}(t, x)$ is therefore analytic in the domain

$$\tilde{\mathscr{D}} = \{ (t, x) : t \in \mathscr{C}^x \} \supseteq \mathscr{D}. \tag{4.13}$$

5. A RIEMANN-HILBERT PROBLEM

The idea of the « Inverse Scattering Method » [*20*] is to trade a solution $\mathrm{P}(x)$ of the non-linear field equation (3.7) for its corresponding « Scattering matrix » $\mathscr{M}(w)$ and vice versa. To go from P to $\mathscr{M}$ requires to solve a differential equation (eq. (3.10)); the opposite direction can be reduced to the solution of a Fredholm integral equation equivalent to the solution of a Riemann-Hilbert problem. The clue to this Riemann-Hilbert problem is the formula (3.29). If we succeed to factorize $\mathscr{M}(t, x)$ in the form $\tau^{(\infty)}(\mathscr{P}^{-1}(t, x))\mathscr{P}(t, x)$ with some « triangular » $\mathscr{P}$ we can reconstruct the solution $\mathrm{P}(x) = \mathscr{P}(0, x)$ from $\mathscr{M}$. This factorization of $\mathscr{M}$ relies on the analytic properties of $\mathscr{P}$ as a function of the spectral parameter t. For reasons already explained above we shall again employ $\tilde{\mathscr{M}}$, although this is not essential.

We can consider $\tilde{\mathscr{M}}(t, x)$ as a family of functions, depending parametrically on x, analytic for $t \in \mathscr{C}^x$. All (possible) singularities lie in the complementary domain $\mathbf{C}\backslash\mathscr{C}^x = \overline{\mathscr{C}}^x_+ \cup \overline{\mathscr{C}}^x_-$. In order to formulate the Riemann-Hilbert problem we first have to study the location of these singularities and in particular their x-dependence. As long as $z < -\mathrm{R}$, the domain $\overline{\mathscr{C}}^x_+$ lies entirely outside the unit circle and to the left of the imaginary axis whereas $\overline{\mathscr{C}}^x_-$ lies inside the unit circle and to the right of the imaginary axis. For these x all singularities of $\tilde{\mathscr{M}}(t, x)$ in $\overline{\mathscr{C}}^x_+$ should be due to $\tilde{\mathscr{P}}(t, x)$ and those in $\overline{\mathscr{C}}^x_-$ due to $\tilde{\mathscr{P}}\left(-\frac{1}{t}, x\right)$ (compare eq. (3.29)). If we vary x, these domains and the singularities of $\tilde{\mathscr{P}}(t, x)$ will move in the complex t plane and will eventually cross the unit circle but the two domains $\overline{\mathscr{C}}^x_\pm$ will never intersect as long as $z + i\rho \in \mathscr{W}$ or equivalently $x \in \mathscr{X}$. As soon as $z > \mathrm{R}$, $\overline{\mathscr{C}}^x_+$ lies entirely inside the unit circle (but still to the left of the imaginary axis) and $\overline{\mathscr{C}}^x_-$ lies outside the unit circle.

We can, therefore, choose a family of contours C^x in the t plane which have, for all $x \in \mathscr{X}$, the following properties:

1. C^x is invariant under the mapping $t \to -\frac{1}{t}$ and will thus pass through the two fixed points $t = \pm i$ of this map.
2. The domain $\mathscr{C}^x$ contains a neighborhood of C^x and the domains $\overline{\mathscr{C}}^x_+$ resp. $\overline{\mathscr{C}}^x_-$ lie in the exterior resp. interior of C^x.
3. The contours C^x depend continuously on x.

There remains a lot of ambiguity in the choice of these contours, and we could e. g. choose the unit circle for $z < -\mathrm{R}$. The solution of the Riemann-Hilbert problem (if such a solution exists at all) will, however, be unique and therefore independent of the particular choice of contours. What we really need is the existence of a contour which separates the domains $\overline{\mathscr{C}}^x_\pm$. No such contour can exist for $x \in \overline{\mathscr{X}}$ because, in this case, both $\overline{\mathscr{C}}^x_+$ and $\overline{\mathscr{C}}^x_-$ will contain the points $t = \pm i$ and this non-existence of a suitable contour will lead to singularities of $\tilde{\mathscr{P}}(t, x)$ and $\tilde{\mathrm{P}}(x)$.

We can now formulate the Riemann-Hilbert problem for the construction of $\tilde{\mathscr{P}}(t, x)$ from $\tilde{\mathscr{M}}_\pm(w)$. Given $\tilde{\mathscr{M}}_\pm(w)$ with the properties stated above we have to factorize $\tilde{\mathscr{M}}(t, x) \equiv \tilde{\mathscr{M}}_\pm(w(t, x))$ in the form

$$\tilde{\mathscr{M}}(t, x) = \mathrm{A}^{\mathrm{T}}_-(t, x)\mathrm{A}_0(x)\mathrm{A}_+(t, x) \tag{5.1}$$

with $\mathrm{A}_\pm(t, x)$ analytic in $\mathscr{D}_\pm$ satisfying $\mathrm{A}_+(0, x) = \mathbf{1} = \mathrm{A}_-(\infty, x)$.

If this factorization does exist then $\mathrm{A}_-(t, x) = \mathrm{A}_+\left(-\frac{1}{t}, x\right)$ and $\mathrm{A}_0(x)$ is symmetric and real. Moreover $\det \mathrm{A}_\pm(t, x) = 1$ and therefore $\mathrm{A}^{-1}_\pm(t, x)$ will be analytic as well. Comparing with eq. (3.29) we can identify

$$\tilde{\mathrm{M}}(x) = \mathrm{A}_0(x), \qquad \tilde{\mathrm{U}}(t, x) = \mathrm{A}_+(t, x), \qquad \tilde{\mathrm{V}}(t, x) = \mathrm{A}_0(x)\mathrm{A}_+(t, x). \tag{5.2}$$

Finally we factorize $\tilde{M}(x)$ in the form

$$\tilde{M}(x) = \tilde{P}^T(x)\tilde{P}(x) \tag{5.3}$$

with a triangular matrix $\tilde{P}(x)$ and define

$$\tilde{\mathscr{P}}(t, x) = \tilde{P}(x)\tilde{U}(t, x). \tag{5.4}$$

There are now two questions to be answered. Is there a solution to the Riemann-Hilbert problem under the given circumstances and, if so, has the solution the right properties to yield a « physically acceptable » solution $\tilde{P}(x)$ as defined in chapter 4?

The first question has been analyzed in [*21*] (compare also [*19*]). There the problem is reduced to a Fredholm integral equation, which has a solution under the present circumstances ($\tilde{\mathscr{M}}$ is holomorphic on the curves C^x!). The only question is about the so-called indices of the Fredholm operator. However, in the present case these (integer) indices vanish for $|x| \to \infty$, since $\tilde{\mathscr{M}}(t, x) \to \mathbf{1}$ for $|x| \to \infty$ and hence they vanish for all $x \in \mathscr{X}$ by continuity.

Let us then turn to the second question about the properties of the solution. Since the parametric dependence of $\tilde{\mathscr{M}}(t, x)$ on x for $x \in \mathscr{X}$ is smooth also the factors in eq. (1) depend smoothly on x there. Furthermore we claim that the functions $\tilde{P}(x)$ resp. $\tilde{M}(x)$ defined by eqs. (2, 3) satisfy the field equations (3.7) resp. (2.10) and $\tilde{\mathscr{P}}(t, x)$, $\tilde{U}(t, x)$ resp. $\tilde{V}(t, x)$ given by eqs. (2, 4) are the solutions of the linear systems (3.10), (3.22) resp. (3.24) for these $\tilde{P}(x)$ and $\tilde{M}(x)$.

We will first prove that $\tilde{U}(t, x)$ and $\tilde{M}(x)$ satisfy eq. (3.22). The field equation (2.10) will then be fulfilled automatically because it is the compatibility condition for the linear system (3.22). The validity of the remaining equations follows by elementary manipulations.

The differential operator (3.20) has poles at $t = \pm i$ but

$$S(t, x) \equiv \left(\frac{1}{t} + t\right)\partial\tilde{U}(t, x)\tilde{U}^{-1}(t, x) \tag{5.5}$$

is analytic in $\mathscr{D}_+$ due to the structure of the differential operator and because $\tilde{U}(0, x) = \mathbf{1}$. Since $\tilde{\mathscr{M}}_{\pm}(w)$ is x-independent and therefore $\partial\tilde{\mathscr{M}}(t, x) = 0$ we find

$$\begin{aligned} 0 &= \left(\frac{1}{t} + t\right)(\tilde{U}^T)^{-1}\left(-\frac{1}{t}, x\right)\partial\tilde{\mathscr{M}}(t, x)\tilde{U}^{-1}(t, x) \\ &= S^T\left(-\frac{1}{t}, x\right)\tilde{M}(x) + \left(\frac{1}{t} + t\right)\partial\tilde{M}(x) + \tilde{M}(x)S(t, x) \end{aligned} \tag{5.6}$$

and thus there is some $G(x)$ such that

$$S(t, x) = \tilde{M}^{-1}(x)(G(x) - t\partial\tilde{M}(x)), \qquad G^T(x) = G(x). \tag{5.7}$$

Finally we observe that

$$*S(\pm i, x) = \pm iS(\pm i, x) \tag{5.8}$$

because $\tilde{U}(t, x)$ is analytic at $t = \pm i$ and thus

$$G(x) = - *\partial\tilde{M}(x). \tag{5.9}$$

This proves that eq. (3.22) is indeed satisfied.

As smooth solutions of the elliptic differential equations (3.7) resp. (2.10) the functions $\tilde{P}(x)$ resp. $\tilde{M}(x)$ are actually analytic in $\mathscr{X}$. The only remaining property to be checked is the asymptotic behaviour (4.2). Yet, this follows from the asymptotic behaviour of $\tilde{\mathscr{M}}(w) \to \mathbf{1}$ for $w \to \pm\infty$ and the analyticity of $\tilde{P}(x)$ in $\mathscr{X}$.

This demonstrates that the process of relating a holomorphic function $\tilde{\mathscr{M}}$ with the specified properties to a « physically acceptable » solution $\tilde{P}(x)$ can in fact be inverted, i. e. $\tilde{P}(x)$ can be reconstructed from $\tilde{\mathscr{M}}(w)$.

6. ACTION OF THE GEROCH GROUP ON $\mathscr{P}$ AND $\mathscr{M}$

In chapter 3 we defined the Kramer-Neugebauer transformation K as the change from (P, λ) to $(\tilde{P}, \tilde{\lambda})$. P resp. $\tilde{P}$ determine two different generating functions $\mathscr{P}(t, x)$ resp. $\tilde{\mathscr{P}}(t, x)$ via eq. (3.10) reducing to $P(x)$ resp. $\tilde{P}(x)$ for $t = 0$. Once we know how K extends to $\mathscr{P}$ we are able to act with both SL(2) groups simultaneously on $\mathscr{P}$, since we can simply transfer the action of the « Ehlers » group $\tilde{G}$ from $(\tilde{P}, \tilde{\lambda})$ to (P, λ) with the (extended) transformation K.

Expanding $\mathscr{P}$ in a Taylor series in t we deduce from eq. (3.10)

$$\mathscr{P}(t, x) = \begin{pmatrix} \sqrt{\dfrac{\rho}{\Delta}} + O(t) & \dfrac{t}{\sqrt{\rho\Delta}}\tilde{\psi} + O(t^2) \\ \sqrt{\dfrac{\Delta}{\rho}}\,\psi + O(t) & \sqrt{\dfrac{\Delta}{\rho}} + O(t) \end{pmatrix} \tag{6.1}$$

and an analogous expression for $\tilde{\mathscr{P}}(t, x)$. Recalling eq. (2.17) we find, to the order given in eq. (1),

$$\tilde{\mathscr{P}}(t, x) \approx K(t)\mathscr{P}(t, x)K^{-1}\left(\frac{t}{\rho}\right) \tag{6.2}$$

where $K(x)$ is the 2×2 matrix

$$K(x) = \begin{pmatrix} 0 & x^{+1/2} \\ x^{-1/2} & 0 \end{pmatrix}. \tag{6.3}$$

The limiting behaviour (3.16) of $t(=t_+)$ suggests to define the mapping

$$\mathrm{K}^{(\infty)}: \mathscr{P}(t, x) \rightarrow \tilde{\mathscr{P}}(t, x) = \mathrm{K}(t)\mathscr{P}(t, x)\mathrm{K}^{-1}(s(t, x)) \tag{6.4}$$

with

$$s(t, x) = \frac{t}{\rho}\left(1 + 2t\frac{z}{\rho} - t^2\right)^{-1} = \frac{t}{\rho} + \mathrm{O}(t^2) \tag{6.5}$$

and one can indeed rather easily verify that this mapping $\mathrm{K}^{(\infty)}$ maps a solution $\mathscr{P}(t, x)$ of eq. (3.10) into a solution $\tilde{\mathscr{P}}(t, x)$ of the corresponding equation. The mapping $\mathrm{K}^{(\infty)}$ is therefore the looked for extension of the Kramer-Neugebauer mapping K (compare eq. (3.1)).

We can now easily determine the action of the Ehlers group $\tilde{\mathrm{G}}$ on $\mathscr{P}(t, x)$. Given an element $\delta\tilde{g}$ from the Lie algebra of $\tilde{\mathrm{G}}$

$$\delta\tilde{\mathscr{P}}(t, x) = \delta\tilde{h}(\tilde{\mathrm{P}}(x), \delta\tilde{g})\tilde{\mathscr{P}}(t, x) - \tilde{\mathscr{P}}(t, x)\delta\tilde{g} \tag{6.6}$$

and therefore

$$\delta\mathscr{P}(t, x) = \delta h(t, x)\mathscr{P}(t, x) - \mathscr{P}(t, x)\delta g(s(t, x)) \tag{6.7}$$

with

$$\begin{aligned} \delta h(t, x) &= \mathrm{K}^{-1}(t)\delta\tilde{h}(\tilde{\mathrm{P}}(x), \delta\tilde{g})\mathrm{K}(t), \\ \delta g(s) &= \mathrm{K}^{-1}(s)\delta\tilde{g}\mathrm{K}(s). \end{aligned} \tag{6.8}$$

Note that although $\mathrm{K}(x)$ contains fractional powers of x the transformations (8) do not introduce branch points. The transformation (4) does, however, introduce branch points at $\rho = 0$, $\rho = \infty$ and $t = t_{\pm}(0, x)$. They are a reflection of the branch points at $\rho = 0$ and $\rho = \infty$ introduced by the Kramer-Neugebauer mapping (3.1).

Choosing in particular

$$\delta\tilde{g} = \begin{pmatrix} 0 & \tilde{b} \\ \tilde{c} & 0 \end{pmatrix} \tag{6.9}$$

we find

$$\delta g(s) = \begin{pmatrix} 0 & s\tilde{c} \\ s^{-1}\tilde{b} & 0 \end{pmatrix} = \sum_{k=-1}^{+1} s^k \delta g_k \tag{6.10}$$

and

$$\delta h(t) = \sum_{k=-1}^{+1} t^k \delta h_k, \qquad \delta h(t) = \tau\left(\delta h\left(-\frac{1}{t}\right)\right) = \tau^{(\infty)}(\delta h(t)). \tag{6.11}$$

Commuting the transformations generated by $\delta g(s)$ (eq. (10)) with the s-independent ones of the Matzner-Misner group G we can generate the infinite dimensional loop algebra related to SL(2) [22].

Let T_a $(a=1,\ldots,3)$ be the generators of sl(2) with commutation relations $[\mathrm{T}_a, \mathrm{T}_b]=f_{ab}{}^c\mathrm{T}_c$ and metric $g_{ab}=\mathrm{Tr}\,(\mathrm{T}_a\mathrm{T}_b)$. The generators $\mathrm{T}_a^{(k)}$ of the loop algebra $(a=1,\ldots,3, k=\ldots,-1,0,+1,\ldots)$ are realized by $\mathrm{T}_a^{(k)} \rightarrow \delta g(s)=s^k\mathrm{T}_a$.

The involutive automorphism $\tau^{(\infty)}: \mathrm{T}_a^{(k)} \to (-)^k(\tau(\mathrm{T}_a))^{(-k)}$ determines a subalgebra $\mathscr{H}^{(\infty)}$ invariant under $\tau^{(\infty)}$. In addition to the right action of $\delta g(s)$ there is the left action of an induced $\delta h(t) \in \mathscr{H}^{(\infty)}$ realized by $\mathrm{T}_a^{(k)} \to t^k \mathrm{T}_a$. The total (non-linear) action of $\delta g(s) \in \mathscr{G}^{(\infty)}$ on $\mathscr{P}(t, x)$ is given by (7) where $\delta h \in \mathscr{H}^{(\infty)}$ $\left(\text{i. e. } \delta h(t) = \tau\left(\delta h\left(-\frac{1}{t}\right)\right)\right)$ is determined such that the resulting $\delta\mathscr{P}(t, x)$ is « triangular » in the sense that it has a Taylor series expansion in t and the t-independent term is triangular. Apart from the x-dependent relation between s and t, which is entirely due to the factor ρ in front of the Lagrangean (2.5), this is just the standard action on the $\mathrm{G}^{(\infty)}/\mathrm{H}^{(\infty)}$ non-linear σ-model parametrized by $\mathscr{P}(t, x)$, which is an element of the « triangular » subgroup of $\mathrm{G}^{(\infty)}$ (see below for a precise definition of $\mathrm{G}^{(\infty)}$). The corresponding action on $\tilde{\mathscr{P}}(t, x)$ is given by

$$\delta\tilde{\mathscr{P}}(t, x) = \delta\tilde{h}(t, x)\tilde{\mathscr{P}}(t, x) - \tilde{\mathscr{P}}(t, x)\delta\tilde{g}(s(t, x)) \tag{6.12}$$

with

$$\begin{aligned} \delta\tilde{h}(t) &= \mathrm{K}(t)\delta h(t)\mathrm{K}(t)^{-1} \\ \delta\tilde{g}(s) &= \mathrm{K}(s)\delta g(s)\mathrm{K}(s)^{-1} \end{aligned} \tag{6.13}$$

and the condition that $\delta\tilde{h}(t)$ is an element of $\mathscr{H}^{(\infty)}$ now takes the form $\delta\tilde{h}(t) = \tilde{\tau}\left(\delta\tilde{h}\left(-\frac{1}{t}\right)\right)$.

The matrix $\mathscr{M}$ introduced in eq. (3.29) transforms as

$$\delta\mathscr{M}(t, x) = -\delta g^{\mathrm{T}}(s(t, x))\mathscr{M}(t, x) - \mathscr{M}(t, x)\delta g(s(t, x)). \tag{6.14}$$

Quite similar to the action of sl(2) on $\mathrm{M}(x)$ this action of the loop algebra $\mathscr{G}^{(\infty)}$ on $\mathscr{M}(t, x)$ is again linear. Note that, apart from a phase due to branch points, $\mathscr{M}(t, x)$ transforms under the Kramer-Neugebauer mapping $\mathrm{K}^{(\infty)}$

$$\mathrm{K}^{(\infty)}: \mathscr{M}(t, x) \to \tilde{\mathscr{M}}(t, x) = (-)^{\frac{1}{2}}(\mathrm{K}^{\mathrm{T}})^{-1}(s(t, x))\mathscr{M}(t, x)\mathrm{K}^{-1}(s(t, x)) \tag{6.15}$$

as expected.

Given the action of the loop algebra $\mathscr{G}^{(\infty)}$ and of the mapping $\mathrm{K}^{(\infty)}$ on $\mathscr{P}(t, x)$ we can deduce the corresponding action on the functions $\mathrm{U}(t, x)$ and $\mathrm{V}(t, x)$ given by eqs. (3.22, 24). This action is, however, less explicit because the old and new values of $\mathrm{P}(x)$ have to be extracted from these functions in a somewhat indirect way.

It is well-known [*22*] that the loop algebra $\mathscr{G}^{(\infty)}$ allows for a central extension with the commutation relations

$$[\mathrm{T}_a^{(k)}, \mathrm{T}_b^{(l)}]_{ce} = f_{ab}{}^c\mathrm{T}_c^{(k+l)} - k\delta_{k+l,0}g_{ab}\mathrm{Z} \tag{6.16}$$

or equivalently

$$[\delta g_1(s), \delta g_2(s)]_{ce} = [\delta g_1(s), \delta g_2(s)] + \frac{1}{2\pi i}\oint \mathrm{Tr}\,(\delta g_1(s)d\,\delta g_2(s))\mathrm{Z}. \tag{6.17}$$

The bilinear skew-form $\omega(\delta g_1, \delta g_2) \equiv \frac{1}{2\pi i}\oint \mathrm{Tr}\,(\delta g_1(s) d\,\delta g_2(s))$ is a so-called Lie algebra 2-cocycle (compare appendix B).

There is a natural extension of the involutive automorphism $\tau^{(\infty)}$ to $\mathcal{G}^{(\infty)}_{ce}$ compatible with (16) given by

$$\tau^{(\infty)}\mathrm{Z} = -\mathrm{Z} \tag{6.18}$$

which we denote again by $\tau^{(\infty)}$ for simplicity.

We see from the transformation law (7) that the generator Z of the central extension does not act on $\mathcal{P}(t, x)$. But, as was first observed by B. Julia [4], this central extension becomes relevant, if we consider the action of $\mathcal{G}^{(\infty)}$ on the conformal factor λ.

From the relation (2.15) we get

$$\frac{\delta\lambda}{\lambda} = -\frac{\delta\Delta}{2\Delta} \tag{6.19}$$

under the action of the Ehlers group (leaving $\tilde{\lambda}$ invariant). Taking the commutator of two elements of $\mathcal{G}^{(\infty)}$ of the form (10) we find

$$[\delta g_1(s), \delta g_2(s)] = \begin{pmatrix} \tilde{c}_1\tilde{b}_2 - \tilde{c}_2\tilde{b}_1 & 0 \\ 0 & \tilde{c}_2\tilde{b}_1 - \tilde{c}_1\tilde{b}_2 \end{pmatrix} \tag{6.20}$$

and hence

$$\frac{\delta\lambda}{\lambda} = -\frac{\delta\Delta}{2\Delta} = -\tilde{c}_1\tilde{b}_2 + \tilde{c}_2\tilde{b}_1 = \frac{1}{2\pi i}\oint ds\,\mathrm{Tr}\begin{pmatrix} 0 & s\tilde{c}_1 \\ \frac{1}{s}\tilde{b}_1 & 0 \end{pmatrix} \frac{d}{ds}\begin{pmatrix} 0 & s\tilde{c}_2 \\ \frac{1}{s}\tilde{b}_2 & 0 \end{pmatrix} \tag{6.21}$$

corresponding exactly to the Z term in eq. (17). This suggests that the action of Z is

$$\mathrm{Z}: \delta\mathcal{P}(t, x) = 0, \qquad \delta\lambda = \lambda\,. \tag{6.22}$$

For the moment we will ignore the central extension, but we will have to come back to it in chapter 7.

Let us study the action of $\delta g(s) = s^k \delta g_k$ on $\mathcal{P}(t, x)$ in some detail:

1. The case $k = 0$ is given by *i*)-*iii*) in chapter 3.

2. For $k > 0$ the induced $\delta h(t)$ vanishes and $\delta\mathcal{P}(t, x) = \Sigma_{l \geq k} t^l \delta \mathrm{P}_l(x)$. This corresponds to the freedom to add constants to the potentials needed to determine $\mathrm{P}_k(x)$.

3. If $k < 0$ then (due to the structure (5) of $s(t, x)$) the corresponding $\delta h(t)$ has the form $\delta h(t) = \Sigma_{l=-|k|}^{+|k|} t^l \delta h_l$ and depends only on the first $|k| + 1$ terms $\mathrm{P}_l(x)$, $l \leq |k|$ of the Taylor series expansion of $\mathcal{P}(t, x)$.

The determination of $\delta h(t, x)$ and $\delta\mathcal{P}(t, x)$ is in all these cases completely straight-forward although tedious.

To the infinitesimal transformations (7) resp. (14) correspond finite ones

$$\mathscr{P}(t,x) \to \mathscr{P}_g(t,x) = h(t,x)\mathscr{P}(t,x)g^{-1}(s(t,x)) \tag{6.23}$$

resp.

$$\mathscr{M}(w) \to \mathscr{M}_g(w) = (g^{\mathrm{T}})^{-1}\left(\frac{1}{2w}\right)\mathscr{M}(w)g^{-1}\left(\frac{1}{2w}\right). \tag{6.24}$$

There are two questions connected with these formulae. The first one is, what are the admissible functions $g(s)$ mapping the class of physically acceptable solutions into itself? The set of these functions, which constitute obviously a group, is our candidate for the Geroch group. Actually we prefer to identify the Geroch group with the central extension of the latter corresponding to eq. (17), since we would like to include the conformal factor λ in the set of dynamical variables. This provides the possibility to compute λ by group-theoretical methods (compare chapter 7). The second question is, how does one construct $\mathscr{P}_g(t,x)$ resp. $h(t,x)$ for a given $g(s)$?

In order to answer these questions we recall from chapter 5 that there is a one-to-one correspondence between $\tilde{\mathscr{P}}$ and $\tilde{\mathscr{M}}$. Therefore it is sufficient to consider the transformations (24) of $\tilde{\mathscr{M}}$. In chapter 4 we showed that $\tilde{\mathscr{M}}$ is holomorphic in a domain $\mathscr{W} = \{w : (\Re w, |\Im w|) \in \mathscr{X}\}$ where $\mathscr{X}$ is that part of the (z, ρ)-plane where $\tilde{\mathrm{P}}(x)$ is analytic. $\tilde{\mathscr{M}}(w)$ is symmetric, real for real w, i. e. $\tilde{\mathscr{M}}(\overline{w}) = \overline{\tilde{\mathscr{M}}}(w)$ and $\det \tilde{\mathscr{M}} = 1$. Furthermore $\tilde{\mathscr{M}} \to \mathbf{1}$ for $|w| \to \infty$. Let $\mathrm{G}^{(\infty)}$ be the set of SL(2)-valued functions $g(s)$ holomorphic in a domain containing a neighbourhood of the origin (remember $s = \frac{1}{2w}\Big)$ with the additional properties $g(\bar{s}) = \bar{g}(s)$ and $g(0) = \mathbf{1}$. It is quite obvious that transforming $\tilde{\mathscr{M}}$ according to (24) with such g's does not change its properties (apart from a possible admissible change of its domain of analyticity). It is the central extension $\mathrm{G}^{(\infty)}_{ce}$ (compare chapter 7) of $\mathrm{G}^{(\infty)}$ that we propose as the Geroch group. For a similar definition (without the central extension) we refer to the work of Ernst and Hauser [*19*]. They also proved, what they called the « Geroch Conjecture », that the group $\mathrm{G}^{(\infty)}$ acts transitively on the set of « physically admissible » solutions. Translating their approach to our formalism we shall try to find some $g(s)$ with values in SL(2) transforming a given $\tilde{\mathscr{M}}$ into $\tilde{\mathscr{M}} = \mathbf{1}$ corresponding to Minkowski space, i. e.

$$\tilde{\mathscr{M}}(w) = g^{\mathrm{T}}\left(\frac{1}{2w}\right)g\left(\frac{1}{2w}\right). \tag{6.25}$$

Writing $\tilde{\mathscr{M}}(w)$ in the form

$$\tilde{\mathscr{M}}(w) = \begin{pmatrix} \dfrac{1+b(w)^2}{a(w)} & b(w) \\ b(w) & a(w) \end{pmatrix}, \tag{6.26}$$

where $a(w) \neq 0$ in $\mathscr{W}$, there is a unique triangular matrix

$$g\left(\frac{1}{2w}\right) = \frac{1}{\sqrt{a(w)}}\begin{pmatrix} 1 & 0 \\ b(w) & a(w) \end{pmatrix} \tag{6.27}$$

fulfilling eq. (25). This function $g(s)$ is also holomorphic for $w = \dfrac{1}{2s}$ in $\mathscr{W}$, has $\det g = 1$, is real for real s and tends to $\mathbf{1}$ for $s \to 0$.

In order to answer the second question about the construction of $\tilde{\mathscr{P}}_g(t, x) = h(t, x)\tilde{\mathscr{P}}(t, x)g^{-1}(s)$ from $\tilde{\mathscr{P}}(t, x)$ for some given $g(s)$ we can proceed as indicated in the following diagram

$$\tilde{\mathscr{P}} \to \tilde{\mathscr{M}} \to \tilde{\mathscr{M}}_g \to \tilde{\mathscr{P}}_g. \tag{6.28}$$

As was shown in chapter 5 the last—and only non-trivial—step reduces to the solution of a Riemann-Hilbert problem. Its solution gives directly $\tilde{\mathscr{P}}_g(t, x)$, from which we can obtain $h(t, x)$ *a posteriori*, if so wanted.

The solution of the Riemann-Hilbert problem is particularly simple if the singularities of $\tilde{\mathscr{M}}_\pm(w)$ are just (simple) poles. For such meromorphic $\tilde{\mathscr{M}}_\pm$ (with N poles) the Riemann-Hilbert problem can be reduced to a system of N linear equations, i. e. it can be solved by inverting an N × N matrix and the conformal factor λ can be determined in closed form [*2*].

It is sometimes convenient to express the new solution $\tilde{\mathscr{P}}^g(t, x)$ as

$$G^{(\infty)} \ni g(s): \tilde{\mathscr{P}}(t, x) \to \tilde{\mathscr{P}}^g(t, x) = Z_+^g(t, x)\tilde{\mathscr{P}}(t, x) \tag{6.29}$$

and use a slightly different Riemann-Hilbert problem in order to determine directly $Z_+^g(t, x)$ [*23*]. We could obviously use similar equations for $U(t, x)$ or $V(t, x)$ but the function $\tilde{\mathscr{P}}(t, x)$ has the advantage of a more direct group theoretical interpretation. Clearly $Z_+^g(t, x)$ is analytic in the domain $\mathscr{D}_+$ (for a suitably chosen $\mathscr{X}$) as are $\tilde{\mathscr{P}}(t, x)$ and $\tilde{\mathscr{P}}^g(t, x)$. We find that we have to construct

$$\begin{aligned} \mathscr{G}(t, x) &= \tilde{\mathscr{P}}(t, x)g^{-1}(s(t, x))\tilde{\mathscr{P}}^{-1}(t, x) \\ \mathscr{Z}^g(t, x) &= \mathscr{G}^{\mathrm{T}}\left(-\frac{1}{t}, x\right)\mathscr{G}(t, x) \end{aligned} \tag{6.30}$$

and decompose $\mathscr{Z}^g(t, x)$ in the form

$$\mathscr{Z}^g(t, x) = Z_-^{g\mathrm{T}}(t, x)Z_+^g(t, x) \tag{6.31}$$

where $Z_\pm^g(t, x)$ is analytic in $\mathscr{D}_\pm$ and $Z_+^g(0, x) = Z_-^g(\infty, x)$ is a triangular matrix.

If the function $\mathscr{Z}^g(t, x)$ in the Riemann-Hilbert problem (31) is meromorphic (e. g. with 2N poles) then the transformation (29) corresponds to an N-fold Bäcklund transformation [*24*] adding N poles to the solution $\tilde{\mathscr{P}}(t, x)$ of the linear system. For a general $\tilde{\mathscr{P}}(t, x)$ it is, unfortunately, practically impossible to find an element $g(s) \in G^{(\infty)}$ such that the corresponding $\mathscr{Z}^g(t, x)$ is meromorphic. One can, however, investigate what are the conditions on a meromorphic $Z^{g}_{+}(t, x)$ in order that $\tilde{\mathscr{P}}^{g}(t, x)$ determined by eq. (29) is again a solution of the linear system (3.10). This is, in fact, the procedure of Belinskiĭ and Sakharov [15] (for $V(t, x)$ instead of $\tilde{\mathscr{P}}(t, x)$). This method yields algebraic equations for the residues of $Z^{g}_{+}(t, x)$ and the conformal factor λ can again be given in closed form. Cosgrove [3] has shown the equivalence of this method with the Bäcklund transformations introduced independently by Harrison and Neugebauer [10].

7. THE CENTRAL EXTENSION OF $G^{(\infty)}$

In chapter 6 we found that the inclusion of the conformal factor λ in the transformations of the Geroch group leads to a central extension $\mathscr{G}^{(\infty)}_{ce}$ of the loop algebra $\mathscr{G}^{(\infty)}$ defined by the Lie algebra 2-cocyle ω. As explained in appendix B there is a corresponding central extension $G^{(\infty)}_{ce}$ of the group $G^{(\infty)}$ defined by a related group 2-cocycle Ω.

According to eq. (6.22) λ transforms as $\delta\lambda = \lambda$ under Z. This suggests to consider the pairs $(\mathscr{P}(t, x), \lambda^{-1}(x))$ as « triangular » elements of the extended group $G^{(\infty)}_{ce}$ representing the coset space $G^{(\infty)}_{ce}/H^{(\infty)}$. Moreover eq. (6.22) suggests to extend the transformation law (6.23) to

$$(\mathscr{P}, \lambda^{-1}) \to (h(\mathscr{P}, g), 1) \circ (\mathscr{P}, \lambda^{-1}) \circ (g, e^{\gamma})^{-1} \quad \text{for} \quad (g, e^{\gamma}) \in G^{(\infty)}_{ce} \tag{7.1}$$

where the multiplication «$\circ$» is defined in appendix B. Let us prove that (1) yields indeed the correct transformation law for λ. We may restrict ourselves to infinitesimal elements of the Matzner-Misner and Ehlers group, since they generate the whole Lie algebra $\mathscr{G}^{(\infty)}_{ce}$. Ignoring the trivial rescaling of λ due to γ and using the results of appendix B we get from eq. (1)

$$\begin{aligned} \lambda^{-1}\delta\lambda &= \Omega'(\mathscr{P}, \delta g) - \Omega'(\mathscr{P}^{-1}, \delta h) \\ &= \Omega'(\mathscr{P}, \mathscr{P}^{-1}\delta h \mathscr{P}) - \Omega'(\mathscr{P}, \mathscr{P}^{-1}\delta\mathscr{P}) - \Omega'(\mathscr{P}^{-1}, \delta h) \\ &= \Omega'(\mathscr{P}, \mathscr{P}^{-1}\delta h \mathscr{P}) - \Omega'(\mathscr{P}^{-1}, \delta h) \end{aligned} \tag{7.2}$$

where we have used that $\Omega'(\mathscr{P}, \mathscr{P}^{-1}\delta\mathscr{P}) = 0$ since both arguments are holomorphic functions of t in some simply connected domain $\mathscr{D}_{+}$ (compare chapter 4). For the Matzner-Misner group δh is constant and we get $\delta\lambda = 0$ as required. On the other hand the only element of the Ehlers group leading to a non-vanishing δh is the Ehlers transformation with (compare eqs. (3.2, 6.8))

$$\delta h(t) = \begin{pmatrix} 0 & -\Delta t \\ \dfrac{\Delta}{t} & 0 \end{pmatrix}. \tag{7.3}$$

From eq. (B.25) we get

$$\lambda^{-1}\delta\lambda = -\frac{1}{2\pi i}\oint dt \, \mathrm{Tr}\left(\begin{pmatrix} 0 & -\Delta t \\ \dfrac{\Delta}{t} & 0 \end{pmatrix}\dot{\mathscr{P}}\mathscr{P}^{-1}\right) \tag{7.4}$$

where we can choose an arbitrary contour in $\mathscr{D}_+$ to evaluate the integral. The only singularity of the integrand is the simple pole at $t = 0$ from δh. Inserting the expansion (6.1) for $\mathscr{P}(t, x)$ we find the correct result (compare eq. (3.3 *iii*))

$$\lambda^{-1}\delta\lambda = -\tilde{\psi}. \tag{7.5}$$

Similar to the construction of $\mathscr{M}$ from $\mathscr{P}$ we can define

$$(\mathscr{M}, \mu) \equiv \tau^{(\infty)}((\mathscr{P}, \lambda^{-1})^{-1}) \circ (\mathscr{P}, \lambda^{-1}) = (\mathscr{M}, \lambda^{-2}e^{\Omega(\tau^{(\infty)}\mathscr{P}^{-1}, \mathscr{P})}) \tag{7.6}$$

where the last expression is obtained using eq. (B.5).

The expression $\Omega(\tau^{(\infty)}\mathscr{P}^{-1}, \mathscr{P})$ requires some words of explanation. In appendix B the 2-cocycle ω and hence also Ω are defined for functions holomorphic in some domain of the complex plane containing a contour encircling the origin. On the other hand $\mathscr{P}$ is (for fixed x) a function defined on a two-sheeted Riemann surface parametrized by $(s, t(s, x))$. The automorphism $\tau^{(\infty)}$ involves the substitution $t \to -\frac{1}{t}$ and hence exchanges the two sheets. For reasons of consistency we have to require

$$\Omega(\tau^{(\infty)}\mathscr{P}^{-1}, \mathscr{P}) = -\Omega(\mathscr{P}^{-1}, \tau^{(\infty)}\mathscr{P}), \qquad \text{i. e.} \quad \tau^{(\infty)}\omega = -\omega$$

(where $\tau^{(\infty)}\omega(\mathrm{A}, \mathrm{B}) \equiv \omega(\tau^{(\infty)}\mathrm{A}, \tau^{(\infty)}\mathrm{B})$). This can be achieved choosing in the formula (B.3) for ω a contour on the Riemann surface, which is invariant under $t \to -\frac{1}{t}$ up to a change of orientation. As discussed in chapter 5 the function $\mathscr{P}(t, x)$ is, for fixed x, holomorphic in a neighbourhood of the contour C^x invariant under $t \to -\frac{1}{t}$. We could choose this contour, yet, it has the disadvantage to pass through the branch points $s = \frac{1}{2(z \pm i\rho)}$ resp. $t = \pm i$, where $\mathscr{P}$ is singular as a function of s. Therefore we prefer to average over two contours (with the same orientation), exchanged and reversed under $t \to -\frac{1}{t}$, avoiding the branch points.

In chapter 3 we concluded that $\mathscr{M}$ was x-independent due to the invariance of $\partial\mathscr{P}\mathscr{P}^{-1}$ under $\tau^{(\infty)}$. The natural extension of this invariance to $\mathscr{G}_{ce}^{(\infty)}$ is

$$\partial(\mathscr{P}, \lambda^{-1}) \circ (\mathscr{P}, \lambda^{-1})^{-1} = (\partial\mathscr{P}\mathscr{P}^{-1}, 0) \tag{7.7}$$

since $\tau^{(\infty)}\mathrm{Z} = -\mathrm{Z}$.

As a consequence of eq. (7) we find $\partial\mu = 0$, i. e. not only $\mathscr{M}$ but the whole pair $(\mathscr{M}, \mu)$ is x-independent. This remarkable fact provides us with a beautiful explicit formula for λ as a function of $\mathscr{P}$

$$\lambda^2 = ce^{\Omega(\tau^{(\infty)}\mathscr{P}^{-1}, \mathscr{P})} \tag{7.8}$$

(where the constant c can be determined from the asymptotic behaviour for $|x| \to \infty$).

There are several ways to prove the validity of eq. (8). The simplest one is to prove the constancy of $\tilde{\mu}$ by relating an arbitrary pair $(\tilde{\mathcal{M}}, \tilde{\mu})$ to $(\mathbf{1}, 1)$ with some $g \in \mathrm{G}_{ce}^{(\infty)}$. In analogy to eq. (6.25) we obtain

$$(\tilde{\mathcal{M}}, \tilde{\mu}) = (g^{\mathrm{T}}, 1) \circ (g, 1) \tag{7.9}$$

and thus

$$\tilde{\mu} = e^{\Omega(g^{\mathrm{T}}, g)} . \tag{7.10}$$

Now g is analytic near $s = 0$ and therefore $\Omega(g^{\mathrm{T}}, g) = 0$ entailing $\tilde{\mu} = 1$.

APPENDIX A

THE SYMMETRIC SPACES SL(2)/SO(2) AND SL(2)/SO(1,1)

The subgroups H = SO(2) resp. SO(1, 1) of G = SL(2) are distinguished by their invariance under some involutive automorphism

$$\tau : g \to \eta^{-1}(g^{\mathrm{T}})^{-1}\eta \qquad \text{with} \qquad \eta = \begin{cases} (++) & \text{for} \quad \mathrm{SO}(2) \\ (-+) & \text{for} \quad \mathrm{SO}(1,1)\,. \end{cases} \tag{A.1}$$

On the triangular representation P from eqs. (2.6, 17) of the cosets the group SL(2) acts in a non-linear fashion through

$$\mathrm{P} \to \mathrm{P}_g = h(\mathrm{P}, g)\mathrm{P}g^{-1} \qquad g \in \mathrm{SL}(2) \tag{A.2}$$

where $h(\mathrm{P}, g) \in \mathrm{SO}(2)$ resp. SO(1,1) restores the triangular form of P_g. In contrast to P the symmetric matrix $\mathrm{M} = \mathrm{P}^{\mathrm{T}}\eta\mathrm{P} = \eta\tau(\mathrm{P}^{-1})\mathrm{P}$ transforms linearly under SL(2):

$$\mathrm{M} \to \mathrm{M}_g = g^{\mathrm{T}}\mathrm{M}g \tag{A.3}$$

On the Lie algebra sl(2) the automorphism τ induces the reflection

$$\tau : \mathrm{X} \to -\eta^{-1}\mathrm{X}^{\mathrm{T}}\eta \tag{A.4}$$

leaving so(2) resp. so(1, 1) invariant.

The Lie algebra valued 1-form $d\mathrm{PP}^{-1}$ can be decomposed according to

$$d\mathrm{PP}^{-1} = \mathscr{A} + \mathscr{I} = (\mathscr{A}_k + \mathscr{I}_k)dx^k \qquad \begin{matrix} \tau(\mathscr{A}) = \mathscr{A} \\ \tau(\mathscr{I}) = -\mathscr{I} \end{matrix} \tag{A.5}$$

with the transformation laws (induced by (2))

$$\mathscr{A} = \frac{1}{2}(d\mathrm{PP}^{-1} + \tau(d\mathrm{PP}^{-1})) \to h\mathscr{A}h^{-1} + dhh^{-1}$$

$$\mathscr{I} = \frac{1}{2}(d\mathrm{PP}^{-1} - \tau(d\mathrm{PP}^{-1})) \to h\mathscr{I}h^{-1} \tag{A.6}$$

i. e. $\mathscr{A}$ can be interpreted as connection for H whereas $\mathscr{I}$ transform H-covariantly and both are G-invariant.

In contrast to $\mathscr{I}$ the « gauge invariant » 1-form

$$\mathrm{J} = \frac{1}{2}\mathrm{M}^{-1}d\mathrm{M} = \mathrm{P}^{-1}\mathscr{I}\mathrm{P} \tag{A.7}$$

is invariant under SO(2) and transforms as

$$\mathrm{J} \to g\mathrm{J}g^{-1} \qquad \text{for} \qquad g \in \mathrm{SL}(2) \tag{A.8}$$

The invariant metric of the symmetric (pseudo-)riemannian space SL(2)/SO(2) resp. SL(2)/SO(1,1) is given by

$$ds^2 = \frac{1}{4}\,\mathrm{Tr}\,(\mathrm{M}^{-1}d\mathrm{M})^2 = \mathrm{Tr}\,\mathrm{J}^2 = \mathrm{Tr}\,\mathscr{I}^2 \tag{A.9}$$

leading to the form of the σ-model Lagrangean (2.8, 14) for M.

APPENDIX B

LIE ALGEBRA AND GROUP 2-COCYCLES

The central extension (6.16) of the Lie algebra $\mathcal{G}^{(\infty)}$ is defined by a so-called 2-cocycle of that algebra. Such a 2-cocycle is given by a bilinear function ω on $\mathcal{G}^{(\infty)}$ with the properties

$$\omega(\mathrm{A}, \mathrm{B}) = -\omega(\mathrm{B}, \mathrm{A}) \tag{B.1}$$

$$\omega([\mathrm{A}, \mathrm{B}], \mathrm{C}) + \omega([\mathrm{B}, \mathrm{C}], \mathrm{A}) + \omega([\mathrm{C}, \mathrm{A}], \mathrm{B}) = 0\,. \tag{B.2}$$

The Lie algebra (6.17) as generated in chapter 6 consists of finite Laurent series $\delta g(s)$. For those a 2-cocycle ω is given by

$$\omega(\delta g_1(s), \delta g_2(s)) = \mathrm{Res}_{s=0}\,\mathrm{Tr}\left(\delta g_1(s)\frac{d}{ds}\,\delta g_2(s)\right) = \frac{1}{2\pi i}\oint \mathrm{Tr}\left(\delta g_1(s)\frac{d}{ds}\,\delta g_2(s)\right)ds \tag{B.3}$$

where the integration path is a closed contour around the origin. The last expression for ω obviously makes sense for a much larger class of functions $\delta g_i(s)$, in particular for functions analytic in a neighbourhood of some contour encircling the origin. It is easily checked that it has the defining properties (1, 2) also in this more general situation.

It is well-known that Lie algebra cocycles are related to inhomogeneous group cocycles [25] [26]. The defining relation for such a group 2-cocycle Ω (which we write additively) is

$$\Omega(b, c) - \Omega(ab, c) + \Omega(a, bc) - \Omega(a, b) = 0 \quad \text{for} \quad a, b, c \in \mathrm{G}^{(\infty)}\,. \tag{B.4}$$

With the help of such a group 2-cocycle it is possible to define a central extension $\mathrm{G}^{(\infty)}_{ce}$ of the group $\mathrm{G}^{(\infty)}$ taking pairs $(a, \alpha) \in \mathrm{G}^{(\infty)} \times \mathbf{C}$ with the multiplication law [27]

$$(a, e^{\alpha}) \circ (b, e^{\beta}) = (ab, e^{\alpha+\beta+\Omega(a,b)}) \tag{B.5}$$

which is associative due to the defining relation (4). This extension depends only on the cohomology class of Ω, because a change of $\Omega(a, b)$ by a coboundary, i. e. an expression of the form $\phi(b) - \phi(ab) + \phi(a)$, can be absorbed into α.

In addition we require tha anti-symmetry conditions

$$\Omega(a, b) = -\Omega(a^{-1}, ab) = -\Omega(b^{-1}, a^{-1}) \tag{B.6}$$

and the additional condition

$$\Omega(e^{\mathrm{A}}, e^{v\mathrm{A}}) = 0 \quad \text{for} \quad \mathrm{A} \in \mathcal{G}^{(\infty)}, \qquad 0 \le v \le 1\,, \tag{B.7}$$

which can always be achieved adding a suitable 2-coboundary, to Ω. From eq. (6) we get

$$(a, e^{\alpha})^{-1} = (a^{-1}, e^{-\alpha})\,. \tag{B.8}$$

From any group 2-cocycle obeying (6) one obtains a Lie algebra 2-cocycle taking a, b of the form $a = 1 + \delta a$, $b = 1 + \delta b$ with δa, δb infinitesimal or equivalently by

$$\omega(\mathrm{A}, \mathrm{B}) = \frac{d^2}{dv^2}\,\Omega(e^{v\mathrm{A}}, e^{v\mathrm{B}})\,|_{v=0}\,. \tag{B.9}$$

On the other hand it is possible to invert this process and construct a group 2-cocycle Ω from a Lie algebra 2-cocycle ω. We can extend the exponential map

$$\mathrm{Exp}: \mathcal{G}^{(\infty)} \ni \mathrm{A} \to e^{\mathrm{A}} \in \mathrm{G}^{(\infty)} \tag{B.10}$$

to the centrally extended Lie algebra by

$$\mathrm{Exp}_{ce}: \mathscr{G}_{ce}^{(\infty)} \ni (\mathrm{A}, \alpha) \rightarrow e^{(\mathrm{A},\alpha)} \equiv (e^{\mathrm{A}}, e^{\alpha}) \in \mathrm{G}_{ce}^{(\infty)}. \tag{B.11}$$

The group multiplication for $\mathrm{G}^{(\infty)}$ in the parametrization (10) is given by the Baker-Campbell-Hausdorff (BCH) formula [*28*]

$$e^{\mathrm{A}} e^{\mathrm{B}} = e^{\langle \mathrm{A}:\mathrm{B} \rangle}, \qquad \mathrm{A}, \mathrm{B} \in \mathscr{G}^{(\infty)} \tag{B.12}$$

and the BCH expression $\langle \mathrm{A}:\mathrm{B} \rangle$ has the form

$$\begin{aligned} \langle \mathrm{A}:\mathrm{B} \rangle &= \mathrm{A} + \mathrm{B} + [\mathrm{A}, f(\mathrm{A},\mathrm{B})] + [-f(-\mathrm{B},-\mathrm{A}), \mathrm{B}] \\ &= \mathrm{A} + \mathrm{B} + \frac{1}{2}[\mathrm{A},\mathrm{B}] + \frac{1}{12}[\mathrm{A},[\mathrm{A},\mathrm{B}]] + \frac{1}{12}[[\mathrm{A},\mathrm{B}],\mathrm{B}] + \mathrm{O}((\mathrm{A},\mathrm{B})^4) \\ &= \mathrm{A} + \frac{\mathrm{ad\,A}}{1 - e^{-\mathrm{ad\,A}}}\mathrm{B} + \mathrm{O}(\mathrm{B}^2). \end{aligned} \tag{B.13}$$

The Lie algebra valued function $f(\mathrm{A}, \mathrm{B})$ can be expressed through iterated commutators of A and B; note that $f(\mathrm{A}, \mathrm{B})$ contains some ambiguity which does, however, not contribute to $\langle \mathrm{A}:\mathrm{B} \rangle$ due to the Jacobi identity. We choose

$$f(\mathrm{A},\mathrm{B}) = \frac{1}{4}\mathrm{B} + \frac{1}{12}[\mathrm{A},\mathrm{B}] + \mathrm{O}((\mathrm{A},\mathrm{B})^3). \tag{B.14}$$

In the same way the group multiplication for $\mathrm{G}_{ce}^{(\infty)}$ in the parametrization (11) is given by

$$(e^{\mathrm{A}}, e^{\alpha}) \circ (e^{\mathrm{B}}, e^{\beta}) = e^{\langle (\mathrm{A},\alpha):(\mathrm{B},\beta) \rangle} \tag{B.15}$$

with the BCH expression for $\mathscr{G}_{ce}^{(\infty)}$

$$\langle (\mathrm{A},\alpha):(\mathrm{B},\beta) \rangle = (\langle \mathrm{A}:\mathrm{B} \rangle, \alpha + \beta + \Omega(e^{\mathrm{A}}, e^{\mathrm{B}})) \tag{B.16}$$

where $\Omega(e^{\mathrm{A}}, e^{\mathrm{B}})$ is uniquely determined by ω and $\langle \mathrm{A}:\mathrm{B} \rangle$

$$\begin{aligned} \Omega(e^{\mathrm{A}}, e^{\mathrm{B}}) &= \omega(\mathrm{A}, f(\mathrm{A},\mathrm{B})) + \omega(-f(-\mathrm{B},-\mathrm{A}), \mathrm{B}) \\ &= \frac{1}{2}\omega(\mathrm{A},\mathrm{B}) + \frac{1}{12}\omega(\mathrm{A},[\mathrm{A},\mathrm{B}]) + \frac{1}{12}\omega([\mathrm{A},\mathrm{B}],\mathrm{B}) + \mathrm{O}((\mathrm{A},\mathrm{B})^4) \\ &= \omega\left(\mathrm{A}, \frac{e^{-\mathrm{ad\,A}} - 1 + \mathrm{ad\,A}}{\mathrm{ad\,A}(1 - e^{-\mathrm{ad\,A}})}\mathrm{B}\right) + \mathrm{O}(\mathrm{B}^2). \end{aligned} \tag{B.17}$$

This group 2-cocycle (17) satisfies the defining relation (4) because the BCH formula (15) satisfies the associative law. The conditions (6, 7) are satisfied as well.

In addition to the group 2-cocycle Ω and the Lie algebra 2-cocycle ω we need a mixed form (compare eq. (9))

$$\begin{aligned} \Omega'(e^{\mathrm{A}}, \mathrm{B}) \equiv \frac{d}{dv}\Omega(e^{\mathrm{A}}, e^{v\mathrm{B}})\,|_{v=0} &= \omega\left(\mathrm{A}, \frac{e^{-\mathrm{ad\,A}} - 1 + \mathrm{ad\,A}}{\mathrm{ad\,A}(1 - e^{-\mathrm{ad\,A}})}\mathrm{B}\right) \\ &= \int_0^1 du\,\omega\left(\mathrm{A}, \frac{1 - e^{-u\,\mathrm{ad\,A}}}{\mathrm{ad\,A}}\langle \mathrm{A}:\mathrm{B} \rangle'\right) \end{aligned} \tag{B.18}$$

with

$$\langle \mathrm{A}:\mathrm{B} \rangle' \equiv \frac{d}{dv}\langle \mathrm{A}:v\mathrm{B} \rangle\,|_{v=0} = \frac{\mathrm{ad\,A}}{1 - e^{-\mathrm{ad\,A}}}\mathrm{B}. \tag{B.19}$$

Using the properties of Ω we find

$$\Omega(e^{\mathrm{A}}, e^{\mathrm{B}}) = \Omega(e^{\mathrm{A}}, e^{v\mathrm{B}}) + \Omega(e^{\langle \mathrm{A}:v\mathrm{B} \rangle}, e^{(1-v)\mathrm{B}}) = \int_0^1 dv\,\Omega'(e^{\langle \mathrm{A}v:\mathrm{B} \rangle}, \mathrm{B}). \tag{B.20}$$

Using eq. (18) we can rewrite this in the form

$$\Omega(e^{A}, e^{B}) = \int_0^1 \int_0^1 dudv\omega(g^{-1}\partial g/\partial u, g^{-1}\partial g/\partial v) \tag{B.21}$$

where

$$g(u, v) = e^{u\langle Av:B\rangle}, \tag{B.22}$$

(with $0 < u, v < 1$) parametrizes the interior of the geodesic triangle with vertices $\{ 1, e^{A}, e^{A}e^{B} \}$. The expression (17) for the group 2-cocycle Ω can therefore be expressed in the equivalent form

$$\Omega(a, b) = \int_\Delta \omega(g^{-1}dg, g^{-1}dg) \tag{B.23}$$

where the domain of integration Δ is a geodesic triangle with vertices $\{ 1, a, ab \}$.

Using the explicit expression (3) for ω and the identity

$$dA = \frac{\operatorname{ad} A}{1 - e^{-\operatorname{ad} A}} e^{-A}de^{A} \tag{B.24}$$

we finally obtain

$$\begin{aligned} \Omega'(e^{A}, B) &= -\frac{1}{2\pi i}\oint \operatorname{Tr}\left(B \frac{e^{\operatorname{ad} A} - 1 - \operatorname{ad} A}{(e^{\operatorname{ad} A} - 1)(1 - e^{-\operatorname{ad} A})}(e^{-A}de^{A})\right) \\ &= -\frac{1}{2\pi i}\oint \operatorname{Tr}\left(B\left(\frac{1}{2} + \chi(\operatorname{ad} A)\right)(e^{-A}de^{A})\right) \end{aligned} \tag{B.25}$$

with the odd function

$$\chi(x) = \frac{e^{x} - e^{-x} - 2x}{2(e^{x} + e^{-x} - 2)}. \tag{B.26}$$

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