ANNALES DE L'I. H. P., SECTION A

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Annales de l'I. H. P., section A, tome 45, nº 2 (1986), p. 173-230

<http://www.numdam.org/item?id=AIHPA_1986__45_2_173_0>

Ann. Inst. Henri Poincaré,

Vol. 45, n° 2, 1986, p. 173-230. *Physique théorique*

Two-particle structure and renormalization

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ABSTRACT. — We present and study renormalized forms of the Bethe-Salpeter equation and of similar two-particle structure equations for the connected N-point Green functions; these equations are implied by the standard BPHZ scheme for renormalized field theories with positive mass; only the scalar neutral case (corresponding to theories of the Φ^4 type) has been considered, but our analysis is applicable to similar non-scalar cases, such as the two-dimensional massive Gross-Neveu model.

Specific asymptotic properties of the renormalized Bethe-Salpeter kernel G and of the connected four-point function F (resp. N-point functions $F^{(N)}$) emerge as natural assumptions and the leading term of the Wilson-Zimmermann expansion of the field is exhibited in a simple way in this renormalized Bethe-Salpeter formalism.

The following two construction problems are then treated (in complex momentum space):

i) Considering F as given in the axiomatic framework, construct G in terms of F.

ii) Reconstruct F in terms of G.

As in the regularized version of the Bethe-Salpeter equation, the renormalized formalism is shown to include the two-particle structural properties of the theory (analyticity, equivalence between the irreducibility of G and an asymptotic completeness equation for F).

RÉSUMÉ. — On présente et on étudie des formes renormalisées de l'équation de Bethe-Salpeter et, plus généralement, des équations de structure à deux particles pour les fonctions de Green connexes à N points; ces équations résultent du formalisme standard de B. P. H. Z. pour les théories de champs renormalisables de masses positives ; seul le cas scalaire neutre (correspondant aux théories de type Φ^4) a été considéré ici, mais l'analyse s'applique à des cas semblables non-scalaires, tels que le modèle de Gross-Neveu massif à deux dimensions.

Des propriétés asymptotiques spécifiques du noyau de Bethe-Salpeter renormalisé G et de la fonction à 4 points connexe F (resp. des fonctions à N points $F^{(N)}$) se dégagent comme des hypothèses naturelles, et le terme dominant du développement de Wilson-Zimmermann du champ est retrouvé de façon simple dans ce formalisme de Bethe-Salpeter renormalisé.

Les problèmes de construction suivants sont alors traités (dans l'espace des impulsions complexes) :

i) Considérant F comme donné dans le cadre axiomatique, déterminer G en fonction de F.

ii) Reconstruire F en fonction de G.

On montre en outre, comme dans la version régularisée de l'équation de Bethe-Salpeter, que le formalisme renormalisé contient les propriétés de structure à deux particules de la théorie (analyticité, équivalence entre l'irréductibilité de G et une équation de complétude asymptotique pour F).

1. INTRODUCTION

Although originating in perturbation theory, the formalism of irreducible (or Bethe-Salpeter type) kernels has a very rich content in the general framework of Quantum Field Theory. One feature of this formalism (first emphasized in [*1*], and rigorously studied in [*2*] [*3*]) is the fact that irreducible kernels are the cornerstone of Many-Particle Structure Analysis (at least for theories with positive masses). A second feature is the implication of these kernels in the asymptotics of renormalizable field theories. Although already discovered and exploited in [*4*] for obtaining original results on the behaviour of the Green functions at the so-called « exceptional momenta », this feature still deserves a systematic general study. The present work aims to investigate the renormalized Bethe-Salpeter equation (and similar equations for the N-point functions) implied by the standard B. P. H. Z. renormalization scheme, and to exhibit therefrom the mergence of two-particle structure and asymptotics, from a general axiomatic viewpoint. In the complex momentum-space formulation of axiomatic field theory, the two features mentioned above correspond to two complementary sets of properties of the analytic N-point Green functions, namely:

i) the local singularity structure near the complex mass shell.

ii) the singularity structure at infinity.

To a large extent, the structural properties *i*) are of universal nature: they express the consequences of the general axiomatic background of Quantum Field Theory (involving a certain formulation of locality and of the spectral condition) together with the property of Asymptotic Completeness of the Fields (the latter implying the unitarity of the S-operator). As it was proved in [*2*], the asymptotic completeness properties are most conveniently expressed in the language of analytic functions through the irreducible character of relevant Bethe-Salpeter type kernels, the general (axiomatic) definition of irreducibility being the vanishing of a certain discontinuity function (or « absorptive part ») of the considered kernel below a certain relevant mass threshold. As a consequence, the Bethe-Salpeter type equations have appeared as a powerful tool for working out the complete analytic and monodromic structure of the Green functions and (by restriction to the complex mass shell) of the multiparticle scattering amplitudes. Among various results, the following outcomes of this approach deserve to be mentioned:

a) the structural equations that one obtains are exact relativistic generalizations of the Faddeev or Weinberg-type equations for multiparticle scattering in potential theory (see [*3*]);

b) the analysis of various terms of these equations introduces at the axiomatic level the notion of Landau singularity, whose status was at first purely attached to the expansion in Feynman graphs (1).

This conceptual success of irreducible kernels in Many-Particle-Structure-Analysis (M. P. S. A.) suggests that similar investigations of axiomatic type of this formalism should also provide an insight on the *singularity structure at infinity* of the Green functions, for the class of renormalizable theories. In fact, such investigations were carried out by Symanzik (see the Appendices of [*4*]), in an approach which indicated an original way of introducing the leading term of the Wilson-Zimmermann expansion of the basic field [*6*]. For a clear understanding of this alternative aspect of the Bethe-Salpeter formalism, it is essential that special

(1) It is only in the context of pure S-matrix theory that such a non-perturbative interpretation of the occurrence of Landau singularities had been proposed (see [*5*] and references therein).

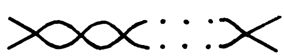
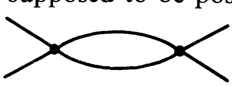
asymptotic properties (suggested by the theory of renormalization) be incorporated in the general axiomatic framework and this corresponds to a crucial distinction in the use of irreducible kernels. The universal character of M. P. S. A. was reflected by the use of *regularized* irreducible kernels, linked to the Green functions by Bethe-Salpeter type equations involving a convergent measure at infinity, and therefore valid (to a large extent) independently of the behaviour at infinity of the Green functions. Mathematically speaking, this regularization procedure played the role of a « compactification » of the integration space, which allowed one to focus the formalism on the *local* analyticity properties of the Green functions.

On the contrary, the study of properties at infinity requires the derivation and the exploitation of a *renormalized version* of the Bethe-Salpeter-type kernels and equations. This study should exhibit in a synthetic way:

i) the geometrical aspects of M. P. S. A., namely (as in the regularized formalism) the relationship between irreducibility and asymptotic completeness, and the two-sheeted analyticity domains of the Green functions around the relevant two-particle thresholds.

ii) the special structural properties at infinity, corresponding to the existence of a Wilson-Zimmermann type leading singularity for the Green functions.

Moreover the links between this renormalized formalism and the (previous) regularized one should also be clarified.

In the present paper, this program is developed under assumptions at infinity that are consistent with the perturbative B. P. H. Z. renormalization scheme. For simplicity, we only consider the case of an even scalar field with positive mass, corresponding to theories of the ϕ^4-type. However, the same analysis is applicable to more general (non-scalar) cases of renormalizable field theories, corresponding to more realistic models (from the point of view of constructive Q. F. T.), such as the massive Gross-Neveu model, recently constructed in [*7*]. In this connection, our study was also motivated by the occurrence of the disturbing « Landau phenomenon » [*8*], which was among the first facts that led people to question about the existence of the ϕ^4-theory in dimension four. Let us recall the argument. The series of Feynman graphs  , renormalized according to the B. P. H. Z. rules, can be summed up as a geometrical series, thus yielding the following contribution to the four-point function: $F_0(k) = \dfrac{g}{1 - gI_0(k)}$; here g denotes the renormalized coupling constant, supposed to be positive, and $I_0(k)$ is the renormalized one-loop integral  , (k being the total energy-momentum of the distinguished channel). Since $I_0(k)$ behaves like $\log(-k^2)$ at infinity, the function F_0 has a pole in the euclidean region (at $k^2 \sim -e^{1/g}$), which is in contradiction with the primitive analytic structure implied by the axioms of field theory.

Although the argument is non-conclusive, since based on a very small part of the perturbation series, it deals with the simplest example of a renormalized Bethe-Salpeter equation (with the kernel G equal to the constant g) and thus indicates that the merging of two-particle structure and asymptotics may generate undesirable ghosts. Thus, the question of the existence of renormalizable field theories (or of the sense in which they exist) has a connection, via analyticity properties, with the formalism of renormalized Bethe-Salpeter equation.

The plan of this article can be summarized as follows: while parts 2, 3 and 4 are devoted to the introduction of the renormalized Bethe-Salpeter formalism through a formal expansion approach based on the B. P. H. Z. renormalization scheme, the subsequent sections 5 to 8 present a « semi-axiomatic » exploitation of this formalism which exhibits simultaneoulsy properties of the types *i*) and *ii*) mentioned above, for the N-point functions of the theory.

Section 2 will fix some notations and give a recall on the use (in axiomatic Q. F. T.) of *regularized* Bethe-Salpeter equations of the form $F = G_\rho + F \circ_\rho G_\rho$, F being the four-point function, $\circ_\rho$ a certain integration operation involving a regularization procedure and G_ρ the corresponding « regularized Bethe-Salpeter » (or « two-particle irreducible ») kernel. This formalism admits a « formal expansion » (or « semi-perturbative ») aspect, which simply consists in considering F as the formal Neumann series $F = \sum_n \underbrace{G_\rho \circ_\rho \ldots \circ_\rho G_\rho}_{n}$. For the derivation of a *renormalized* Bethe-Salpeter (in short B. S.) formalism, one is led to investigate the analog of the previous formal expansion of F, G_ρ being now replaced by a « genuine » renormalized B. S. kernel, G, and each term $G_\rho \circ_\rho \ldots \circ_\rho G_\rho$ being replaced by a certain « renormalized $\mathscr{G}$-convolution » product (in the sense of [9]) which we denote by $\underbrace{[G \circ \ldots \circ G]_r}_{n}$.

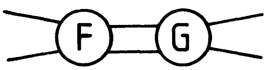
The justification of the existence of such an expansion relies on the perturbative approach of field theory, and specially on the B. P. H. Z. scheme. The various properties of the terms $\underbrace{[G \circ \ldots \circ G]_r}_{n}$ of the expansion can be studied, however, at a more general level, namely as consequences of postulated properties for G and for the two-point function $H^{(2)}$ of the theory, without referring to the subjacent perturbative level. These results will be described in section 3, supplemented by Appendices A and B. As a matter of fact, a more complete study of the asymptotic and analytic properties of the terms $[\underbrace{G \circ \ldots \circ G}_{n}]_r$, based on a precise formulation of asymptotic properties to be satisfied by G and $H^{(2)}$ will be given in separate papers [10]: our appendix A will only present a sketch of the proof of [10]. while the appendix B is devoted to a derivation (of purely algebraic nature) of « asymptotic completeness equations » for the various terms $[G \circ \ldots \circ G]_r$ of the expansion of F; these equations were first presented in [11] (in the latter, they were established for similar quantities $[G \circ_\rho \ldots \circ_\rho G]_r$ in a regularized version of the renormalized B. S. formalism).

In section 4, two kinds of Bethe-Salpeter-type identities will be derived from the previous formal expansion approach. On the one hand a « renormalized Bethe-Salpeter equation » will be obtained for the four-point function F: it contains an extra-term that involves an auxiliary function Λ, defined through a formal expansion in terms of G which is similar to that of F. On the other hand, various relations which link F to the increments of G between couples of points of momentum space are also derived (for making our presentation simpler, the proof of one of these has been rejected in Appendix C). These relations are independent of Λ and turn out to be formally identical with relations that can be derived from the « naive » or regularized B. S. equation. This fact had been essentially taken as an ansatz in the approach of [4], where all the equations involving « G » only through its increments had been considered as reliable equations for the renormalized quantities. Section 4 also gives a set of renormalized equations for the N-point Green functions in terms of the corresponding renormalized *two-particle irreducible* N-point functions; as in the four-point case, auxiliary functions Λ_N (of the same nature as Λ) are involved.

The subsequent sections of our article, devoted to the study of the renormalized two-particle structure equations from a general viewpoint, are based on the following considerations.

The problem of solving the renormalized B. S. equation either with respect to F (considering G as given in a constructive approach) or conversely with respect to G (considering F as given in an axiomatic approach) is not obviously tractable via the Fredholm theory, as it was the case for regularized B. S. equations.

As a matter of fact, the difference between the asymptotic properties of F and G (in euclidean directions of momentum space) turns out to break the symmetry of their respective roles in the B. S. equation. The regularized equations, unsensitive to the asymptotic properties of F and G_ρ, were pure Fredholm resolvent equations, algebraically symmetric with respect to F and $(-G_\rho)$. On the contrary, a complete dissymetry appears between F and G in the renormalized B. S. equation, with the presence of the auxiliary function Λ: this dissymetry is precisely supposed to express the discrepancy between the asymptotic behaviours of F and G, taken into account in the renormalized equation. This analysis of the consistency of relevant asymptotic properties of F and G with the algebraic form of the renormalized B. S. equation is carried out in section 5.

To be more specific, we can say that the existence of a renormalized B. S. formalism is linked to the possibility of extracting from F and G suitable « regular parts » having integrability properties at infinity (in euclidean directions of momentum space) in theories where F and G themselves are bounded, but *not* integrable (namely, the two particle $\mathcal{G}$-convolution integral $F \circ G =$  does not exist).

As far as G is concerned, our basic asymptotic assumption (suggested by the perturbative framework and formulated in section 3 as « property B ») asserts that the *derivatives* of G with respect to four-momentum coordinates have decrease properties at infinity which imply integrability (It is indeed, under assumptions of this type, formulated with appropriate inequalities, that the results of the formal expansion approach of section 3 and [*10*] can be established). An immediate consequence of this assumption is that various *increments* of G between couples of points of momentum space have themselves integrability properties at infinity and can play the roles of *regular parts* of G in the renormalized equations.

The asymptotic behaviour of F turns out to be completely different from that of G, and one of the purposes of sections 5 and 6 is to exhibit various four-point kernels, functionals of F (or F and G) which must enjoy regularity conditions at infinity, if the renormalized equations of section 4 are reliable.

In section 5, two asymptotic properties of F (resp. of the N-point connected Green functions $F^{(N)}$), called properties A and A′ (resp. A and A'), are inferred from the renormalized B. S. equation (resp. from the two-particle structure equations for N-point functions). Property A (resp. A) amounts to define suitable *regular parts* of F (resp. $F^{(N)}$) by subtracting from the latter a *singular factorized contribution* involving the function Λ (resp. $\Lambda^{(N)}$). Property A′ (resp. A') expresses Λ (resp. $\Lambda^{(N)}$) in terms of F (resp. $F^{(N)}$) via an appropriate limiting procedure. Moreover, this singular part of F (resp. $F^{(N)}$) involving Λ (resp. $\Lambda^{(N)}$) is shown to be interpretable as the Wilson-Zimmermann leading singularity of F (resp. $F^{(N)}$); a clear account of the connection of the latter with the renormalized B. S. formalism is thus given.

In section 6, the incremental equations derived in section 4 are exploited, and various « regular kernels », functionals of F and G, are introduced. In this study (of algebraic nature), we use as an « ansatz » the fact that each four-point function with appropriate integrability properties at infinity admits a resolvent kernel which enjoys the same integrability properties. Various relations are then established, between the regular parts of F and the resolvents of appropriate increments of G; a connection with the formalism of [*4*] is displayed in the course of this analysis (see § 6.2). On the other hand, a regularized version of the renormalized B. S. formalism (introduced by D. Iagolnitzer during the course of the present work, in [*11*]) is summarized (see § 6.3); this regularized version is shown to give a simple account of the property (used implicity in [*4*]) according to which the kernel G involved in the incremental equations can be indifferently considered as a renormalized or non-renormalized B. S. kernel (before the regularization is removed).

The various results obtained in sections 5 and 6 then allow us to treat, respectively in sections 7 and 8, the following two problems:

i) Considering F (resp. all the N-point functions $F^{(N)}$) as given in the axiomatic framework, and such that the asymptotic properties A and A′ (resp. *A* and *A*′) hold, construct the renormalized B. S. kernel G (resp. a complete set of irreducible functions $G^{(N_1,N_2)}$) in terms of F (resp. of the functions $F^{(N)}$), and show that all the requisite properties of G (resp. of the functions $G^{(N_1,N_2)}$) are satisfied, namely: analyticity, irreducibility and asymptotic properties of the type B.

ii) G and $H^{(2)}$ being considered as given, with all the requisite properties (in particular with asymptotic properties B and A_0), reconstruct F and show that it satisfies the two-particle structural properties (analyticity and A. C. equations) together with the asymptotic properties A and A′.

Each of these problems is given a unique solution, whose validity is however submitted to the « ansatz » that the resolvent kernels of « regular parts » of F or G (or of more general functionals with regularity properties at infinity) are well-defined: here, the « Landau phenomenon » may occur...

Moreover, in the computation of G in terms of F, given in section 7, an additional asymptotic assumption has to be satisfied by F: it is expressed as the existence of the limit of a certain functional of F, when a suitable regularization parameter is removed.

On the other hand, it is worth noticing that in this general approach, a proof is given (in sections 7 and 8) of the algebraic equivalence between the irreducibility of G (resp. $G^{(N_1,N_2)}$) and the validity of two-particle asymptotic completeness equations for F (resp. $F^{(N)}$). As a by-product of this proof, the function Λ (resp. $\Lambda^{(N)}$) and the regular parts of F (resp. $F^{(N)}$) are shown to satisfy themselves appropriate asymptotic completeness (or unitarity-type) equations.

2. THE BETHE-SALPETER EQUATION: REGULARIZED VERSIONS AND THE PROBLEM OF RENORMALIZATION

§ 2.1. — Let us fix some notations. In the following, F will always denote the amputated analytic four-point function, of a certain even scalar field, expressed in the variables of a given channel (1 2, 1′ 2′). More precisely, let k_1, k_2 (resp. k'_1, k'_2) be the incoming (resp. outgoing) four-momenta of this channel, we put:

$$k = k_1 + k_2 = k'_1 + k'_2, \qquad z = \frac{k_1 - k_2}{2}, \qquad z' = \frac{k'_1 - k'_2}{2} \tag{1}$$

$$k_i \equiv \hat{k}_i(k, z) = \frac{k}{2} + \varepsilon_i z, \quad k'_i \equiv \hat{k}_i(k, z'), \quad i = 1, 2, \quad \varepsilon_1 = 1, \quad \varepsilon_2 = -1 \tag{2}$$

Then if $H^{(2)}$ and $H^{(4)}$ denote respectively the analytic two-point and four-point functions (see [2*a*, *b*]), F is defined as:

$$F(k; z, z') = H^{(4)}(k_1, k_2, -k'_1, -k'_2)x \prod_{i=1,2} \{ H^{(2)}(\hat{k}_i(k, z))H^{(2)}(\hat{k}_i(k, z')) \}^{-1} \tag{3}$$

We also define for convenience the « two-line propagator function »:

$$\omega(k; z) = \frac{1}{2} H^{(2)}(\hat{k}_1(k, z))H^{(2)}(\hat{k}_2(k, z)) \tag{4}$$

In the axiomatic framework, F is analytic in a standard domain D_4 of complex (k, z, z')-space [2], which contains in particular the euclidean subspace relative to any given frame; D_4 is specified by the spectrum of the theory, which is here supposed to contain a single positive mass μ in its discrete part.

If the field satisfies the Asymptotic Completeness property (A. C.), the discontinuity ΔF of F across the two-particle cut σ: $(2\mu) \leqslant k^2 < (4\mu)^2$ (with $k^2 = k^{(0)2} - \vec{k}^2$, $k = (k^{(0)}, \vec{k})$) real, $k^{(0)} > 0$) fulfils the following equation:

$$\Delta F(k; z, z') = \frac{1}{2} \frac{Z^2}{(2\pi)^2} \int F^+(k; z, \zeta)F^-(k; \zeta, z')\delta^+_\mu(\hat{k}_1(k, \zeta))\delta^+_\mu(\hat{k}_2(k, \zeta))d_4\zeta \tag{5}$$

where $\delta^+_\mu(k_i) = \theta(k_i^{(0)})\delta(k_i^2 - \mu^2)$, the constant Z is the residue of $H^{(2)}(k_i)$ at $k_i^2 = \mu^2$, and F^+, F^- respectively refer to the boundary values of F from the sides $\operatorname{Im} k^{(0)} > 0$, $\operatorname{Im} k^{(0)} < 0$ in its primitive analyticity domain (ΔF being by definition $F^+ - F^-$).

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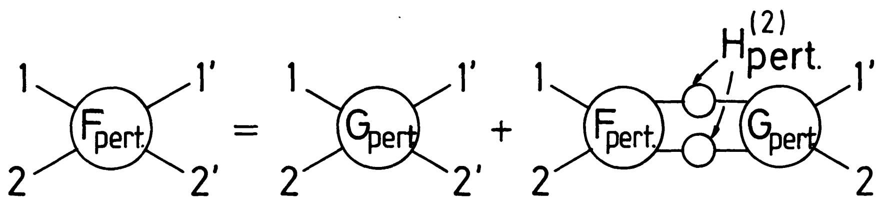
(2) See e. g. [2*a*] [*19a*] [*19b*] and the original references therein.

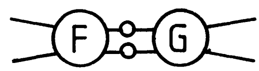
The mass shell integral at the r. h. s. of (5) runs over the S_2-sphere: $k \cdot \zeta = 0,\ -\zeta^2 = \frac{k^2}{4} - \mu^2 > 0$; as in [2d], we denote by $*$ this « mass shell convolution » and rewrite (5) in short notation:

$$\Delta F = F^+ * F^- \tag{5'}$$

§ 2.2. — The primary idea concerning the Bethe-Salpeter equation [1] expressed the ambition of considering in a general (axiomatic) sense the following identity between formal series of Feynman graphs



F_{pert} denoting the perturbative series of connected graphs and G_{pert} the corresponding subseries of two-particle irreducible graphs in the channel (12, 1′2′). If one substitutes F to F_{pert} and a hypothetic kernel G to G_{pert},

and if one applies formally the Feynman rules to 

one obtains a Fredholm-type relation between F and G, depending on the parameter k, namely:

$$F(k; z; z') = G(k; z, z') + \int F(k; z, \zeta)\omega(k; \zeta)G(k; \zeta, z')d_4\zeta \tag{7}$$

or in short:

$$F = G + F \circ G \tag{7'}$$

However, for theories involving renormalization (of the type ϕ^4 in dimension 4), equation (7) is meaningless as it is written; the integral at the r. h. s. is indeed supposed to present a logarithmic divergence, since all the graphs underlying F and G produce a degree 0 in the power counting at infinity, while $\omega = \frac{1}{2} H^{(2)}(k_1) \times H^{(2)}(k_2)$ would contribute with a degree -4. This fact was taken into account in [1] and [4], where it was said that the identity (7) had to be understood as pertaining to a « regularized » version of perturbation theory.

In the rigorous approach of Many-Particle Structure Analysis, this difficulty was overcome as follows [2]: a new measure $\omega_\rho(k, \zeta)d\zeta$ is substituted to $\omega(k, \zeta)d\zeta$, where:

$$\omega_\rho(k, \zeta) = \omega(k, \zeta) \times \chi_\rho(\hat{k}_1(k, \zeta)) \times \chi_\rho(\hat{k}_2(k, \zeta)) \tag{8}$$

and the function χ_ρ is an analytic cut-off, which has sufficient decrease properties at infinity in euclidean space and is equal to 1 on the (complex) mass shell. A choice which offers a large possibility of behaviour at infinity (it applies as well to Wightman-type theories as to local observable theories of the Araki-Haag-Kastler type) is: $\chi_\rho(k) = e^{\rho(k^2-\mu^2)}$, with $\rho > 0$.

The formal Bethe-Salpeter equation (7) is then replaced by the following « regularized Bethe-Salpeter equation » (where $\circ_\rho$ denotes the integration operation similar to that involved in Eq. (7), with ω replaced by ω_ρ):

$$\mathrm{F} = \hat{\mathrm{G}}_\rho + \mathrm{F} \circ_\rho \hat{\mathrm{G}}_\rho = \hat{\mathrm{G}}_\rho + \hat{\mathrm{G}}_\rho \circ_\rho \mathrm{F}, \tag{9}$$

The latter is a genuine k-dependent Fredholm equation which allows one to define $\hat{\mathrm{G}}_\rho(k; z, z')$ as a ratio of analytic functions $\dfrac{\mathrm{N}_\rho(k, z, z')}{\mathrm{D}_\rho(k)}$, where N_ρ and D_ρ are Fredholm series with respect to F. N_ρ has been proved to be analytic, and $\hat{\mathrm{G}}_\rho$ meromorphic, in the domain of F, the crucial property being the following one [*2*]:

PROPERTY (I). — *Let $\hat{\mathrm{G}}_\rho^+$ and $\hat{\mathrm{G}}_\rho^-$ be the boundary values of $\hat{\mathrm{G}}_\rho$, at k real, from the respective sides* $\mathrm{Im}\, k^{(0)} > 0$ *and* $\mathrm{Im}\, k^{(0)} < 0$ *in the domain* D_4 *; and let $\Delta\hat{\mathrm{G}}_\rho = \hat{\mathrm{G}}_\rho^+ - \hat{\mathrm{G}}_\rho^-$ be the corresponding discontinuity. Then the asymptotic completeness equation* (5) *for* F *is algebraically equivalent to the irreducibility of $\hat{\mathrm{G}}_\rho$, namely to the following condition*:

$$\Delta\hat{\mathrm{G}}_\rho(k; z, z') = 0 \qquad \text{for } k \text{ real such that: } (2\mu)^2 < k^2 < (4\mu)^2 \tag{10}$$

It is worth noticing that this property is shared by *all* kernels $\hat{\mathrm{G}}_\rho$ defined by (9) for any choice of the function χ_ρ. As a matter of fact, any such irreducible kernel $\hat{\mathrm{G}}_\rho$ is suitable for studying the two-particle structure of F, namely the analytic continuation properties of F around the two-particle threshold $k^2 = (2\mu)^2$ (see [*2*]). We emphasize that such a family of regularized Bethe-Salpeter kernels $\hat{\mathrm{G}}_\rho$ can be associated with *any* theory in the general axiomatic framework mentioned above. Now, it is *only* in theories without u. v. divergences that the regularized equation (9) has a meaningful limit (7) when the regularization is removed: the limit ($\rho \to 0$) of the regularized Bethe-Salpeter kernel is then uniquely defined and can be called the « exact two-particle irreducible kernel ». As shown in the $\mathrm{P}(\phi)$-models in dimension 2, this exact kernel is even directly accessible in the functional formalism [*12*], and its irreducible character can be used to establish (for these models), the property of asymptotic completeness in the two-particle region [*13*]: the property (I) quoted above is indeed applicable to this limiting equation as well.

In theories involving renormalization, one can set the following problem: define suitable conditions so that a renormalized Bethe-Salpeter equation exists, namely a relation between the connected (amputated) four-point function F, the two-point function $H^{(2)}$ and a « renormalized two-particle irreducible kernel » G. The fact that F is a functional of $H^{(2)}$ and G (only) in renormalizable theories of a certain restricted type will be established in section 3, in the framework of perturbation theory (see proposition 1).

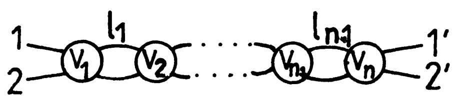
3. THE FORMAL EXPANSION OF F WITH RESPECT TO A RENORMALIZED BETHE-SALPETER KERNEL G

§ 3.1. — For any theory involving a single field with even interaction [3], the perturbative expansion of the connected and amputated four-point function F admits the following decomposition, with respect to a given channel (1, 2; 1′2′):

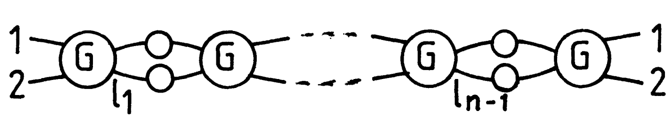
$$F = \sum_{n \geqslant 1} F_n = G + \sum_{n \geqslant 2} F_n \tag{11}$$

where the F_n correspond to the following classification of Feynman graphs.

$F_1 \equiv G$ denotes the perturbative subseries corresponding to all Feynman graphs which are « two-particle irreducible » with respect to the channel (1, 2, 1′2′); and for each $n \geqslant 2$, F_n denotes the perturbative subseries corresponding to all graphs which are « two-particle reducible of order n » with respect to the channel (1, 2; 1′2′): by definition, such a graph $\mathcal{G}$ admits a structural chain, which is a graph

$\mathcal{G}_n =$ 

containing $(n-1)$ two-lined loops $l_1, \ldots, l_{n-1}$ and n « bubble vertices » $v_1, \ldots, v_n$; at each vertex v_i a two-particle irreducible subgraph γ_i is sitting, and arbitrary propagator subgraphs are carried by the internal lines of $\mathcal{G}_n$. Considered as a formal series of Feynman graphs, F_n has obviously the factorized structure



however, it is only in a regularized formalism (or in a theory without renormalization) that F_n is genuinely a « $\mathcal{G}$-convolution integral » [*2*] [*14*], namely the n^{th}-iterated kernel associated with G:

$$F_n = \overbrace{G \circ G \ldots \circ G}^{n}; \tag{12}$$

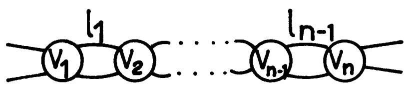
---

(3) All the N-point Green functions with N odd thus vanish.

in this case, F_n is simply the n^{th} term of the Neumann series of the Fredholm equation (7); it is purely a functional of G and (through ω) of the complete propagator $H^{(2)}$.

In the framework of renormalized perturbation theory, each F_n is defined as the formal series of all *renormalized* Feynman amplitudes corresponding to the relevant graphs (i. e. irreducible for $n = 1$, reducible of order n for $n \geqslant 2$); each individual amplitude is given by the (convergent) integral of the corresponding renormalized integrand, in the sense of [*15*]. In particular $F_1 \equiv G$ is the renormalized Bethe-Salpeter kernel (in the perturbative sense).

We now state the following:

PROPOSITION 1. — *A sufficient condition for each term* F_n $(n \geqslant 2)$ *to be purely a functional of* G *and* $H^{(2)}$ *(defined in the sense of perturbation series) is that the « renormalization parts » of the theory (in the sense of* [*15*]*) are all the proper* (4) *graphs of the two-point and four-point functions. Under this condition,* F_n *is formally a renormalized* $\mathcal{G}$*-convolution integral associated with the graph* $\mathcal{G}_n =$ 

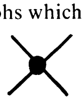
whose n vertex factors are equal to G *and whose line factors are all equal to* $H^{(2)}$.

Proof. — Let γ be an arbitrary reducible graph of order n, $\gamma_1, \dots, \gamma_n$ its two-particle irreducible components, sitting at the vertices $v_1, \dots, v_n$ of the associated chain $\mathcal{G}_n$; $\sigma_1^{(\alpha_1)}, \dots, \sigma_{n-1}^{(\alpha_{n-1})}$ will denote the propagator subgraphs carried respectively by the lines α_i of the loops l_i $(1 \leq i \leq n-1)$ of $\mathcal{G}_n$ (each α_i being a two-valued index : $\alpha_i = 1$ or 2).

The corresponding Feynman amplitude $F_n[\gamma]$ which contributes to F_n admits a renormalized integrand (see [*15*]) of the form

$$R_\gamma = \sum_U X_\gamma^{(U)}, \tag{13}$$

where each term $X_\gamma^{(U)}$ is labelled by a « forest » U of γ, namely a set of subgraphs of γ which are non-overlapping (i. e. either nested or disjoint) renormalization parts; the forest corresponding to the empty set labels the term which is the non-renormalized integrand of $F_n[\gamma]$. If the renormalization parts of the theory are supposed to be all the proper graphs

(4) We mean all the graphs which are completely one-particle irreducible, at the exclusion of the single vertex graph 

with two or four external lines and no others, then the renormalization parts in $\mathcal{G}_n$ are ([5]):

i) all proper subgraphs of $\gamma_1, \ldots, \gamma_n$ with two or four external lines (including the graphs γ_i $(1 \leqslant i \leqslant n)$ which are not single-vertex graphs).

ii) all proper subgraphs with two or four external lines of all the propagator subgraphs $\sigma_1^{(\alpha_1)}, \sigma_2^{(\alpha_2)}, \ldots, \sigma_{n-1}^{(\alpha_{n-1})}$ (including possibly the latter).

iii) All proper subgraphs of γ with four external lines which contain p consecutive components $\gamma_r, \gamma_{r+1}, \ldots, \gamma_{r+p-1}$ $(2 \leqslant p \leqslant n - r + 1)$; any such graph admits a structural chain isomorphic to $\mathcal{G}_p$, namely the subgraph $\mathcal{G}_p^{(r)}$ of $\mathcal{G}_n$ with vertices $v_r, \ldots, v_{r+p-1}$ (with $1 \leq r \leq n-1$).

It then follows that each forest U of γ can be seen as a certain forest $\mathrm{U}(\mathcal{G}_n)$ of $\mathcal{G}_n$ (namely the subset of elements of U which are of type *iii*)), completed by internal forests $\mathrm{U}_{\mathrm{int}}$ of the subgraphs $\gamma_1, \ldots, \gamma_n$ and $\sigma_1^{(\alpha_1)}, \ldots, \sigma_{n-1}^{(\alpha_{n-1})}$ (i. e. elements of U of types *i*) and *ii*)).

By writing $\sum\limits_{\mathrm{U}} = \sum\limits_{\mathrm{U}(\mathrm{G}_n)} \cdot \sum\limits_{\mathrm{U}_{\mathrm{int}}}$ in (13), and performing first the summation $\sum\limits_{\mathrm{U}_{\mathrm{int}}}$, one obtains partial renormalized integrands for the subgraphs $\gamma_1 \ldots \gamma_n, \sigma_1^{(\alpha_1)} \ldots \sigma_{n-1}^{(\alpha_{n-1})}$, which can be integrated in the corresponding internal variables and yield vertex factors $\mathrm{F}(\gamma_1), \ldots, \mathrm{F}(\gamma_n)$, and line factors $\pi(\sigma_1^{(\alpha_1)}) \ldots \pi(\sigma_{n-1}^{(\alpha_{n-1})})$.

The amplitude $\mathrm{F}_n[\gamma] = \int \mathrm{R}_\gamma$ then reduces to an integral over the loop momenta $\zeta = (\zeta_1, \ldots, \zeta_{n-1})$ of $\mathcal{G}_n$ of a certain renormalized $\mathcal{G}$-convolution integrand (see [*9*]) associated with $\mathcal{G}_n$ (namely a certain sum of terms labelled by the forests $\mathrm{U}(\mathcal{G}_n)$ of $\mathcal{G}_n$ and defined by a direct generalization of Zimmermann's prescription [*15*], in which the various propagators and vertex coupling constants are respectively replaced by general two-point and four-point functions):

$$\mathrm{F}_n[\gamma] = \int \mathcal{R}_{\mathcal{G}_n}[\mathrm{F}(\gamma_1), \ldots, \mathrm{F}(\gamma_n); \omega(\sigma_1), \ldots, \omega(\sigma_n)] d^{4(n-1)}\zeta \qquad (14)$$

in the r. h. s. of (14), the notation expresses the fact that the renormalized integrand $\mathcal{R}_{\mathcal{G}_n}$ is a (multilinear) functional of the vertex factors $\mathrm{F}(\gamma_1), \ldots, \mathrm{F}(\gamma_n)$ and of the loop factors:

$$\omega(\sigma_i) = \pi(\sigma_i^{(1)}) \cdot \pi(\sigma_i^{(2)}), \qquad 1 \leqslant i \leqslant n-1 .$$

([5]) Note that no four-legged subgraph can be overlapping with $\gamma_1, \ldots, \gamma_n$, since the latter are two-particle irreducible.

By multilinearity, it follows that summing over all individual contributions $F_n[\gamma]$ to F_n yields a similar global expression, which is the renormalized $\mathcal{G}$-convolution integral:

$$F_n = \int \mathcal{R}_{\mathcal{G}_n}[G, \ldots, G; \omega, \ldots, \omega]; \tag{15}$$

in the latter, each factor G sitting at a vertex v_i $(1 \leqslant i \leqslant n)$ of $\mathcal{G}_n$ is the perturbative series including all the two-particle irreducible amplitudes $F(\gamma_i)$, namely the Bethe-Salpeter kernel, in the perturbative sense.

Similarly, each loop factor ω is the perturbative series including all propagator amplitudes $\omega(\sigma_i)$ carried by the two lines of the loop l_i of $\mathcal{G}_n$ $(1 \leqslant i \leqslant n - 1)$, namely the « two-line complete propagator »

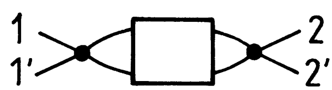
$$H^{(2)}\left(\frac{k}{2} + \zeta_i\right)H^{(2)}\left(\frac{k}{2} - \zeta_i\right)$$

(also in the perturbative sense).

§ 3.2. — We now leave the perturbative framework and will assume that there exists in the theory a four-point function G (playing the role of a « renormalized Bethe-Salpeter kernel ») and that G and the two-point function $H^{(2)}$ enjoy asymptotic properties which ensure the *convergence of the integrals* (15) *and* therefore *the existence of the functions* F_N (also denoted in section 1 by $[\underbrace{G \circ G \ldots \circ G}_{n}]_r$). If this is the case, the series (11) will then be interpretable as a formal expansion of the four-point function in terms of G and $H^{(2)}$ (i. e. a « renormalized substitute » to the Neumann series given by (12)). Adopting this viewpoint is justified to a certain extent by the result of [*9*]; in [*9*], conditions have been given for the asymptotic behaviour of vertex functions and line factors of a general renormalized integrand associated with an arbitrary graph $\mathcal{G}$, which ensure the convergence of the corresponding integral in Euclidean momentum-space. This result applies in particular to the graphs $\mathcal{G}_n$ and to the integrals (15), provided $H^{(2)}$ and G satisfy conditions of the type specified in [*9*]. These conditions prescribe power-type majorizations, not only for the functions G and $H^{(2)}$ themselves, but also for their derivatives with respect to any energy-momentum coordinate, *each derivative producing a lowering of the degree of increase at infinity by one unit.* Conditions of this type are seen to be consistent with the behaviour of individual Feynman amplitudes, provided some care is taken in their formulation.

Concerning $H^{(2)}$, the condition given in [*9*] can be adopted without modification, except that it is useful to consider it not only in the euclidean region, but in a tube-shaped neighbourhood of the latter that contains points on the real mass shell ($k^2 = \mu^2$). This condition is specified below as *property A_0*.

Concerning G, the conditions given in [*9*] turn out to be too restrictive, because the degree of decrease that they provide for the derivatives of $G(k, z, z')$ is « isotropic » with respect to all the directions of (k, z, z')-space. Such a behaviour may be appropriate for the part of G which is two-particle irreducible with respect to the three two-particle channels (12, 1′2′), (11′, 22′), (12′, 1′2). However it it not suitable for other contributions to G, such as those corresponding to all Feynman graphs of the form



,

which are pure functions of $z - z'$, and therefore have a *strongly anisotropic* behaviour with respect to the set of variables (k, z, z'). For further use throughout this paper, we are thus led to postulate below, under the name of « property B » a more general form of asymptotic behaviour for the derivatives of G, which takes into account the previous fact.

Let us first give some notations. For any complex vector k, $|k|_E$ denotes the « euclidean projection norm » of k, defined as follows: let $k = p + iq \in \mathbb{C}^4$, $p = (p^{(0)}, \vec{p})$, $q = (q^{(0)}, \vec{q})$;

$$|k|_E = (q^{(0)2} + \vec{p}^2)^{1/2} \tag{16}$$

We also introduce the tube-shaped regions:

$$T_{m,\eta} = \{ k ; \vec{q}^2 \leqslant \eta^2, |p^{(0)}| \leqslant m \} \tag{17}$$

($T_{0,0}$ being the euclidean subspace). If η is sufficiently small, conditions of the type $k_i \in T_{m_i,\eta}$ satisfied by a suitable set of vectors k_i involved in the argument of any N-point Green function will always ensure (for suitable values of m_i) that the corresponding argument stays inside the axiomatic analyticity domain of this function [*2b*] [*19*]. Finally, the notation $D_\alpha^{(\nu)} f$ refers to a derivative of total order ν with respect to an arbitrary set of components of the vectors involved in the argument of the function f.

We postulate :

PROPERTY A_0. — The two-point function $H^{(2)}$ satisfies bounds of the following type:

$$|H^{(2)}(k)| \leqslant C_0 \chi (1 + |k|_E^2)(1 + |k|_2^2)^{-1} \tag{18}$$

$$|D_\alpha^{(\nu)} H^{(2)}(k)| \leqslant C_0^{(\nu)} \chi (1 + |k|_E^2)(1 + |k|_E^2)^{-1-\nu/2} \tag{19}$$

These bounds are valid in a region of the form $T_{3m,3\eta}$ (with $m < \mu$, $\eta < \mu$), outside a bounded neighbourhood of the polar set ($k^2 = \mu^2$) of $H^{(2)}$, and for an appropriate choice of the constants C_0, $C_0^{(\nu)}$ and of the function χ; the latter is supposed to be dominated by every power $|k|_E^\varepsilon(\varepsilon > 0)$ for $|k|_E \to \infty$.

PROPERTY B. — The Bethe-Salpeter kernel G and its derivatives $D_\alpha^{(\nu_k,\nu_z,\nu_{z'})}G$ of respective total orders ν_k, ν_z, $\nu_{z'}$ with respect to the components of k, z and z' satisfy bounds of the following form:

$$|G| \leq C\underline{\chi} \tag{20a}$$

$$|D_\alpha^{(\nu_k,\nu_z,\nu_{z'})}G| \leq C_\alpha\underline{\chi}[1 + h_k]^{-\nu_k}[1 + h_z]^{-\nu_z}[1 + h_{z'}]^{-\nu_{z'}} \tag{20b}$$

In these formulae, h_k, h_z and $h_{z'}$ denote positive homogeneous functions of degree 1 of (k, z, z'); the restrictions of the latter at (k, z) (resp. (k, z')) fixed are supposed to be non constant, namely to behave at infinity like $C \times z'$ (resp. $C \times z$); $\underline{\chi}$ is a function of degree zero, which can be chosen as follows:

$$\underline{\chi}(k, z, z') = \chi(1 + |k|_E^2)\chi(1 + |z|_E^2)\chi(1 + |z'|_E^2), \tag{21}$$

χ being of the same type as in property A_0;

C and $\{C_\alpha\}$ are suitable constants.

Moreover, the bounds (20a) and (20b) are assumed to hold for (k, z, z') varying in the union D of a family of domains D_ε of the form:

$$D_\varepsilon = T_{4\mu-\varepsilon,\eta} \times T_{\mu,\eta} \times T_{\mu,\eta},$$

with (for instance) $0 < \varepsilon < \mu$ and $\eta = \eta(\varepsilon)$ ($\lim_{\varepsilon\to 0} \eta(\varepsilon) = 0$).

Remarks. — *i*) The functions h_k, h_z, $h_{z'}$ involved in this property remain unspecified in the rest of the paper. It is only for the results described in this section, that more precise assumptions on their asymptotic behaviour are necessary (see [*10*]).

ii) All the bounds expressed by formulae (18) ... (21) are supposed to reproduce reasonable properties of a scalar renormalizable theory of the ϕ^4-type: in perturbation theory, all Feynman amplitudes of the four-point (resp. two-point) function have a degree of increase equal to zero (resp. -2) up to powers of logarithms; these corrections, which are « of degree 0 », are represented by the function χ in formulae (18) ... (21), and will not play any specific role in the following results; so by a slight abuse of language, we shall summarize formulae (18) ... (21) by saying that G, $D_\alpha^{(1)}G$, $H^{(2)}$, $D_\alpha^{(1)}H^{(2)}$, etc... are respectively of degree 0, -1, -2, -3 etc...

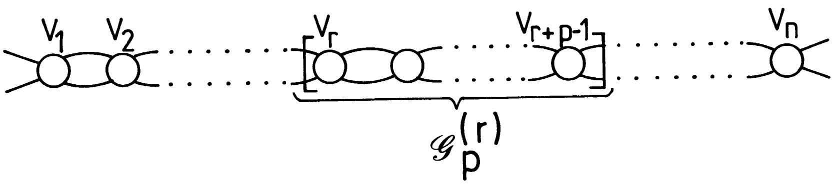
Concerning the integrals (15), one first obtains the following result in euclidean space [*9*] [*10*]:

PROPOSITION 2. — *Let* $H^{(2)}$ *and* G *satisfy properties* A_0 *and* B. *Then,* *under suitable specification of the functions h_k, h_z, $h_{z'}$ of property* B *(corresponding to the assumptions of* [*9*] *or of* [*10*]*), all the functions:*

$$\mathrm{F}_n(k, z, z') = \int \mathcal{R}_{\mathcal{G}_n}[\underbrace{\mathrm{G}, \ldots, \mathrm{G}}_{n}; \underbrace{\omega, \ldots, \omega}_{n-1}](kzz'; \zeta_1 \ldots \zeta_{n-1})d\zeta_1 \ldots d\zeta_{n-1} \tag{22}$$

are well-defined as renormalized $\mathcal{G}$-convolution integrals in euclidean space.

In the latter the renormalized integrand $\mathcal{R}_{\mathcal{G}_n}$ is specified by attributing the dimension zero to each (four-point) renormalization part, namely to each subgraph $\mathcal{G}_p^{(r)}$ of $\mathcal{G}_n$ $(2 \leqslant p \leqslant n; 1 \leqslant r \leqslant n - p + 1)$ of the following form:



We recall that (as for the perturbative integrand of Zimmermann [*15*]), in the definition of the renormalized integrand, a renormalization prescription is fixed, namely the choice of a « renormalization point » (k_0, z_0, z'_0), at which the Taylor expansion is taken. By construction, the functions F_n thus obtained satisfy the conditions:

$$\forall n \geqslant 2, \qquad \mathrm{F}_n(k_0; z_0, z'_0) = 0\,. \tag{23}$$

If we choose k_0, z_0, z'_0 equal to zero, the integral (22) has the following form for the lowest values of n ($n = 2$ and 3)

$$\mathrm{F}_2(k; z, z') = \int d\zeta\, [\mathrm{G}(k; z, \zeta)\omega(k, \zeta)\mathrm{G}(k; \zeta, z') - \mathrm{G}(o; o, \zeta)\omega(o, \zeta)\mathrm{G}(o; \zeta, o)]$$

$$\begin{aligned}\mathrm{F}_3(k, z, z') = \int d\zeta_1 d\zeta_2 \{ & \mathrm{G}(k, z, \zeta_1)\omega(k; \zeta_1)\mathrm{G}(k; \zeta_1, \zeta_2)\omega(k, \zeta_2)\mathrm{G}(k; \zeta_2, z') \\ & - \mathrm{G}(o; o, \zeta_1)\omega(o, \zeta_1)\mathrm{G}(o; \zeta_1, \zeta_2)\omega(o, \zeta_2)\mathrm{G}(o; \zeta_2, o) \\ & - [\mathrm{G}(k; z, \zeta_1)\omega(k, \zeta_1) - \mathrm{G}(o; o, \zeta_1)\omega(o, \zeta_1)]\mathrm{G}(o; o, \zeta_2)\omega(o, \zeta_2)\mathrm{G}(o; \zeta_2, o) \\ & \mathrm{G}(o; o, \zeta_1)\omega(o, \zeta_1)\mathrm{G}(o; \zeta_1, o)[\omega(k, \zeta_2)\mathrm{G}(k; \zeta_2, z') - \omega(o, \zeta_2)\mathrm{G}(o; \zeta_2, o)] \}\end{aligned}$$

For F_2, the two terms in the integrand correspond respectively to the forests ϕ and $\{ \mathcal{G}_2 \}$. For F_3, the six terms in the integrand correspond respectively to the forests ϕ, $\{ \mathcal{G}_3 \}$, $\{ \mathcal{G}_2^{(2)} \}$, $\{ \mathcal{G}_3, \mathcal{G}_2^{(2)} \}$, $\{ \mathcal{G}_2^{(1)} \}$, $\{ \mathcal{G}_3, \mathcal{G}_2^{(1)} \}$; each term is preceded by the sign $(-1)^{|\mathrm{U}|}$, where $|\mathrm{U}|$ denotes the number of subgraphs in the corresponding forest U.

Since each function F_n is a homogeneous functional of G with degree equal to n, the expression (11) of F can now be considered as a « formal expansion with respect to G », namely (with the choice of the renormalization point at $(k_0; z_0, z'_0)$) a formal series with respect to the parameter $g = G(k_0; z_0, z'_0)$. We note that, in view of (23), this series reduces to its first term at the renormalization point:

$$F(k_0; z_0, z'_0) = G(k_0; z_0, z'_0) = g \tag{24}$$

§ 3.3. — The study of the functions F_n can be extended from the euclidean space to a complex domain containing the « elastic » two-particle region, and yields the following.

PROPOSITION 3. — *Under the asymptotic conditions of proposition 2, assumed to be valid in the domains specified in properties* A_0 *and* B, *the two-particle irreducibility of* G *implies the following properties for the functions* F_n:

i) Analyticity in a ramified domain $\tilde{D}$ *containing the two-particle physical region*: $(2\mu)^2 < k^2 < (4\mu)^2$, *and two-sheeted around the threshold* $k^2 = (2\mu)^2$, *the first sheet of* $\tilde{D}$ *being the « cut domain »* $D \backslash \{ (k, z, z'); k^2 = (2\mu)^2 + \rho, \rho \geq 0 \}$.

ii) Asymptotic behaviour at infinity in D: *each* F_n *has the degree of increase zero* [6] *in the variables* z, z'.

iii) « Asymptotic completeness equations » in the sense of the formal expansion (11) *of* F, *namely the following set of discontinuity formulae*:

$$\forall n \geq 2, \qquad \Delta F_n = \sum_{1 \leqslant p \leqslant n-1} [(F_p)^+ * (F_{n-p})^-] \tag{25}$$

The proof of the points *i*) and *ii*) of this statement will only be sketched in this paper; as a matter of fact, the result of [*16*] concerning the analyticity of renormalized $\mathcal{G}$-convolution integrals in the primitive axiomatic domain yields only a part of *i*), since it does not allow analytic continuation in the neighbourhood of the two-particle physical region. It turns out that a more explicit treatment of the renormalized integrals (22) can be given which applies to all geometric situations involving a *local* contour deformation, as it is the case in the present exploration of the two-particle region. This improved treatment, which yields properties *i*) and *ii*) is fully presented in [*10*] and outlined here in Appendix A; the method also allows a derivation of Eq. (25) to be given: this proof of *iii*) is presented in Appendix B.

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(6) With the same abuse of language as for G, i. e. up to subdominant factors of the same type as the function χ occuring in properties A_0 and B.

4. IDENTITIES BETWEEN FORMAL SERIES

§. 4.1. The four-point function.

Throughout this section, the renormalization point is supposed to be fixed at the origin, so that Eqs. (23) (24) become:

$$\forall n \geqslant 2, \qquad F_n(o; o, o) = 0 \tag{23'}$$

$$F(o; o, o) = G(o; o, o) = g \tag{24'}$$

Together with the formal series of F (see Eqs. (11) (15)), we now introduce the following auxiliary series:

$$\Lambda(k, z) = 1 + \sum_{n \geqslant 1} \Lambda_n(k, z) \tag{26}$$

where:

$$\Lambda_n(k, z) = \int \mathscr{R}_{\mathscr{G}_{n+1}}[\underbrace{G, \dots, G}_{n}, 1; \underbrace{\omega, \dots, \omega}_{n}](k, z; \zeta_1, \dots, \zeta_n) d\zeta_1 \dots d\zeta_n \tag{27}$$

is a renormalized $\mathscr{G}$-convolution integral associated with the graph $\mathscr{G}_{n+1}$, and involving only n vertex factors G, the $(n+1)^{\text{th}}$ vertex factor (sitting at the right side) being constant and equal to 1. The existence and analyticity properties of these integrals are established in the same way as those of the functions F_n, under the same assumptions on $H^{(2)}$ and G (namely those of propositions 2 and 3). We notice that the functions Λ_n do not depend on z', since their right-side vertex factor is a constant. The renormalization prescription which led to Eq. (23), also yields:

$$\forall n \geqslant 1, \qquad \Lambda_n(o, o) = 0 \tag{28}$$

and

$$\Lambda(o, o) = 1 . \tag{29}$$

We shall now prove:

PROPOSITION 4. — *The following two identities hold in the sense of formal expansions with respect to* G:

$$F(k; z, z') = G(k; z, z') + \int [F(k; z, \zeta)\omega(k, \zeta)G(k; \zeta, z') - \Lambda(k, z)F(o; o, \zeta)\omega(o, \zeta)G(o; \zeta, o)] d\zeta \tag{30}$$

$$\Lambda(k, z) = 1 + \int [F(k; z, \zeta)\omega(k, \zeta) - \Lambda(k, z)F(o; o, \zeta)\omega(o, \zeta)] d\zeta \tag{31}$$

Proof. — In view of the definitions (11), (26) and of the homogeneity (of degree n) of the functions F_n and Λ_n with respect to G, Eqs. (30), (31) are respectively equivalent to the following sets of formulae:

$$F_1(k; z, z') = G(k; z, z'),$$

$$\forall n \geqslant 2, \quad F_n(k; z, z') = \int \Big[F_{n-1}(k; z; \zeta)\omega(k, \zeta)G(k; \zeta, z') - \sum_{1 \leqslant p \leqslant n-1} \Lambda_{n-p-1}(k, z)F_p(o; o, \zeta)\omega(o, \zeta)G(o; \zeta, o) \Big] d\zeta \quad (32)$$

$$\forall n \geqslant 1, \quad \Lambda_n(k, z) = \int \Big[F_n(k; z, \zeta)\omega(k, \zeta) - \sum_{1 \leqslant p \leqslant n} \Lambda_{n-p}(k, z)F_p(o; o, \zeta)\omega(o, \zeta) \Big] d\zeta \quad (33)$$

(in these formulae, the notation $\Lambda_0 = 1$ has been adopted).

To establish Eq. (32) for a given n, one makes a suitable partition of the set of terms of the integrand (7) $\mathcal{R}_{\mathcal{G}_n}$ of F_n, these terms being labelled by the forests of $\mathcal{G}_n$. The latter can be classified according to the situation of the right-side loop l_{n-1} of $\mathcal{G}_n$, whose momentum variable ζ_{n-1} is now called ζ. A first subset U_0 consists of all the forests of $\mathcal{G}_n$ whose *no* subgraph contains the loop l_{n-1}. The part of the integrand $\mathcal{R}_{\mathcal{G}_n}$ which corresponds to U_0 can clearly be written globally as:

$$\mathcal{R}_{\mathcal{G}_{n-1}}(k; z, \zeta; \zeta_1, \ldots, \zeta_{n-2})\omega(k, \zeta)G(k; \zeta, z'),$$

which yields (by integration on $\zeta_1, \ldots, \zeta_{n-2}$) the first term of the integrand in (32).

Let now U_p ($1 \leqslant p \leqslant n-1$) be the set of forests of $\mathcal{G}_n$ whose smallest subgraph containing the loop l_{n-1} is fixed and contains exactly the p loops $l_{n-p}, l_{n-p+1}, \ldots, l_{n-1}$; this subgraph is $\mathcal{G}_{p+1}^{(n-p)}$. The part of the integrand $\mathcal{R}_{\mathcal{G}_n}$ which corresponds to U_p can then be globally written as follows:

$$\lambda_{n-p-1}(k, z; \zeta_1, \ldots, \zeta_{n-p-1})\mathcal{R}_{\mathcal{G}_p}(o; o, \zeta; \zeta_{n-p}, \ldots, \zeta_{n-2})\omega(o, \zeta)G(o; \zeta, o) \quad (34)$$

where the factor λ_{n-p-1} corresponds to a sum over all the forests of the reduced subgraph $\mathcal{G}_n/\mathcal{G}_{p+1}^{(n-p)}$; each term of this sum involves a factor equal to 1 at the vertex formed by the contraction of $\mathcal{G}_{p+1}^{(n-p)}$ and $(n-p-1)$ vertex factors G elsewhere; so λ_{n-p-1} is a renormalized $\mathcal{G}$ convolution integrand of the form (27) (except that n is replaced by $n-p-1$); its integral over $\zeta_1, \ldots, \zeta_{n-p-1}$ exists and is equal to $\Lambda_{n-p-1}(k, z)$. Thus the integral of (34) over all the variables $\zeta_1, \ldots, \zeta_{n-2}$ yields the term of order p of the integrand in formula (32) (note that in the special case $p = n-1$, λ_{n-p-1} is replaced by 1). Finally, the sum of all the terms under the integration sign in (32) is integrable in ζ in view of Fubini's theorem, since

(7) For simplicity, we have replaced the complete notation $\mathcal{R}_{G_n}[G \ldots G; \omega \ldots \omega]$ by $\mathcal{R}_{G_n}$.

this sum is by construction equal to $\int \mathscr{R}_{\mathscr{G}_n}(k; z, z'; \zeta_1, \ldots, \zeta_{n-2}, \zeta) d\zeta_1 \ldots d\zeta_{n-2}$, and since $\mathscr{R}_{\mathscr{G}_n}$ is integrable in $(\zeta_1, \ldots, \zeta_{n-2}, \zeta)$. A quite similar analysis gives the proof of formulae (33). Q. E. D.

By subtracting from Eq. (32) the restriction of this equation at a special value z'_0 of z', one immediately obtains the following.

COROLLARY. — *The following identity (independent of* Λ*) holds, for any value* z'_0 *(in the sense of formal expansions with respect to* G*):*

$$\mathrm{F}(k; z, z') - \mathrm{F}(k; z, z'_0) = \mathrm{G}(k; z, z') - \mathrm{G}(k; z, z'_0) + \int \mathrm{F}(k; z, \zeta)\omega(k, \zeta)[\mathrm{G}(k; \zeta, z') - \mathrm{G}(k; \zeta, z'_0]d\zeta \quad (35)$$

The following proposition also holds; its proof is given in Appendix C.

PROPOSITION 5. — *The following identity (independent of* Λ*) holds, for any value* k_0 *of the variable* k *(in the sense of formal expansions with respect to* G*):*

$$\begin{aligned}
\mathrm{F}(k; z, z') - \mathrm{F}(k_0; z, z') &= \mathrm{G}(k; z, z') - \mathrm{G}(k_0; z, z') \\
&+ \int \mathrm{F}(k_0; z, \zeta)[\omega(k, \zeta) - \omega(k_0, \zeta)]\mathrm{F}(k; \zeta, z')d\zeta \\
&+ \int \mathrm{F}(k_0; z, \zeta_1)\omega(k_0, \zeta_1)[\mathrm{G}(k; \zeta_1, \zeta_2) - \mathrm{G}(k_0; \zeta_1, \zeta_2)]\omega(k, \zeta_2) \times \mathrm{F}(k; \zeta_2, z')d\zeta_1 d\zeta_2 \\
&+ \int \mathrm{F}(k_0; z, \zeta)\omega(k_0, \zeta)[\mathrm{G}(k; \zeta, z') - \mathrm{G}(k_0; \zeta, z')]d\zeta \\
&+ \int [\mathrm{G}(k; z, \zeta) - \mathrm{G}(k_0; z, \zeta)]\omega(k, \zeta)\mathrm{F}(k; \zeta, z')d\zeta. \quad (36)
\end{aligned}$$

§. 4.2. The n-point functions.

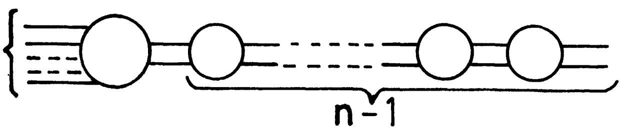
We shall now establish relations similar to Eqs. (30) (31), which link together the N-point connected and amputated functions $\mathrm{F}^{(\mathrm{N})}$(N even), appropriate two-particle irreducible N-point kernels $\mathrm{G}^{(\mathrm{N}_1, \mathrm{N}_2)}$, with $\mathrm{N}_1 + \mathrm{N}_2 = \mathrm{N}$, auxiliary functions $\Lambda^{(\mathrm{N})}$ (similar to Λ), and the previous four-point kernels F and G. These relations will be the « renormalized counterparts » of the regularized equations studied in [2*b*], namely:

$$\forall \mathrm{N} \neq 4, \qquad \mathrm{F}^{(\mathrm{N})} = \mathrm{G}_\rho^{(2, \mathrm{N}-2)} + \mathrm{F} \circ_\rho \mathrm{G}_\rho^{(2, \mathrm{N}-2)} = \mathrm{G}_\rho^{(2, \mathrm{N}-2)} + \hat{\mathrm{G}}_\rho \circ_\rho \mathrm{F}^{(\mathrm{N})} \quad (37)$$

$$\forall \mathrm{N}_1, \qquad \mathrm{N}_2 \neq 2, \qquad \mathrm{F}^{(\mathrm{N})} = \mathrm{G}_\rho^{(\mathrm{N}_1, \mathrm{N}_2)} + \mathrm{F}^{(\mathrm{N}_1 + 2)} \circ_\rho \mathrm{G}_\rho^{(2, \mathrm{N}_2)} \quad (38)$$

(these relations define recurrently the regularized kernels $G_\rho^{(N_1,N_2)}$, once the Bethe-Salpeter kernel $\hat{G}_\rho$ has been defined).

Let us introduce the following set of graphs:

$\mathscr{G}_n^{N_1} = N_1$  ,

with $n \geq 2$, and call $z^{(N_1)}$ the set of internal momenta corresponding to the N_1 incoming lines. By an argument of renormalized perturbation theory similar to the proof of proposition 1, one shows that the following formulae hold formally:

$$F_1^{(N_1+2)} = G^{(N_1,2)}, \tag{39}$$

$$\forall n \geq 2, \qquad F_n^{(N_1+2)}(k; z^{(N_1)}, z')$$

$$= \int \mathscr{R}_{\mathscr{G}_n^{N_1}}[G^{(N_1,2)}, \underbrace{G, \ldots, G}_{n-1}; \underbrace{\omega, \ldots, \omega}_{n-1}](k; z^{(N_1)}, z'; \zeta_1, \ldots, \zeta_{n-1})d\zeta_1 \ldots d\zeta_{n-1} \tag{39'}$$

where $F_n^{(N_1+2)}$ denotes the formal series of all Feynman (N_1+2)-point amplitudes associated with two-particle reducible graphs of order n with respect to a certain $(N_1, 2)$-channel, and $G^{(N_1,2)}$ represents the corresponding formal series associated with all two-particle irreducible graphs; k still denotes the total energy-momentum of the channel.

The convergence of the renormalized $\mathscr{G}$-convolution integrals (39') as well as the analyticity properties and Asymptotic Completeness relations satisfied by the functions $F_n^{(N_1+2)}$ could be obtained as in propositions 2 and 3, on the basis of the same asymptotic properties, the functions $G^{(N_1,2)}(k; z^{(N_1)}, z')$ being assumed to be regular at infinity in the two-particle internal momentum z'.

Together with the formal series:

$$F^{(N_1+2)} = G^{(N_1,2)} + \sum_{n\geq 2} F_n^{(N_1+2)} \tag{40}$$

We also define:

$$\Lambda^{(N_1+1)} = \sum_{n\geq 1} \Lambda_n^{(N_1+1)} \tag{41}$$

where:

$$\Lambda_n^{(N_1+1)}(k, z^{(N_1)})$$

$$= \int \mathscr{R}_{\mathscr{G}_{n+1}^{N_1}}[G^{(N_1,2)}, \underbrace{G, \ldots, G}_{n-1}, 1; \underbrace{\omega, \ldots, \omega}_{n}](k; z^{(N_1)}; \zeta_1 \ldots \zeta_n)d\zeta_1 \ldots d\zeta_n \tag{42}$$

We can now state:

PROPOSITION 6. — *The following sets of identities hold in the sense of formal expansions with respect to* G:

$\forall N_1$ *even*, $N_1 > 2$:

$$F^{(N_1+2)}(k; z^{(N_1)}, z') = G^{(N_1,2)}(k; z^{(N_1)}, z') + \int [F^{(N_1+2)}(k; z^{(N_1)}, \zeta)\omega(k, \zeta)G(k; \zeta, z') - \Lambda^{(N_1+1)}(k; z^{(N_1)})F(o; o, \zeta)\omega(o, \zeta)G(o; \zeta, o)]d\zeta \quad (43)$$

$$\Lambda^{(N_1+1)}(k; z^{(N_1)}) = \int [F^{(N_1+2)}(k; z^{(N_1)}, \zeta)\omega(k, \zeta) - \Lambda^{(N_1+1)}(k; z^{(N_1)})F(o; o, \zeta)\omega(o, \zeta)]d\zeta \quad (44)$$

The proof is similar to that of proposition 4.

Finally, we quote for completeness the identities obtained for the functions $G^{(N_1,N_2)}$, with $N_1 > 2$, $N_2 > 2$; as it is checked in the renormalized perturbative framework, these identities are identical with those of the regularized formalism (no extra-function $\Lambda^{(N)}$ is needed), namely:

$$F^{(N_1+N_2)}(k; z^{(N_1)}, z'^{(N_2)}) = G^{(N_1,N_2)}(k; z^{(N_1)}, z'^{(N_2)}) + \int F^{(N_1+2)}(k; z^{(N_1)}, \zeta)\omega(k, \zeta)G^{(2,N_2)}(k; \zeta, z'^{(N_2)})d\zeta \quad (45)$$

5. RENORMALIZED TWO-PARTICLE STRUCTURE EQUATIONS

§. 5.1. The renormalized Bethe-Salpeter equation.

The identities (30), (31) of section 4 have been established in the sense of formal expansions with respect to G under the conditions of propositions 1, 2, 3 (including in particular the asymptotic properties (A_0) and (B) of $H^{(2)}$ and G). We shall now assume that these identities are genuine relations between F, G and an auxiliary function Λ; in particular the integrals of formulae (30), (31) are supposed to converge, as this is the case (see proposition 4) for the integrals (32), (33) of the corresponding formal expansions. In this subsection, all vectors are in euclidean space and the integrals of formulae (30), (31) are taken over euclidean space $E_{(\zeta)}$.

We will show that if $H^{(2)}$ and G satisfy properties (A_0) and (B), formulae (30), (31) make sense provided F satisfies a special asymptotic property (A), involving the auxiliary function Λ (which can be checked in the formal expansions of section 4). Moreover a natural definition of Λ in terms of F can be given, provided F satisfies another asymptotic property (A′).

We first notice that properties (A_0) and (B) imply bounds of the following type for the various increments of the functions $H^{(2)}$ and G in euclidean space:

$$\left| H^{(2)}\left(\frac{k}{2}+z\right)-H^{(2)}\left(\frac{k_0}{2}+z\right)\right| \leqslant C^{(1)}_{k,k_0}(1+|z|^2_E)^{-3/2}\hat{\chi}(1+|z|^2_E) \quad (46)$$

$$\begin{aligned}&| G(k; z, z') - G(k_0, z, z') | \\ &\quad \leqslant \inf\left[C^{(2)}_{k,k_0,z}(1+|z'|^2_E)^{-1/2}\hat{\chi}(1+|z'|^2_E), C^{(2)}_{k,k_0,z'}(1+|z|^2_E)^{-1/2}\hat{\chi}(1+|z|^2_E)\right]\end{aligned} \quad (47)$$

$$| G(k; z, z')-G(k; z, z'_0) | \leqslant C^{(3)}_{k,z',z'_0}[1+|z|^2_E]^{-1/2}\hat{\chi}(1+|z|^2_E) \quad (48)$$

$$| G(k; z, z')-G(k; z_0, z') | \leqslant C^{(3)}_{k,z,z_0}[1+|z'|^2_E]^{-1/2}\hat{\chi}(1+|z'|^2_E) \quad (49)$$

where the « constants » $C^{(i)}\ldots$ depend on the subscript variables, and the function $\hat{\chi}$ denotes a function « of degree 0 », similar to χ. The inequality (46) follows directly from the bound (19) (for $\nu = 1$) by integration. Similarly the inequalities (47), (48), (49) follow from the bounds (20 b) (written respectively for $(\nu_k, \nu_z, \nu_{z'}) = (1,0,0)$, $(0,1,0)$, $(0,0,1)$) by integrating the latter in the respective variables, k, z, z' and taking into account the properties of the functions h_k, h_z, $h_{z'}$ at fixed values of (k, z) and (k, z'). In view of the definition (4) of ω, the following inequalities follow respectively from the bounds (18) and (46):

$$| \omega(k, z) | \leqslant C^{(4)}_k(1+|z|^2_E)^{-2}\hat{\chi}(1+|z|^2_E) \quad (50)$$

and

$$| \omega(k, z) - \omega(k_0, z) | \leqslant C^{(5)}_{k,k_0}(1+|z|^2_E)^{-5/2}\hat{\chi}(1+|z|^2_E) \quad (51)$$

Finally, the inequalities (47) (48) and (51) imply (in view of (20) and (50)):

$$\begin{aligned}&| \omega(k, z)G(k; z, z') - \omega(k_0, z)G(k_0; z, z'_0) | \\ &\qquad < C_{k,k_0,z',z'_0}(1+|z|^2_E)^{-5/2}\hat{\chi}(1+|z|^2_E)\end{aligned} \quad (52)$$

For simplicity, we use the following terminology:

i) A measure of the form $\nu(k, z)d_4z$, with ν analytic in k, z, is said to be of degree 0 (at infinity) in z if there exist constants $C_{k\varepsilon}$ such that:

$$\forall\varepsilon > 0, \qquad |\nu(k, z)| \leqslant C_{k\varepsilon}(1+|z|_E)^{-4+\varepsilon}.$$

ii) A k-dependent kernel $K(k; z, z')$ is said to satisfy the « right-sided regularity property » (resp. « left-sided regularity ») if, for k and z fixed (resp. k and z'), K is integrable over euclidean space with respect to any measure of degree 0 in the variable z' (resp. z). K is said to be both-sided-regular if it both satisfies the right-sided and the left-sided regularity properties. In view of the inequalities (50) and (47), (48), (49), the measure $\underline{\omega}(k, \zeta) = \omega(k, \zeta)d\zeta$ is of degree zero in ζ, and various increment kernels associated with G satisfy accordingly the left-sided, right-sided and both sided regularity properties.

We can rewrite the integrand occurring at the r. h. s. of Eq. (30) in either one of the following two forms:

$$[\mathrm{F}(k; z, \zeta) - \Lambda(k, z)\mathrm{F}(o; o, \zeta)]\omega(k, \zeta)\mathrm{G}(k; \zeta, z')d\zeta \\ + \Lambda(k, z)\mathrm{F}(o; o, \zeta)[\omega(k, \zeta)\mathrm{G}(k; \zeta, z') - \omega(o, \zeta)\mathrm{G}(o; \zeta, o)]d\zeta \quad (53)$$

and

$$\left[\mathrm{F}(k; z, \zeta) - \Lambda(k, z)\mathrm{F}(o; o, \zeta)\frac{\omega(o, \zeta)}{\omega(\mathrm{k}, \zeta)}\right]\omega(k, \zeta)\mathrm{G}(k; \zeta, z')d\zeta \\ + \Lambda(k, z)\mathrm{F}(o; o, \zeta)\omega(o, \zeta)[\mathrm{G}(k; \zeta, z') - \mathrm{G}(o; \zeta, o)]d\zeta \quad (54)$$

In either form, the second term is (in view of the inequalities (52) and (47), (48) respectively) integrable with respect to ζ, provided F is at most of degree zero; this property (satisfied by all the corresponding perturbative contributions) will be our first assumption on F. Now, the convergence of the integral of Eq. (30) implies in turn that the first term in each one of the expressions (53) (54) is integrable with respect to ζ. Since the measure $\omega(k, \zeta)$ and the function $\mathrm{G}(k; \zeta, z')$ are of degree zero (in ζ), this suggests the following asymptotic property of F to hold:

PROPERTY A. — *a) The function* $\mathrm{F}(k; z, z')$ *is of degree zero with respect to* (z, z').

b) There exists a function $\Lambda(k, z)$ *of degree zero with respect to z, satisfying* $\Lambda(o, o) = 1$, *such that the associated k-dependent kernels*:

$$F(k; z, z') = \mathrm{F}(k; z, z') - \Lambda(k, z)\mathrm{F}(o; o, z') \quad (55)$$

$$N(k; z, z') = \mathrm{F}(k; z, z') - \Lambda(k, z)\frac{\mathrm{F}(o; o, z')\omega(o, z')}{\omega(k, z')} \quad (56)$$

satisfy the right-sided regularity property.

In the following, it is convenient to adopt the operator notation of [*4*]: with any k-dependent kernel $\mathrm{K} \equiv \mathrm{K}(k; z, z')$, one associates the following special kernels, for any fixed values k_0, z_0, z_0':

$$\overset{k_0}{\mathrm{K}} \equiv \mathrm{K}(k_0; z, z'),$$
$$_{z_0}\mathrm{K} \equiv \mathrm{K}(k; z_0, z'),$$
$$\mathrm{K}_{z_0'} \equiv \mathrm{K}(k; z, z_0'),$$

any other combination $_{z_0}\overset{k_0}{\mathrm{K}}$, etc... being defined similarly (including the constant kernel $_{z_0}\overset{k_0}{\mathrm{K}}_{z_0'} = \mathrm{K}(k_0; z_0, z_0')$). The symbols $\underline{\omega}$ and $\underline{\overset{k_0}{\omega}}$ will respectively represent integration (over euclidean space) with respect to the measures $\omega(k, \zeta)d\zeta$ and $\omega(k_0, \zeta)d\zeta$.

By taking into account the forms (53) and (54) of the integrand of formula (30), we can then rewrite the latter in either one of the following two forms which we call « *renormalized Bethe-Salpeter equation* »:

$$\mathrm{F} = \mathrm{G} + F\underline{\omega}\mathrm{G} + \Lambda.{}_{\circ}\mathring{\mathrm{F}}(\underline{\omega}\mathrm{G} - \underline{\omega}\mathrm{G}_{\circ}) \tag{57}$$

$$\mathrm{F} = \mathrm{G} + N\underline{\omega}\mathrm{G} + \Lambda.{}_{\circ}\mathring{\mathrm{F}}\underline{\mathring{\omega}}(\mathrm{G} - \mathring{\mathrm{G}}_{\circ}) \tag{58}$$

with F and N respectively defined by formulae (55), (56).

Remark. — From the condition ${}_{\circ}\Lambda \equiv \Lambda(o, o) = 1$, it follows that ${}_{\circ}\mathring{F} = {}_{\circ}\mathring{N} = 0$, and therefore that: ${}_{\circ}\mathring{\mathrm{F}}_{\circ} = {}_{\circ}\mathring{\mathrm{G}}_{\circ} = g$.

By making a similar analysis for Eq. (31), and by taking into account the right-sided regularity of F and N, one also obtains the following expressions for Λ:

$$\Lambda = \frac{1 + F\underline{\omega}1}{1 - {}_{\circ}\mathring{\mathrm{F}}(\underline{\omega} - \underline{\mathring{\omega}})1} = 1 + N\underline{\omega}1 \tag{59}$$

$\left(\text{where the notation } K\underline{\omega}1 \text{ means } \int K(k; z, \zeta)\omega(k, \zeta)d\zeta\right)$.

The interest of Eq. (59) will be displayed in § 5.3. As a matter of fact, these equations do not yield an explicit definition of Λ in terms of F, since Λ itself is involved in the definitions (55), (56) of F and N. In order to obtain an explicit definition of Λ in terms of F, we reconsider the identity (31), and replace it by a regularized form in which the function $\omega(k, z)$ is replaced by a regularized function $\omega_\rho(k, z)$ (as specified in section 2) which tends to ω, when ρ tends to zero. This regularized form of Eq. (31), involving a function Λ_ρ, instead of Λ, reads:

$$\Lambda_\rho(k, z) = 1 + \int [\mathrm{F}(k; z, \zeta)\omega_\rho(k, \zeta) - \Lambda_\rho(k, z)\mathrm{F}(o; o, \zeta)\omega_\rho(o, \zeta)]d\zeta\,,$$

but since $\mathrm{F}\underline{\omega_\rho}1 \equiv \int \mathrm{F}(k; z, \zeta)\omega_\rho(k, \zeta)d\zeta$ is meaningful, the latter can be solved for Λ_ρ in terms of F, and yields:

$$\Lambda_\rho = \frac{1 + \mathrm{F}\underline{\omega_\rho}1}{1 + {}_{\circ}\mathring{\mathrm{F}}\underline{\mathring{\omega}_\rho}1} \tag{60}$$

When $\rho \to 0$, both integrals $\mathrm{F}\underline{\omega_\rho}1$ and ${}_{\circ}\mathring{\mathrm{F}}\underline{\mathring{\omega}_\rho}1$ become divergent, but it is natural to postulate the

PROPERTY A′. — *The function* $\Lambda(k, z)$ *is defined in terms of* F, *as the following limit, which is assumed to exist*:

$$\Lambda = \lim_{\rho \to 0} \frac{1 + \mathrm{F}\underline{\omega_\rho}1}{1 + {}_{\circ}\mathring{\mathrm{F}}\underline{\mathring{\omega}_\rho}1} \tag{61}$$

Remark. — Let us define (for each ρ): $N_\rho = \mathrm{F} - \Lambda_\rho \cdot {}_{\circ}\mathring{\mathrm{F}}\, \dfrac{\mathring{\omega}_\rho}{\omega_\rho}$. It follows from property A′ that, when ρ tends to zero, N_ρ tends to N. Moreover, by taking Eq. (60) into account, one easily checks that:

$$\Lambda_\rho = 1 + N_\rho \omega_\rho 1 .$$

Then, in view of property A the latter yields Eq. (59) in the limit $\rho \to 0$. We conclude that Eq. (59) can be considered as a consequence of properties A and A′.

In view of the symmetry of the four-point function F with respect to the variables z and z', one would obtain the following similar relations:

$$\mathrm{F} = \mathrm{G} + \mathrm{G}\underline{\omega}F^t + (\mathrm{G}\underline{\omega} - {}_{\circ}\mathring{\mathrm{G}}\underline{\mathring{\omega}})\mathring{\mathrm{F}}_{\circ} \cdot \Lambda^t = \mathrm{G} + \mathrm{G}\underline{\omega}N^t + (\mathrm{G} - {}_{\circ}\mathring{\mathrm{G}})\underline{\mathring{\omega}}\mathring{\mathrm{F}}_{\circ} \cdot \Lambda^t \quad (62)$$

with:

$$F^t = \mathrm{F} - \mathring{\mathrm{F}}_{\circ} \cdot \Lambda^t \quad (63)$$

$$N^t = \mathrm{F} - \frac{\mathring{\omega}}{\omega} \mathring{\mathrm{F}}_{\circ} \cdot \Lambda^t \quad (64)$$

and:

$$\Lambda^t = \frac{1 + 1\underline{\omega}F^t}{1 - 1(\underline{\omega} - \underline{\mathring{\omega}})\mathring{\mathrm{F}}_{\circ}} = 1 + 1\underline{\omega}N^t = \lim_{\rho \to 0} \frac{1 + 1\underline{\omega}_\rho F}{1 + 1\underline{\mathring{\omega}}_\rho \mathring{\mathrm{F}}_{\circ}} \quad (65)$$

Note that F^t, N^t, Λ^t are respectively the transposed quantities of F, N, Λ (i. e. $F^t(k; z, z') = F(k; z', z), \ldots, \Lambda^t(k, z) \equiv \Lambda(k, z)$), so that F^t, N^t satisfy the left-sided regularity property.

We will now show that the set of relations [(55)-(58)] can be replaced by a single integral relation linking G to the following auxiliary function:

$$\Phi(k; z, z') = \frac{\mathrm{F}(k; z, z')}{\Lambda(k, z)} . \quad (66)$$

We first notice that (since $\Lambda(o, o) = 1$):

$$\Phi - {}_{\circ}\mathring{\Phi} = \frac{F}{\Lambda} \quad \text{and} \quad \Phi\omega - {}_{\circ}\mathring{\Phi}\mathring{\omega} = \frac{N\omega}{\Lambda} . \quad (67)$$

Property A then implies that the function $\Phi - {}_{\circ}\mathring{\Phi}$ satisfies the right-sided regularity property. From Eqs. (59) and (67), we deduce that:

$$\Lambda = \frac{1}{1 - (\Phi - {}_{\circ}\mathring{\Phi})\underline{\omega}1 - {}_{\circ}\mathring{\Phi}(\underline{\omega} - \underline{\mathring{\omega}})1} = \frac{1}{1 - (\Phi\underline{\omega} - {}_{\circ}\mathring{\Phi}\underline{\mathring{\omega}})1} \quad (68)$$

and (in view of Eq. (66)):

$$F = \frac{\Phi}{1 - (\Phi\underline{\omega} - {}_{\circ}\overset{\circ}{\Phi}\overset{\circ}{\underline{\omega}})1}. \tag{69}$$

Inserting the expressions (68), (69) of Λ and F in Eq. (58) then yields (in view of Eq. (67)):

$$\Phi = [1 - (\Phi\underline{\omega} - {}_{\circ}\overset{\circ}{\Phi}\overset{\circ}{\underline{\omega}})1].G + (\Phi\underline{\omega} - {}_{\circ}\overset{\circ}{\Phi}\overset{\circ}{\underline{\omega}})G + {}_{\circ}\overset{\circ}{\Phi}\overset{\circ}{\underline{\omega}}(G - \overset{\circ}{G}_{\circ}) \tag{70}$$

(in view of the right-sided regularity of $\Phi\omega - {}_{\circ}\overset{\circ}{\Phi}\overset{\circ}{\omega}$, each term is meaningful in formulae (68), (69), (70)).

§. 5.2. **Equations for the N-point functions.**

We now consider the formal identities (43), (44) for the N-point functions and treat them as genuine integral relations, as we did for the four-point function in the previous subsection. We are thus led to define kernels $F^{(N)}(k; z^{(N-2)}, z')$ and $N^{(N)}(k; z^{(N-2)}, z')$, similar to F and N, namely (in the operator notation of § 5.1):

$$F^{(N)} = \mathrm{F}^{(N)} - \Lambda^{(N-1)}.{}_{\circ}\overset{\circ}{\mathrm{F}} \tag{71}$$

$$N^{(N)} = \mathrm{F}^{(N)} - \Lambda^{(N-1)}.{}_{\circ}\overset{\circ}{\mathrm{F}}\,\frac{\overset{\circ}{\omega}}{\omega} \tag{72}$$

Formulae (43), (44) can then be rewritten in a form similar to (57), (58), (59) namely:

$$\mathrm{F}^{(N)} = \mathrm{G}^{(N-2,2)} + F^{(N)}\underline{\omega}\mathrm{G} + \Lambda^{(N-1)}.{}_{\circ}\overset{\circ}{\mathrm{F}}(\underline{\omega}\mathrm{G} - \overset{\circ}{\underline{\omega}}\overset{\circ}{\mathrm{G}}_{\circ}) \tag{73}$$

or

$$\mathrm{F}^{(N)} = \mathrm{G}^{(N-2,2)} + N^{(N)}\underline{\omega}\mathrm{G} + \Lambda^{(N-1)}.{}_{\circ}\overset{\circ}{\mathrm{F}}\overset{\circ}{\underline{\omega}}(\mathrm{G} - \overset{\circ}{\mathrm{G}}_{\circ}) \tag{74}$$

with:

$$\Lambda^{(N-1)} = \frac{F^{(N)}\underline{\omega}1}{1 - {}_{\circ}\overset{\circ}{\mathrm{F}}(\underline{\omega} - \overset{\circ}{\underline{\omega}})1} = N^{(N)}\underline{\omega}1\,. \tag{75}$$

The fact that each term at the r. h. s. of formulae (73), (74) is well-defined follows from the (left-sided) regularity of the function $\mathrm{G} - \overset{\circ}{\mathrm{G}}_{\circ}$ and of the measure $\underline{\omega} - \overset{\circ}{\underline{\omega}}$ (as for the corresponding terms of formulae [(57)-(59)] in § 5.1).

We are thus led to postulate the following properties A and A':

PROPERTY *A.* — *i) The functions* $\mathrm{F}^{(N)}(k; z^{(N-2)}, z')$ (N *even* $\geqslant 6$), *are of degree zero with respect to* z'.

ii) There exist functions $\Lambda^{(N-1)}(k, z^{(N-2)})$ *such that the auxiliary func-*

tions $F^{(N)} = \mathrm{F}^{(N)} - \Lambda^{(N-1)} \cdot {}_{\circ}\mathring{\mathrm{F}}$ *and* $N^{(N)} = \mathrm{F}^{(N)} - \Lambda^{(N-1)} \cdot {}_{\circ}\mathring{\mathrm{F}} \frac{\mathring{\omega}}{\omega}$ *satisfy the right-sided regularity property.*

PROPERTY *A'*. — *Each function* $\Lambda^{(N-1)}$ (N *even* $\geqslant 6$) *is defined in terms of the corresponding function* $\mathrm{F}^{(N)}$ *through the following limiting procedure*:

$$\Lambda^{(N-1)} = \lim_{\rho \to 0} \frac{\mathrm{F}^{(N)} \underline{\omega_\rho} 1}{1 + {}_{\circ}\mathring{\mathrm{F}} \underline{\mathring{\omega}_\rho} 1}. \tag{76}$$

Property *A'* is justified (as property A'), by considering the regularized form of Eq. (44), solving for $\Lambda_\rho^{(N-1)}$ and then taking the limit $\rho \to 0$. Moreover, formula (75) can be shown to be a consequence of properties *A* and *A'* (the argument is similar to that given in the remark after property A', in § 5.1).

§. 5.3. Connection with the formalism of local operator products.

Let us fix some notations, concerning the passage from the N-point momentum variables:

$$\{ (k_1, k_2, \dots, k_{N-2}; k_1', k_2'); k_1' + k_2' = k = k_1 + \dots + k_{N-2} \}$$

to the corresponding kernel variables:

$$\left\{ (k; z^{(N-2)}, z'); z^{(N-2)} = (z_i = k_i - \frac{k}{N-2}, 1 \leqslant i \leqslant N-2), z' = \frac{k_1' - k_2'}{2} \right\}$$

(note that $z_1 + \dots + z_{N-2} = 0$; for N = 4, $z^{(N-2)}$ is simply represented by: $z_1 = - z_2 \equiv z = k_1 - \frac{k}{2} = \frac{k_1 - k_2}{2}\Big)$.

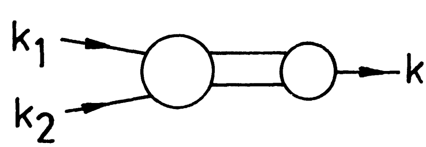
A general N-point function will be denoted by $\hat{K}_N$ if the corresponding k-dependent kernel is K_N, namely:

$$\hat{K}_N(-k_1, -k_2, \dots, -k_{N-2}; k_1', k_2') = K_N(k; z^{(N-2)}, z').$$

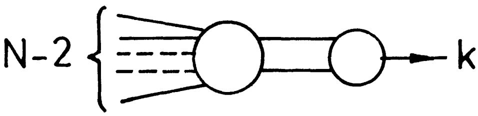
Moreover, in this subsection, the notations $\mathrm{F}^{(N)}$, $F^{(N)}$, $N^{(N)}$, $\Lambda^{(N-1)}$ will be extended (for simplicity) to the case N = 4, with

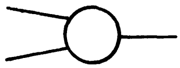
$$\mathrm{F}^{(4)} = \mathrm{F}, \qquad F^{(4)} = F, \qquad N^{(4)} = N, \qquad \Lambda^{(3)} = \Lambda.$$

In formulae (61) and (76), the expressions $\mathrm{F}\underline{\omega_\rho} 1$ and $\mathrm{F}^{(N)}\underline{\omega_\rho} 1$ can be considered as regular $\mathscr{G}$-convolution integrals (in the sense of [*14*]), respectively associated with the graphs



and



(the vertex factor associated with  being equal to 1).

Therefore, in view of the main theorem of [*14*], $F\underline{\omega}_\rho 1$ and $F^{(N)}\underline{\omega}_\rho 1$ are analytic functions satisfying the primitive analytic structure of axiomatic field theory, namely $\widehat{(F\underline{\omega}_\rho 1)}(-k_1, -k_2; k)$ is a general three-point function and (for each N even $\geqslant 6$) $(\widehat{F^{(N)}\underline{\omega}_\rho 1})(-k_1, \ldots, -k_{N-2}, k)$ is a general (N $-$ 1)-point function. Since $1 + {}_\rho\overset{\circ}{F}\overset{\circ}{\underline{\omega}}_\rho 1$ is a constant, it follows from (61) (76) that (for each N even $\geqslant 4$), $\hat{\Lambda}^{(N-1)}(-k_1, \ldots, -k_{N-2}, k) \equiv \Lambda^{(N-1)}(k, z^{(N-2)})$ is a general (N $-$ 1)-point function, provided property A' (resp. A$'$ for N $=$ 4) is now postulated to hold, not only in euclidean space, but in the primitive analyticity domain of (N $-$ 1)-point functions.

For each general N-point function $\hat{\mathcal{U}}^{(N)}$, there exists a corresponding « τ-boundary value » $\hat{\mathcal{U}}_\tau^{(N)}$ on real Minkowski space, obtained from the primitive axiomatic analyticity domain by the general « $i\varepsilon$-prescription » defined in [*18*] (see also [*2*], [*19*]). For $\hat{F}^{(N)}$, the corresponding τ-boundary value $\hat{F}_\tau^{(N)}$ is linked to the Fourier transform of the time-ordered T. V. E. V. of the subjacent field operator by the following formula:

$$\delta(k'_1 + k'_2 - k_1 \ldots - k_{N-2})\hat{F}_\tau^{(N)}(-k_1, \ldots, -k_{N-2}, k'_1, k'_2)$$
$$= \Big[\prod_{1 \leq i \leq N-2} H_\tau^{(2)}(k_i) \Big]^{-1} \times \ldots$$
$$\langle T\tilde{A}(-k_1) \ldots \tilde{A}(-k_{N-2})\tilde{A}(k'_1)\tilde{A}(k'_2) \rangle^{CONN} [\omega_\tau(k, z')]^{-1}.$$

The boundary values $\hat{\Lambda}_\tau^{(N-1)}$, $\hat{N}_\tau^{(N)}$ of the corresponding analytic functions introduced above have the following connection with the formalism of local operator products developed in [*6*]. First, one defines the matrix elements of a composite field $N\{A(x)^2\}$ (see [*6*]) by the following formulae:

$$(N \geqslant 4)\Big[\prod_{1 \leqslant i \leqslant N-2} H_\tau^{(2)}(k_i) \Big]^{-1} . \langle T\tilde{A}(-k_1) \ldots \tilde{A}(-k_{N-2}) N\{A(x)^2\} \rangle^{CONN}$$
$$= 2\int e^{-ik.x}\delta(k_1 + \ldots + k_{N-2} - k)\hat{\Lambda}_\tau^{(N-1)}(-k_1, \ldots, -k_{N-2}, k)dk$$
$$= 2e^{-ik.x}\Lambda_\tau^{(N-1)}(k, z^{(N-2)})\,|_{k=k_1+\ldots+k_{N-2}} \tag{77}$$

One introduces similarly a normal operator product $N\{A(x+\xi)A(x-\xi)\}$

through the following formulae for its matrix elements [8] for $N \geqslant 4$,

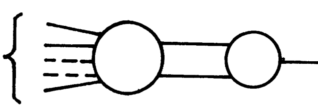
$$\left[\prod_{1 \leqslant i \leqslant N-2} H_\tau^{(2)}(k_i)\right]^{-1} \langle T\tilde{A}(-k_1)\ldots\tilde{A}(-k_{N-2})N\{A(x+\xi)A(x-\xi)\}\rangle^{CONN}$$

$$= 2\int e^{-ik.x-2iz'.\xi}\omega(k, z')\delta(k_1+\ldots+k_{N-2}-k)\hat{N}_\tau^N(-k_1,\ldots,-k_{N-2}, k_1', k_2')dkdz' \tag{78}$$

$\left(\text{where: } k = k_1' + k_2', z' = \dfrac{k_1' - k_2'}{2}\right)$.

These formulae must be supplemented by the following one:

$$\begin{aligned}\langle T\tilde{A}(-k_1)\tilde{A}(-k_2)N\{A(x+\xi)A(x-\xi)\}\rangle &= \langle T\tilde{A}(-k_1)\tilde{A}(-k_2)N\{A(x+\xi)A(x-\xi)\rangle^{CONN} \\ &+ H_\tau^{(2)}(k_1)H_\tau^{(2)}(k_2)e^{-i(k_1+k_2)x}[e^{-i(k_1-k_2)\xi} + e^{i(k_1-k_2)\xi}]\end{aligned} \tag{79}$$

in order to yield a complete definition of the operator $N\{A(x+\xi)A(x-\xi)\}$ (see our previous footnote). We now make use of the right-sided regularity of the kernels $N^{(N)}$ ($N \geqslant 4$) postulated in property A (resp. A for $N = 4$); according to the latter, each kernel $N^{(N)}$ can be integrated (on euclidean space) with the measure $e^{-2iz'\xi}\omega(z')$, if ξ is a real euclidean (i. e. space-like) vector, possibly equal to zero. One can then consider the $\mathscr{G}$-convolution integrals (associated with the graphs N-2 )

$$e^{-ik.x}N^{(N)}\underline{\omega}(e^{-2iz'.\xi}) \equiv e^{-ik.x}\int N^{(N)}(k; z^{(N-2)}, z')\omega(k, z')e^{-2iz'.\xi}dz'; \tag{80}$$

the latter defines a general $(N-1)$-point function and a corresponding τ-boundary value whose dependence on $\xi = (0, \vec{\xi})$ is continuous; this τ boundary value coincides with the restriction at $\xi^{(0)} = 0$ of the integral at the r. h. s. of Eq. (78): the regularity of the matrix elements of

$$N\{A(x+\xi)A(x-\xi)\}$$

on the set $\{\xi = (0, \vec{\xi})\}$ is thus obtained. Moreover, the limit of

$$N\{A(x+\xi)A(x-\xi)\} \qquad \text{when} \qquad \xi = (0, \vec{\xi})$$

---

[8] As in [6], the set of distributions (77) (resp. (78), (79) define the operator valued distribution $N\{A(x)^2\}$ (resp. $N\{A(x+\xi)A(x-\xi)\}$ through an argument based on asymptotic completeness which makes use of the reduction formulae.

tends to zero can be seen to be equal to $N\{A(x)^2\}$. In fact, in view of Eq. (79), this amounts to check that:

$$
\begin{aligned}
(N \geqslant 4) \quad & \lim_{\xi \to 0} \left[\prod_{1 \leqslant i \leqslant N-2} H_\tau^{(2)}(k_i) \right]^{-1} \times \ldots \\
& \langle T\tilde{A}(-k_1)\ldots\tilde{A}(-k_{N-2}) N\{A(x+\xi)A(x-\xi)\} \rangle^{CONN} \\
&= \left[\prod_{1 \leqslant i \leqslant N-2} H_\tau^2(k_i) \right]^{-1} \times \ldots \\
& \langle T\tilde{A}(-k_1)\ldots\tilde{A}(-k_{N-2}) N\{A(x)^2\} \rangle^{CONN} - 2e^{-ik.x}\delta_{N4} \qquad (81)
\end{aligned}
$$

but in view of Eqs. (77) (78) (80), Eqs. (81) are equivalent to the following ones:

$$
(N \geqslant 4) \qquad \Lambda^{(N-1)} = \delta_{N4} + N^{(N)} \underline{\omega} 1 \qquad (82)
$$

These formulae coincide with Eqs. (75) (for general N) and (59) (for $N=4$), which are themselves implied by the postulated properties A and A' (resp. A and A′, for $N = 4$).

In order to achieve the connection of properties A, A with the normal operator product formalism, it remains to take the (k, z')-Fourier transform of the equality deduced from formula (72) after passing to the τ-boundary values and multiplying both sides by $2\omega(k, z')\delta(k_1 + \ldots + k_{N-2} - k)$. This yields:

$$
\begin{aligned}
& \langle T\tilde{A}(-k_1)\ldots\tilde{A}(-k_{N-2}) N\{A(x+\xi)A(x-\xi)\} \rangle^{CONN} \\
&\qquad = \langle T\tilde{A}(-k_1)\ldots\tilde{A}(-k_{N-2}) A(x+\xi)A(x-\xi) \rangle^{CONN} \\
& - \frac{1}{2} \langle T\tilde{A}(0)\tilde{A}(0)A(\xi)A(-\xi) \rangle^{CONN} [H^{(2)}(0)]^{-2} \\
&\qquad \times \langle T\tilde{A}(-k_1)\ldots\tilde{A}(-k_{N-2}) N\{A(x)^2\} \rangle^{CONN} \qquad (83))
\end{aligned}
$$

The set of Eq. (83) (for all $N \geqslant 4$) corresponds to the lowest order expansion of the local operator product (see [*6*], formulae III.110, III.111), namely:

$$
\begin{aligned}
:A(x+\xi)A(x-\xi): &= N\{A(x+\xi)A(x-\xi)\} \\
& + \frac{1}{2} \langle T\tilde{A}(0)\tilde{A}(0)A(\xi)A(-\xi) \rangle^{CONN} [H^{(2)}(0)]^{-2} . N\{A(x)^2\} \qquad (84)
\end{aligned}
$$

6. SYMANZIK'S INCREMENTAL FORMS OF THE BETHE-SALPETER EQUATION

In all the following, we shall make the following technical assumption: whenever a k-dependent kernel K satisfies the left-sided (resp. right-sided) regularity property, we assume that all the iterated kernels $K\underline{\omega}K \ldots \underline{\omega}K$

of K and the corresponding resolvent kernel R(K) (with formal expansion $\left. K + \sum_{n\geq 2} \underbrace{K\underline{\omega}K \ldots \underline{\omega}K}_{n} \right)$ exist and also satisfy the left-sided (resp. right-sided) regularity property. The validity of this assumption would follow from bounds of the form (47), (48), (49) in which the constants $C_{k,k_0,z'}$ etc..., would be replaced by more refined estimates.

§ 6.1. — We first consider the identities (35), which are independent of Λ and relate F and $F_{z'_0}$ to the increment $G - G_{z'_0}$ of the kernel G. It is sufficient to consider the identity obtained for $z'_0 = 0$ (which generates all the others by subtraction), namely:

$$F - F_0 = G - G_0 + F\underline{\omega}(G - G_0). \tag{85}$$

In view of the left-sided regularity of $G - G_0$ (formula (48)), we can introduce the resolvent L of $G - G_0$ which satisfies the equation:

$$L = G - G_0 + L\underline{\omega}(G - G_0) = G - G_0 + (G - G_0)\underline{\omega}L \tag{86}$$

or equivalently (for the composition of kernels):

$$\text{-----}.\,[1 - \underline{\omega}(G - G_0)].\,[1 + \underline{\omega}L] = \text{---}.1. \tag{86'}$$

We can then compute F in terms of L and F_0. From Eqs. (85) and (86) (86'), we deduce successively:

$$F[1 - \underline{\omega}(G - G_0)] = F_0 + G - G_0$$

and

$$F = (F_0 + G - G_0)(1 + \underline{\omega}L) = L + F_0(1 + \underline{\omega}L). \tag{88}$$

Each term of these equations is well-defined in view of the (assumed) left-sided regularity of L and we notice that:

$$F_0\underline{\omega}L = F(k; z, 0).\int \omega(k; \zeta)L(k; \zeta, z')d\zeta = F_0.1\underline{\omega}L.$$

This allows one to rewrite Eq. (88) as follows:

$$F = F_0.(1 + 1\underline{\omega}L) + L. \tag{89}$$

On the other hand, property (A) provides the following connection between $1\underline{\omega}L$ and Λ^t; from Eq. (89), we obtain (for any choice of the regularized measure ω_ρ):

$$1\underline{\omega}_\rho(F - F_0) = (1 + 1\underline{\omega}_\rho F_0)(1\underline{\omega}_\rho L) + (1\underline{\omega}_\rho F_0)[1(\underline{\omega} - \underline{\omega}_\rho)L] \tag{90}$$

But in view of property (A), written for Λ^t, we have:

$$\lim_{\rho\to 0} \frac{1\underline{\omega}_\rho(F - F_0)}{1 + 1\underline{\omega}_\rho F_0} = \frac{\Lambda^t - \Lambda_0^t}{\Lambda_0^t}, \tag{91}$$

and since $\lim_{\rho \to 0} 1\underline{\omega}_\rho \mathrm{L} = 1\underline{\omega}\mathrm{L}$, Eq. (90) yields (in view of formula (91)):

$$1\underline{\omega}\mathrm{L} = \frac{\Lambda^t - \Lambda_0^t}{\Lambda_0^t} - \lim_{\rho \to 0} \left[\frac{1\underline{\omega}_\rho \mathrm{F}_0}{1 + 1\underline{\omega}_\rho \mathrm{F}_0} \right] \times [1(\underline{\omega} - \underline{\omega}_\rho)\mathrm{L}] \tag{92}$$

Now, the second term at the r. h. s. of Eq. (92) is equal to zero, under the additional technical assumption that -1 is not an adherence point for the set $\{ 1\underline{\omega}_\rho \mathrm{F}_0 ; \rho \to 0 \}$.

We then have:

$$1 + 1\underline{\omega}\mathrm{L} = \frac{\Lambda^t}{\Lambda_0^t} \tag{93}$$

and formula (89) can thus be rewritten as well:

$$\mathrm{L} = \mathrm{F} - \mathrm{F}_0 . \frac{\Lambda^t}{\Lambda_0^t} . \tag{94}$$

If we now use the kernel F^t (introduced in Eqs. (61) (62)), we have:

$$F^t(k; z, z') = \mathrm{F}(k; z, z') - \mathrm{F}(o; z, o)\Lambda(k, z')$$
$$F^t(k; z, o) = \mathrm{F}(k; z, o) - \mathrm{F}(o; z, o)\Lambda(k, o)$$

and the latter allow us to replace Eq. (94) by the alternative form:

$$\mathrm{L} = F^t - F_0^t . \frac{\Lambda^t}{\Lambda_0^t} . \tag{95}$$

The interest of formula (95) is that it exhibits the left-sided regularity of L, as being a consequence of that of F^t, namely of property (A′).

We summarize these results in the following

PROPOSITION 7. — *a) If* $\mathrm{G} - \mathrm{G}_0$ *is given and satisfies the left-sided regularity property, then* F *can be constructed in terms of* F_0 *and of the resolvent* L *of* $\mathrm{G} - \mathrm{G}_0$ *through formula* (89).

b) If F *is given and satisfies properties* A *and* A′ *then* $\mathrm{G} - \mathrm{G}_0$ *can be constructed as the inverse resolvent of the kernel* L, *the latter being defined in terms of* F *(or* F*) by formula* (94) *(or* (95)*); the left-sided regularity of* $\mathrm{G} - \mathrm{G}_0$ *results (through the regularity of* L*) from property* A.

A similar result can of course be given for the connections between $\mathrm{G} - {}_0\mathrm{G}$, its resolvent L^t and the kernels F, ${}_0\mathrm{F}$, F, ${}_0F$.

§ 6.2. — We now consider the incremental identities (36) and establish the connection between the previous kernels Λ^t and N^t and Symanzik's formalism of [4*a*] (App. A) and [*4b*] (App. A and B). We rewrite the identity (36) for $k_0 = 0$, as follows:

$$\mathrm{F} - \breve{\mathrm{F}} - \breve{\mathrm{F}}(\underline{\omega} - \underline{\omega})\mathrm{F} = (1 + \breve{\mathrm{F}}\underline{\mathring{\omega}})(\mathrm{G} - \mathrm{G})(1 + \underline{\omega}\mathrm{F}) . \tag{96}$$

Let us introduce the following kernel:

$$W = (\mathrm{G} - \overset{\circ}{\mathrm{G}})(1 + \underline{\omega}\mathrm{F}) + \frac{(\omega - \overset{\circ}{\omega})}{\overset{\circ}{\omega}}\mathrm{F}. \tag{97}$$

According to the latter, the left-sided regularity of $\mathrm{G} - \overset{\circ}{\mathrm{G}}$ is equivalent to that of W. From Eq. (96) we then deduce the following relation between F, $\overset{\circ}{\mathrm{F}}$ and W:

$$\frac{\omega}{\overset{\circ}{\omega}}\mathrm{F} = \overset{\circ}{\mathrm{F}}(1 + \underline{\overset{\circ}{\omega}}W) + W. \tag{98}$$

For any regularized measure $\underline{\omega}_\rho$, Eq. (98) yields (by applying the operation $1\underline{\overset{\circ}{\omega}}_\rho$ to both sides of the latter, from the left):

$$\begin{aligned}(1 + 1\underline{\omega}_\rho\mathrm{F}) + 1(\underline{\omega} - \underline{\overset{\circ}{\omega}})\frac{\omega_\rho}{\overset{\circ}{\omega}}\mathrm{F} + 1(\underline{\overset{\circ}{\omega}}_\rho - \underline{\omega}_\rho)\mathrm{F}& \\ = (1 + 1\underline{\overset{\circ}{\omega}}_\rho\overset{\circ}{\mathrm{F}})(1 + \underline{\overset{\circ}{\omega}}W) + 1(\underline{\overset{\circ}{\omega}}_\rho - \underline{\overset{\circ}{\omega}})W&\end{aligned} \tag{99}$$

By dividing both sides of Eq. (99) by $1 + 1\underline{\overset{\circ}{\omega}}_\rho\overset{\circ}{\mathrm{F}}_\circ$, and taking the limit $\rho \to 0$, we obtain, in view of the definition (65) of Λ^t:

$$\Lambda^t = \overset{\circ t}{\Lambda}(1 + \underline{\overset{\circ}{\omega}}W) \tag{100}$$

(in this derivation, we used the following regularity properties:

$$\lim_{\rho\to 0} 1(\underline{\omega}_\rho - \underline{\overset{\circ}{\omega}}_\rho)\mathrm{F} = 1(\underline{\omega} - \underline{\overset{\circ}{\omega}})\mathrm{F}, \qquad \lim_{\rho\to 0} 1\underline{\overset{\circ}{\omega}}_\rho W = 1\underline{\overset{\circ}{\omega}}W,$$

and the assumption that -1 is not an adherence point for the set $\{1\underline{\overset{\circ}{\omega}}_\rho\overset{\circ}{\mathrm{F}}_\circ, \rho \to 0\}$).

Let us now consider the kernel $N^t \equiv N^{(4)t}$, defined in formula (64), namely:

$$N^t = \mathrm{F} - \frac{\overset{\circ}{\omega}}{\omega}\overset{\circ}{\mathrm{F}}_\circ . \Lambda^t.$$

From Eqs. (98) and (100), we then deduce the following expression for ωN^t:

$$\omega N^t = \overset{\circ}{\omega}(\overset{\circ}{\mathrm{F}} - \overset{\circ}{F}_\circ . \Lambda^t)(1 + \underline{\overset{\circ}{\omega}}W) + \overset{\circ}{\omega}W, \tag{101}$$

or (in view of Eq. (64), written at $k = 0$):

$$\omega N^t = \overset{\circ}{\omega}\overset{\circ}{N}^t(1 + \underline{\overset{\circ}{\omega}}W) + \overset{\circ}{\omega}W. \tag{102}$$

The latter can also be rewritten as follows:

$$.(1 + \underline{\omega}N^t) = .(1 + \underline{\overset{\circ}{\omega}}\overset{\circ}{N}^t)(1 + \underline{\overset{\circ}{\omega}}W) \tag{103}$$

and since (in view of Eqs. (64), (94) and (86′), written at $k = 0$):

$$.(1 + \underline{\mathring{\omega}}\mathring{N}^t) = .(1 + \underline{\mathring{\omega}}\mathring{L}) = .\,[1 - \underline{\mathring{\omega}}(\mathring{G} - \mathring{G}_\circ)]^{-1}, \tag{104}$$

we also have:

$$.\underline{\omega}N^t = .\{\,[1 - \underline{\mathring{\omega}}(\mathring{G} - \mathring{G}_\circ)]^{-1}(1 + \underline{\mathring{\omega}}W) - 1\,\}. \tag{105}$$

In view of the definition (97) of W, this gives the following expression for ωN^t, as a functional of F and G:

$$.\underline{\omega}N^t = .\{\,[1 - \underline{\mathring{\omega}}(\mathring{G} - \mathring{G}_\circ)]^{-1}[1 + \underline{\mathring{\omega}}(G - \mathring{G})(1 + \underline{\omega}F) + (\underline{\omega} - \underline{\mathring{\omega}})F] - 1\,\} \tag{106}$$

Remark. — The right-hand side of Eq. (106) can be seen to coincide formally with the definition of Symanzik's operator W, introduced in [*4a*] (formula A-9*b*) and used by this author to recover the local operator product formalism.

§ 6.3. — The following regularized version of the previous renormalized formalism has been introduced in [*11*] by D. Iagolnitzer. One considers the family of regularized Bethe-Salpeter kernels $\hat{G}_\rho$ defined as in [*2*] (see section 2) by the regular Fredholm equations:

$$F = \hat{G}_\rho + F\underline{\omega}_\rho\hat{G}_\rho = \hat{G}_\rho + \hat{G}_\rho\underline{\omega}_\rho F. \tag{107}$$

With each kernel $\hat{G}_\rho$, one associates a « renormalized kernel »:

$$G_\rho = \hat{G}_\rho - C_\rho, \tag{108}$$

where C_ρ is a constant, chosen in such a way that:

$$G_\rho(o; o, o) = F(o; o, o). \tag{109}$$

Eq. (107) can then be rewritten as follows:

$$F = G_\rho + F\underline{\omega}_\rho G_\rho + C_\rho(1 + F\underline{\omega}_\rho 1) \tag{110}$$

with:

$$C_\rho = \hat{G}_\rho(o; o, o) - F(o; o, o) = -{}_\circ\mathring{F}\underline{\mathring{\omega}}_\rho\mathring{G}_{\rho\circ} = -\frac{{}_\circ\mathring{F}\underline{\mathring{\omega}}_\rho\mathring{G}_{\rho\circ}}{1 + {}_\circ\mathring{F}\underline{\mathring{\omega}}_\rho 1}. \tag{111}$$

Eqs. (110), (111) then yield:

$$F = G_\rho + F\underline{\omega}_\rho G_\rho - \Lambda_\rho\,.\,{}_\circ\mathring{F}\underline{\mathring{\omega}}_\rho\mathring{G}_{\rho\circ} \tag{112}$$

with:

$$\Lambda_\rho = \frac{1 + F\underline{\omega}_\rho 1}{1 + {}_\circ\mathring{F}\underline{\mathring{\omega}}_\rho 1} \tag{113}$$

or equivalently:

$$F = G_\rho + F_\rho\underline{\omega}_\rho G_\rho + \Lambda_\rho\,.\,{}_\circ\mathring{F}(\underline{\omega}_\rho G_\rho - \underline{\mathring{\omega}}_\rho\mathring{G}_{\rho\circ}) \tag{114}$$

$$F = G_\rho + N_\rho\underline{\omega}_\rho G_\rho + \Lambda_\rho\,.\,{}_\circ\mathring{F}\underline{\mathring{\omega}}_\rho(G_\rho - \mathring{G}_{\rho\circ}) \tag{115}$$

if one puts:

$$F_\rho = \mathrm{F} - \Lambda_\rho \cdot {}_\circ\overset{\circ}{\mathrm{F}} \tag{116}$$

$$N_\rho = \mathrm{F} - \Lambda_\rho \cdot {}_\circ\overset{\circ}{\mathrm{F}}\frac{\overset{\circ}{\omega}_\rho}{\omega_\rho}. \tag{117}$$

Moreover, Eq. (108) yields:

$$\hat{\mathrm{G}}_\rho - \hat{\mathrm{G}}_{\rho\circ} = \mathrm{G}_\rho - \mathrm{G}_{\rho\circ}, \quad \hat{\mathrm{G}}_\rho - {}_\circ\hat{\mathrm{G}}_\rho = \mathrm{G}_\rho - {}_\circ\mathrm{G}_\rho, \quad \hat{\mathrm{G}}_\rho - \overset{\circ}{\hat{\mathrm{G}}}_\rho = \mathrm{G}_\rho - \overset{\circ}{\mathrm{G}}_\rho \tag{118}$$

It follows from Eqs. (118) that all the incremental identities implied by the regular Bethe-Salpeter equations (107), also hold for corresponding renormalized kernels G_ρ; in particular:

$$\mathrm{F} - \mathrm{F}_\circ = \mathrm{G}_\rho - \mathrm{G}_{\rho\circ} + \mathrm{F}\underline{\omega}_\rho(\mathrm{G}_\rho - \mathrm{G}_{\rho\circ}) \tag{119}$$

$$\mathrm{F} - {}_\circ\mathrm{F} = \mathrm{G}_\rho - {}_\circ\mathrm{G}_\rho + (\mathrm{G}_\rho - {}_\circ\mathrm{G}_\rho)\underline{\omega}_\rho\mathrm{F} \tag{120}$$

$$\mathrm{F} - \overset{\circ}{\mathrm{F}} - \overset{\circ}{\mathrm{F}}(\underline{\omega}_\rho - \underline{\overset{\circ}{\omega}}_\rho)\mathrm{F} = (1 + \overset{\circ}{\mathrm{F}}\underline{\overset{\circ}{\omega}}_\rho)(\mathrm{G}_\rho - \overset{\circ}{\mathrm{G}}_\rho)(1 + \underline{\omega}_\rho\mathrm{F}) \tag{121}$$

In the limit $\rho \to 0$, the constants C_ρ, ${}_\circ\overset{\circ}{\mathrm{F}}\overset{\circ}{\omega}_\rho 1$ diverge and the kernel $\overset{\circ}{\mathrm{G}}_\rho$ has no limit. However, it is a reasonable conjecture (as postulated in [*11*]) that the kernel G_ρ should tend to a kernel G which satisfies the renormalized equation (57) (or (58)) with: $\Lambda = \lim_{\rho\to 0} \Lambda_\rho, F = \lim_{\rho\to 0} F_\rho$ and $N = \lim_{\rho\to 0} N_\rho$. Moreover, all the incremental equations such as Eqs. (119) (121) should also pass to the limit and yield correspondingly Eqs. (85) (86). In spite of the difference of this approach from the one of [*4*] (where the kernels F, G and the Bethe-Salpeter equation are considered at first in a regularized perturbative approach). the analysis of [*11*] thus gives a simple account of the statement (taken as reliable on the basis of perturbation theory in [*4*]) according to which all incremental equations remain valid in the limit when the regularization is removed, the kernel G involved being reinterpreted as the renormalized Bethe-Salpeter kernel.

In the axiomatic study given just below (section 7), we shall make some use of the renormalized regularized formalism and will give a proof of the previous conjecture, on the basis of properties A_0, A, A′ supplemented by an extra-assumption of the same nature as property A′.

7. RECONSTRUCTION OF THE IRREDUCIBLE KERNELS IN THE AXIOMATIC APPROACH

In this section, we consider as given the set of n-point amputated Green functions $\hat{\mathrm{F}}_n(k_1, \ldots, k_n)$ which are supposed to satisfy:

a) the primitive analytic structure of the axiomatic approach

b) the asymptotic properties A_0, A, A′, *A*, *A′*

c) the system of two-particle asymptotic completeness relations.

Then we wish to show that the renormalized two-particle structure equations (57), (58), (73), (75) allow one to introduce the renormalized irreducible kernels $G^{(N_1,N_2)}$ (with $G = G^{(2,2)}$). This includes the following steps:

i) introduce these kernels in euclidean space

ii) define their analytic continuation in the primitive axiomatic domain

iii) show that these kernels are two-particle irreducible, namely that their analytic continuation satisfy the following discontinuity formulae:

$$\forall N_1, N_2, \qquad \Delta G^{(N_1,N_2)}(k; z^{(N_1)}, z^{(N_2)}) = 0 \tag{122}$$

for $k^2 < (4\mu)^2$.

The reconstruction and study of the Bethe-Salpeter kernel G will be treated in § 7.1 (for step *i*)) and § 7.2 (for steps *ii*) and *iii*)); § 7.3 is devoted to the reconstruction and study of the other kernels $G^{(N_1,N_2)}$.

§ 7.1. — We start from the renormalized Bethe-Salpeter equation (58), namely:

$$F = G + N\underline{\omega}G + \Lambda.{}_{\circ}\mathring{F}\mathring{\underline{\omega}}(G - \mathring{G}_{\circ}) \tag{58}$$

with N given by Eq. (56), which we rewrite:

$$N = F - \Lambda.{}_{\circ}\mathring{F}\frac{\mathring{\omega}}{\omega}; \tag{123}$$

Λ is expressed in terms of F by Eq. (61) (property A′) and satisfies Eq. (59), as a consequence of properties A, A′ (see the remark after property A′ in § 5.1).

Since (in view of property A) N satisfies the right-sided regularity property, we can introduce its resolvent R (see our comment at the top of section 6) such that

$$(1 - R\underline{\omega}). = (1 + N\underline{\omega})^{-1}. \tag{124}$$

Eq. (59) yields:

$\Lambda = 1 + N\underline{\omega}1$, and therefore (in view of (124))

$$(1 - R\underline{\omega}).\Lambda = 1. \tag{125}$$

We also need the following formula:

$$(1 - R\underline{\omega}).(1 + F\underline{\omega}). = (1 + {}_{\circ}\mathring{F}\mathring{\underline{\omega}}). \tag{126}$$

which is a direct consequence of Eqs. (123), (124) and (125). From (126), we then deduce:

$$(1 - R\underline{\omega}).F = \frac{1}{\omega} + {}_{\circ}\mathring{F}\frac{\mathring{\omega}}{\omega} - (1 - R\omega)\frac{1}{\omega} = R + {}_{\circ}\mathring{F}\frac{\mathring{\omega}}{\omega}. \tag{127}$$

By multiplying both sides of Eq. (58) by $(1 - R\underline{\omega})$ and taking Eq. (125) into account, we then obtain:

$$\begin{aligned}(1 - R\underline{\omega})\mathrm{F} &= \mathrm{G} + (1 - R\underline{\omega})\Lambda . {}_{\circ}\mathring{\mathrm{F}}\underline{\mathring{\omega}}(\mathrm{G} - \mathring{\mathrm{G}}_{\circ}) \\ &= \mathrm{G} + {}_{\circ}\mathring{\mathrm{F}}\underline{\mathring{\omega}}(\mathrm{G} - \mathring{\mathrm{G}}_{\circ}) \qquad (128)\end{aligned}$$

or (in view of Eq. (127)):

$$R + {}_{\circ}\mathring{\mathrm{F}}\frac{\mathring{\omega}}{\omega} = \mathrm{G} + {}_{\circ}\mathring{\mathrm{F}}\underline{\mathring{\omega}}(\mathrm{G} - \mathring{\mathrm{G}}_{\circ}) \qquad (129)$$

We now distinguish two steps in the derivation of G, namely: *a*) the computation of $\mathring{\mathrm{G}}$ which is straightforward, *b*) the computation of $\mathrm{G} - \mathring{\mathrm{G}}$ which involves a limiting process requiring a stronger assumption than the postulated properties A_0, A, A′.

a) By putting $k=0$ and $z'=0$ in Eq. (128) (or (129)), we obtain:

$$\mathring{\mathrm{G}}_{\circ} = (1 - \mathring{R}\underline{\mathring{\omega}})\mathring{\mathrm{F}}_{\circ} = \mathring{R}_{\circ} + g \qquad (130)$$

with

$$g = {}_{\circ}\mathring{\mathrm{F}}_{\circ} = {}_{\circ}\mathring{\mathrm{G}}_{\circ}. \qquad (131)$$

We can now apply the results of § 6.1, since they are consequences of the incremental equation (85) which is itself implied by Eqs. (58), (123) (by subtracting from Eq. (58) the restriction of the latter at $z'_0 = 0$). We thus have in view of formulae (86) and (94):

$$\mathring{\mathrm{G}} - \mathring{\mathrm{G}}_{\circ} = (1 + \mathring{\mathrm{L}}\underline{\mathring{\omega}})^{-1}\mathring{\mathrm{L}} \qquad (132)$$

with

$$\mathring{\mathrm{L}} = \mathring{\mathrm{F}} - \mathring{\mathrm{F}}_{\circ} . \mathring{\Lambda}^t = \mathring{N}^t \quad (\text{since } \mathring{\Lambda}^t_{\circ} = 1). \qquad (133)$$

So, by introducing the resolvent R^t of N^t (namely the transposed kernel of R), we deduce from Eqs. (132) and (133) that:

$$\mathring{\mathrm{G}} - \mathring{\mathrm{G}}_{\circ} = (1 + N^t\underline{\omega})^{-1}N^t = R^t. \qquad (134)$$

In view of Eq. (130), we thus obtain:

$$\mathring{\mathrm{G}} = \mathring{R}^t + \mathring{R}_{\circ} + g. \qquad (135)$$

b) At $k = 0$, Eq. (129) yields:

$$\mathring{R} + {}_{\circ}\mathring{\mathrm{F}} = \mathring{\mathrm{G}} + {}_{\circ}\mathring{\mathrm{F}}\underline{\mathring{\omega}}(\mathring{\mathrm{G}} - \mathring{\mathrm{G}}_{\circ}). \qquad (136)$$

By subtracting the latter from eq. (129), we thus have:

$$R - \mathring{R} + {}_{\circ}\mathring{\mathrm{F}}\frac{(\mathring{\omega} - \omega)}{\omega} = (1 + {}_{\circ}\mathring{\mathrm{F}}\underline{\mathring{\omega}})(\mathrm{G} - \mathring{\mathrm{G}}). \qquad (137)$$

Since $R - \mathring{R}$ is regular from the right, but not from the left, computing

$\mathrm{G} - \mathring{\mathrm{G}}$ involves a special limiting procedure: by applying the kernel ${}_{\circ}\mathring{\mathrm{F}}\mathring{\omega}_\rho$ on both sides of Eq. (137) from the left, we obtain:

$$
\begin{aligned}
&{}_{\circ}\mathring{\mathrm{F}}\underline{\mathring{\omega}}_\rho(R - \mathring{R}) + ({}_{\circ}\mathring{\mathrm{F}}\underline{\mathring{\omega}}_\rho 1){}_{\circ}\mathring{\mathrm{F}}\frac{(\mathring{\omega} - \omega)}{\omega} \\
&\qquad = {}_{\circ}\mathring{\mathrm{F}}(\underline{\mathring{\omega}}_\rho - \underline{\mathring{\omega}})(\mathrm{G} - \mathring{\mathrm{G}}) + (1 + {}_{\circ}\mathring{\mathrm{F}}\underline{\mathring{\omega}}_\rho 1)\,.\,{}_{\circ}\mathring{\mathrm{F}}\underline{\mathring{\omega}}(\mathrm{G} - \mathring{\mathrm{G}}) \qquad (138)
\end{aligned}
$$

and therefore:

$$
{}_{\circ}\mathring{\mathrm{F}}\underline{\mathring{\omega}}(\mathrm{G} - \mathring{\mathrm{G}}) = {}_{\circ}\mathring{\mathrm{F}}\frac{(\mathring{\omega} - \omega)}{\omega} + \lim_{\rho\to 0}\frac{{}_{\circ}\mathring{\mathrm{F}}\underline{\mathring{\omega}}_\rho(R - \mathring{R})}{1 + {}_{\circ}\mathring{\mathrm{F}}\underline{\mathring{\omega}}_\rho 1}. \qquad (139)
$$

Eq. (137) thus yields, in view of the latter:

$$
\mathrm{G} - \mathring{\mathrm{G}} = R - \mathring{R} - \lim_{\rho\to 0}\frac{{}_{\circ}\mathring{\mathrm{F}}\underline{\mathring{\omega}}_\rho(R - \mathring{R})}{1 + {}_{\circ}\mathring{\mathrm{F}}\underline{\mathring{\omega}}_\rho 1}. \qquad (140)
$$

The validity of this computation is of course submitted to the existence of the limit in the second term of the r.h.s. of Eq. (140)); it was also implicitly assumed (in the passage from Eqs. (138) to (139)) that $\mathrm{G} - \mathring{\mathrm{G}}$ satisfies the left-sided regularity property (thus implying that $\lim_{\rho\to 0} {}_{\circ}\mathring{\mathrm{F}}(\underline{\mathring{\omega}}_\rho - \underline{\mathring{\omega}})(\mathrm{G} - \mathring{\mathrm{G}}) = 0$), while eq. (137) only exhibits the right-sided regularity of $\mathrm{G} - \mathring{\mathrm{G}}$.

We present below in *c*) some improvement of the previous derivation.

c) In § 6.3, we introduced (after [*11*]) the family of regularized renormalized kernels G_ρ; each kernel G_ρ is defined by Eqs. (108), (111) in terms of the auxiliary kernel $\hat{\mathrm{G}}_\rho$, which is the solution of the regular Fredholm equation (107) with given kernel F.

Since, in view of property A′, $\Lambda = \lim_{\rho\to 0} \Lambda_\rho$, one also has (in view of Eq. (117) : $\lim_{\rho\to 0} N_\rho = N$, and we can express the right-sided regularity property of N by saying that for each function $f(z')$ of degree zero, one has:

$$
\lim_{\rho\to 0} N_\rho \underline{\omega} f = N \underline{\omega} f. \qquad (141)
$$

It follows that each iterated kernel satisfies:

$$
\lim_{\rho\to 0} \underbrace{(N_\rho \underline{\omega}_\rho \ldots \underline{\omega}_\rho N_\rho)}_{n} \underline{\omega}_\rho f = \underbrace{(N \underline{\omega} \ldots \underline{\omega} N)}_{n} \underline{\omega} f
$$

and that the resolvent R_ρ of N_ρ satisfies similarly:

$$
\lim_{\rho\to 0} R_\rho \underline{\omega}_\rho f = R \underline{\omega} f. \qquad (142)
$$

Since F, N_ρ and G_ρ are linked by the (regularized) renormalized equation (115), the algebraic developments given above in *a)* and *b)* apply to the corresponding regularized kernels. In particular, one obtains:

$$\overset{\circ}{\mathrm{G}}_\rho = \overset{\circ}{R}{}^t_\rho + \overset{\circ}{R}_{\rho\circ} + g,$$

and, in view of Eq. (135) and (142):

$$\overset{\circ}{\mathrm{G}} = \lim_{\rho\to 0} \overset{\circ}{\mathrm{G}}_\rho . \tag{143}$$

One also obtains the following formula, similar to Eq. (137):

$$(1 + {}_\circ\overset{\circ}{\mathrm{F}}\underline{\overset{\circ}{\omega}_\rho})(\mathrm{G}_\rho - \overset{\circ}{\mathrm{G}}_\rho) = R_\rho - \overset{\circ}{R}_\rho + {}_\circ\overset{\circ}{\mathrm{F}}\,\frac{(\overset{\circ}{\omega}_\rho - \omega_\rho)}{\omega_\rho}. \tag{144}$$

Now, in view of the permutability of F and $\hat{\mathrm{G}}_\rho$ in Eq. (107), G_ρ also satisfies the transposed renormalized equation:

$$\mathrm{F} = \mathrm{G}_\rho + \mathrm{G}_\rho\underline{\omega_\rho}N^t_\rho + (\mathrm{G}_\rho - \overset{\circ}{\mathrm{G}}_\rho)\underline{\overset{\circ}{\omega}_\rho}\overset{\circ}{\mathrm{F}}_\rho . \Lambda^t_\rho \tag{145}$$

from which the transposed counterpart of Eq. (144) follows, namely:

$$(\mathrm{G}_\rho - \overset{\circ}{\mathrm{G}}_\rho)(1 + \underline{\overset{\circ}{\omega}_\rho}\overset{\circ}{\mathrm{F}}_\circ) = R^t_\rho - \overset{\circ}{R}{}^t_\rho + \frac{(\overset{\circ}{\omega}_\rho - \omega_\rho)}{\omega_\rho}\,\overset{\circ}{\mathrm{F}}_\circ . \tag{146}$$

From Eqs. (144) and (146), we deduce the following two expressions of $\mathrm{G}_\rho - \overset{\circ}{\mathrm{G}}_\rho$ (respectively by computing ${}_\circ\overset{\circ}{\mathrm{F}}\underline{\omega_\rho}(\mathrm{G}_\rho - \overset{\circ}{\mathrm{G}}_\rho)$ and $(\mathrm{G}_\rho - \overset{\circ}{\mathrm{G}}_\rho)\underline{\overset{\circ}{\omega}_\rho}\overset{\circ}{\mathrm{F}}_\circ$):

$$\mathrm{G}_\rho - \overset{\circ}{\mathrm{G}}_\rho = R_\rho - \overset{\circ}{R}_\rho - \frac{{}_\circ\overset{\circ}{\mathrm{F}}\underline{\overset{\circ}{\omega}_\rho}(R_\rho - \overset{\circ}{R}_\rho)}{1 + {}_\circ\overset{\circ}{\mathrm{F}}\underline{\overset{\circ}{\omega}_\rho}1} + \frac{1}{(1 + {}_\circ\overset{\circ}{\mathrm{F}}\underline{\overset{\circ}{\omega}_\rho}1)} \cdot {}_\circ\overset{\circ}{\mathrm{F}}\,\frac{(\overset{\circ}{\omega}_\rho - \omega_\rho)}{\omega_\rho} \tag{147}$$

$$= R^t_\rho - \overset{\circ}{R}{}^t_\rho - \frac{(R^t_\rho - \overset{\circ}{R}{}^t_\rho)\underline{\overset{\circ}{\omega}_\rho}\overset{\circ}{\mathrm{F}}_\circ}{1 + 1\underline{\overset{\circ}{\omega}_\rho}\overset{\circ}{\mathrm{F}}_\circ} + \frac{1}{(1 + 1\underline{\overset{\circ}{\omega}_\rho}\overset{\circ}{\mathrm{F}}_\circ)} \cdot \frac{(\overset{\circ}{\omega}_\rho - \omega_\rho)}{\omega_\rho}\,\overset{\circ}{\mathrm{F}}_\circ . \tag{148}$$

Under the assumption that the following limiting kernel exists,

$$\lim_{\rho\to 0} \frac{{}_\circ\overset{\circ}{\mathrm{F}}\underline{\overset{\circ}{\omega}_\rho}(R_\rho - \overset{\circ}{R}_\rho)}{1 + {}_\circ\overset{\circ}{F}\underline{\overset{\circ}{\omega}_\rho}1} = \lim_{\rho\to 0} \frac{{}_\circ\overset{\circ}{\mathrm{F}}\underline{\overset{\circ}{\omega}_\rho}(R - \overset{\circ}{R})}{1 + {}_\circ\overset{\circ}{\mathrm{F}}\underline{\overset{\circ}{\omega}_\rho}1}, \tag{149}$$

there exists a kernel: $\mathrm{G} - \overset{\circ}{\mathrm{G}} = \lim_{\rho\to 0} (\mathrm{G}_\rho - \overset{\circ}{\mathrm{G}}_\rho)$, which, in view of Eq. (147), is expressed by Eq. (140) and also by the transposed equation (via Eq. (148)):

$$\mathrm{G} - \overset{\circ}{\mathrm{G}} = R^t - \overset{\circ}{R}{}^t - \lim_{\rho\to 0} \frac{(R^t - \overset{\circ}{R}{}^t)\underline{\overset{\circ}{\omega}_\rho}\overset{\circ}{\mathrm{F}}_\circ}{1 + 1\underline{\overset{\circ}{\omega}_\rho}\overset{\circ}{\mathrm{F}}_\circ}. \tag{150}$$

In view of the right-sided (resp. left-sided) regularity of R (resp. R^t), formulae (140) and (150) imply that $\mathrm{G} - \overset{\circ}{\mathrm{G}}$ satisfies both-sided regularity.

Finally by taking Eq. (135) into account (and the transposed version of it, equally valid) we deduce from Eqs. (140) and (150) the following:

THEOREM 1. — *If the amputated (connected) four-point function* F *satisfies the (euclidean) asymptotic properties* A, A′ *and if the limiting kernels involved in the expression below exist, then there exists a unique Bethe-Salpeter kernel G solution of the renormalized Bethe-Salpeter equation* (58). *G admits the following expressions in terms of the resolvent R (resp. R^t) of the normal product kernel N (resp. N^t) and of* ${}_{\circ}\overset{\circ}{\mathrm{F}}$ *(resp.* $\overset{\circ}{\mathrm{F}}_{\circ}$*)*

$$\mathrm{G} = R + {}_{\circ}\overset{\circ}{R}{}^t + g - \lim_{\rho\to 0} \frac{{}_{\circ}\overset{\circ}{\mathrm{F}}\underline{\overset{\circ}{\omega}_\rho}(R - \overset{\circ}{R})}{1 + {}_{\circ}\overset{\circ}{\mathrm{F}}\underline{\overset{\circ}{\omega}_\rho}1}$$

$$= R^t + \overset{\circ}{R}_{\circ} + g - \lim_{\rho\to 0} \frac{(R^t - \overset{\circ}{R}{}^t)\underline{\overset{\circ}{\omega}_\rho}\overset{\circ}{\mathrm{F}}_{\circ}}{1 + 1\underline{\overset{\circ}{\omega}_\rho}\overset{\circ}{\mathrm{F}}_{\circ}}. \tag{151}$$

Moreover, the regularization procedure has a limit, namely: if G_ρ *is defined by Eqs.* (107), (108), (111), *then*

$$\mathrm{G} = \lim_{\rho\to 0} \mathrm{G}_\rho. \tag{152}$$

§ 7.2. — We now consider the four-point functions F, N and N^t in the primitive analyticity domain D_4 prescribed by axiomatic field theory and we assume that the asymptotic properties A, A′ are valid (as well as A_0) not only in the euclidean region itself, but on any integration cycle which is asymptotically parallel to the euclidean region. Then, the resolvent R (resp. R^t) of N (resp. N^t) can be analytically continued (9) in D_4 as it results from the technique of contour deformation used in [*2-a*]; according to the latter, it is indeed possible to define a four-dimensional cycle $\Gamma(k; z, z')$ in complex ζ-space, depending continuously on k, z, z' and admitting the euclidean subspace as its initial configuration (obtained when k, z, z' are themselves in euclidean space). Eq. (58) and all subsequent equations in the algebraic analysis of § 7.1 can then be understood as integral relations on the space $\Gamma(k; z, z')$; in particular any term of the form $\mathrm{K}_1\underline{\omega}\mathrm{K}_2$ or ${}_{\circ}\overset{\circ}{\mathrm{F}}\underline{\overset{\circ}{\omega}}\mathrm{K}$ (K, K_1, K_2 denoting general four-point functions) now has the following meaning:

$$(\mathrm{K}_1\underline{\omega}\mathrm{K}_2)(k; z, z') = \int_{\Gamma(k,z,z')} \mathrm{K}_1(k; z, \zeta)\mathrm{K}_2(k, \zeta, z')\omega(k, \zeta)d\zeta \tag{153}$$

$$({}_{\circ}\overset{\circ}{\mathrm{F}}\underline{\overset{\circ}{\omega}}\mathrm{K})(k, z') = \int_{\Gamma(k,z')} \mathrm{F}(o; o, \zeta)\mathrm{K}(k; \zeta, z')\omega(o, \zeta)d\zeta \tag{154}$$

(9) Up to possible poles in the variable k^2, as it is always the case when a k-dependent Fredholm resolvent is taken (see [*2*]).

Remark. — A function of the form (154) (being constant with respect to z) is analytic in a larger domain than D_4, namely in the following domain: $\{ (k, z, z'); z \in \mathbb{C}^4, (k, z') \in D_3 \}$, where D_3 is the primitive domain of a general three-point function, with $k = k'_1 + k'_2, z' = \frac{k'_1 - k'_2}{2}\Big)$.

From formula (151), we then deduce that G admits an analytic continuation (up to possible poles in the variables k^2) in the domain D_4; similarly Eq. (152) also extends to D_4. This leads to a short proof (indicated in [*11*]) of the fact that G is two-particle irreducible: each regular kernel $\hat{G}_\rho$ being irreducible in view of [*2*], one has $\Delta\hat{G}_\rho = 0$ (for $k^2 < (4\mu)^2$), and therefore $\Delta G_\rho = 0$ (since $\hat{G}_\rho = G_\rho - C_\rho$); this yields:

$$\Delta G = \lim_{\rho \to 0} \Delta G_\rho = 0. \tag{155}$$

It is however interesting to give a direct proof of the irreducibility of G, starting from the asymptotic completeness equation satisfied by F, namely: for $k^2 < (4\mu)^2$,

$$\Delta F = F^+ * F^- \tag{156}$$

(see section 2, formulae (5) (5')).

The main ingredient of this proof is the discontinuity formula for $\mathcal{G}$-convolution products of the form (153) or (154), which we now recall. The geometrical fact that matters is that when k tends to a real point such that $k^2 < (4\mu)^2$, from the respective sides $\operatorname{Im} k^{(0)} > 0$, $\operatorname{Im} k^{(0)} < 0$, the cycle $\Gamma(k, z, z')$ of formulae (157) (154) tends to two limiting positions, called respectively $\Gamma^+(k)$ and $\Gamma^-(k)$ (the dependence of Γ on the variables (z, z') can indeed be suppressed, when (k, z, z') vary in the domain D, which is sufficient for reaching the mass shell); these two cycles satisfy the basic relation:

$$\Gamma^+(k) - \Gamma^-(k) = \tilde{e}(k), \tag{157}$$

here $\tilde{e}(k)$ denotes a small cycle surrounding the « mass shell manifold »: $e(k) = \left\{ \zeta; \left(\frac{k}{2} + \zeta\right)^2 = \mu^2, \left(\frac{k}{2} - \zeta\right)^2 = \mu^2 \right\}$, which is such that any integral over $\tilde{e}(k)$ is equal to the integral over $e(k)$ of the (double) residue of the corresponding integrand: this « mass shell integration » corresponds precisely to the operation denoted by $*$, introduced in section 2 (see formulae (5) (5')). By using these facts, one derives the following discontinuity formula for a $\mathcal{G}$-convolution product of the form (153) (see [*2b*]):

$$\Delta(K_1 \underline{\omega} K_2) = (K_1^+ \underline{\omega}^+ K_2^+) - (K_1^- \underline{\omega}^- K_2^-) = K_1^+ \underline{\omega}^+ \Delta K_2 + \Delta K_1 \underline{\omega}^- K_2^- + K_1^+ * K_2^-, \tag{158}$$

where the notation $K_i^\pm$ refers to the side ($\operatorname{Im} k^{(0)} > 0$ or < 0) from which the boundary value of K_i is taken, while the notation $\underline{\omega}^+$ (resp. $\underline{\omega}^-$) indicates that the prescribed integration cycle is $\Gamma^+(k)$ (resp. $\Gamma^-(k)$) (note that $\Gamma^+(k)$ and $\Gamma^-(k)$ differ precisely by the way they go round the double pole manifold, on which ω is singular). On the other hand, the discontinuity formula for (154) reduces to:

$$\Delta(\underline{F\omega K}) = \underline{F\omega}\Delta K \tag{159}$$

since the measure $\underline{\omega(\zeta)} = [H^{(2)}(\zeta)]^2 d\zeta$ is regular on the double pole manifold mentioned above, thus implying that the corresponding residue on $e(k)$ is equal to zero; in (159) the integration cycle can be chosen indifferently to be $\Gamma^+(k)$ or $\Gamma^-(k)$.

We are now in a position to derive from the A. C. equation (156):

i) similar discontinuity formulae for Λ and N

ii) the irreducibility formula: $\Delta G = 0$.

i) By applying formula (61), we first obtain:

$$\Delta\Lambda = \lim_{\rho\to 0} \frac{1}{1 + {}_{\circ}\overset{\circ}{F}\underline{\overset{\circ}{\omega}_\rho} 1} \Delta(F\underline{\omega_\rho} 1),$$

and by making use of formula (158), and then taking Eq. (156) into account:

$$\begin{aligned}\Delta(F\underline{\omega_\rho}1) &= \Delta F\underline{\omega_\rho^-}1 + F^+ * 1\\ &= F^+ * (1 + F^-\underline{\omega_\rho^-}1).\end{aligned}$$

which thus yields (for $k^2 < (4\mu)^2$):

$$\Delta\Lambda = \lim_{\rho\to 0} F^+ * \frac{(1 + F^-\underline{\omega_\rho^-}1)}{(1 + {}_{\circ}\overset{\circ}{F}\underline{\overset{\circ}{\omega}_\rho}1)} = F^+ * \Lambda^- . \tag{160}$$

From Eq. (56), we now deduce:

$$\Delta N = \Delta F - \Delta\Lambda . {}_{\circ}\overset{\circ}{F} \frac{\overset{\circ}{\omega}}{\omega},$$

which, in view of Eqs. (156), (160), yields:

$$\Delta N = F^+ * \left(F^- - \Lambda^- . {}_{\circ}\overset{\circ}{F} \frac{\overset{\circ}{\omega}}{\omega}\right) = F^+ * N^- . \tag{161}$$

ii) Let us take the discontinuities of both sides of Eq. (58); by making use of formulae (158) and (159), we obtain:

$$\begin{aligned}\Delta F &= \Delta G + N^+\underline{\omega^+}\Delta G + \Delta N\underline{\omega^-}G^- + N^+ * G^-\\ &+ \Delta\Lambda . {}_{\circ}\overset{\circ}{F}\underline{\overset{\circ}{\omega}}(G^- - \overset{\circ}{G}_{\circ}) + \Lambda^+ . {}_{\circ}\overset{\circ}{F}\underline{\overset{\circ}{\omega}}\Delta G\end{aligned} \tag{162}$$

(note that in the last term ${}_{\circ}\overset{\circ}{F}\underline{\overset{\circ}{\omega}}\Delta G$ makes sense, since the increment kernel $\Delta G = G^+ - G^- = (G^+ - \overset{\circ}{G}) - (G^- - \overset{\circ}{G})$ is regular at infinity from both sides).

By now taking Eqs. (156), (160) and (161) into account, we can rewrite Eq. (162) as follows:

$$\begin{aligned}\Delta F = F^+ * F^- &= (1 + N^+ \underline{\omega}^+)\Delta G + N^+ * G^- \\ &+ F^+ * [N^- \underline{\omega}^- G^- + \Lambda^- . {}_{\circ}\mathring{F}\underline{\mathring{\omega}}(G^- - \mathring{G}_{\circ}) + \Lambda^+ . {}_{\circ}\mathring{F}\underline{\mathring{\omega}}\Delta G, \end{aligned} \tag{163}$$

We now notice that (in view of Eq. (56)):

$$N^+ * G^- = F^+ * G^- - \Lambda^+ . \left({}_{\circ}\mathring{F} \frac{\mathring{\omega}}{\omega} \right) * G^-, \tag{164}$$

but since ${}_{\circ}\mathring{F}\mathring{\omega}$ is analytic on $e(k)$, the last term in Eq. (164) vanishes, and we can thus rewrite Eq. (163), as follows:

$$\begin{aligned} F^+ * [F^- - G^- - N^- \underline{\omega}^- G^- . - \Lambda^- . {}_{\circ}\mathring{F}\underline{\mathring{\omega}}(G^- - \mathring{G}_{\circ})] \\ = [1 + N^+ \underline{\omega}^+ + \Lambda^+ . {}_{\circ}\mathring{F}\underline{\mathring{\omega}}]\Delta G \end{aligned} \tag{165}$$

By using again Eq. (58) for the boundary value F^- of F, and Eq. (56), we deduce from Eq. (165):

$$(1 + F^+ \underline{\omega}^+)\Delta G = 0 \tag{166}$$

or, by applying the operator $(1 - R^+ \underline{\omega}^+)$ from the left and taking Eq. (126) into account (more precisely, its analytic continuation):

$$(1 + {}_{\circ}\mathring{F}\underline{\mathring{\omega}})\Delta G = 0. \tag{167}$$

Integrating the l. h. s. of the latter with ${}_{\circ}\mathring{F}\underline{\mathring{\omega}}_{\rho}$ yields:

$$\forall \rho, \qquad {}_{\circ}\mathring{F}(\underline{\mathring{\omega}}_{\rho} - \underline{\mathring{\omega}})\Delta G + (1 + {}_{\circ}\mathring{F}\underline{\mathring{\omega}}_{\rho}1){}_{\circ}\mathring{F}\underline{\mathring{\omega}}\Delta G = 0 \tag{168}$$

and therefore

$${}_{\circ}\mathring{F}\underline{\mathring{\omega}}\Delta G = \lim_{\rho \to 0} \frac{{}_{\circ}\mathring{F}(\underline{\mathring{\omega}} - \underline{\mathring{\omega}}_{\rho})\Delta G}{1 + {}_{\circ}\mathring{F}\underline{\mathring{\omega}}_{\rho}1} = 0. \tag{169}$$

Eqs (167) and (169) then imply: $\Delta G = 0$.

§ 7.3. — We now consider the equations (74) for the N-point functions; namely:

$$F^{(N)} = G^{(N-2,2)} + N^{(N)}\underline{\omega}G + \Lambda^{(N-1)} . {}_{\circ}\mathring{F}\underline{\mathring{\omega}}(G - \mathring{G}_{\circ}). \tag{74}$$

These equations define directly the kernels $G^{(N-2,2)}$ in terms of the functions $F^{(N)}$ and of the Bethe-Salpeter kernel $G \equiv G^{(2,2)}$, previously constructed; this definition holds in the whole primitive analyticity domain of N-point functions.

We now indicate briefly the proof of the two-particle irreducibility of the kernels $G^{(N-2,2)}$ (similar to the proof of the irreducibility of G, given in § 7.2).

i) The Asymptotic Completeness equations for the functions $F^{(N)}$ in the $(N-2, 2)$ channels are (see e. g. [2*b*]):

$$\Delta F^{(N)} = F^{(N)+} * F^{-} \qquad \text{for} \qquad k^2 < (4\mu)^2 . \tag{170}$$

By applying formula (76), we then deduce from the latter (in view of (61)):

$$\Delta\Lambda^{(N-1)} = F^{(N)+} * \Lambda^{-} . \tag{171}$$

Then, in view of Eq. (71), we also obtain:

$$\Delta N^{(N)} = F^{(N)+} * N^{-} . \tag{172}$$

ii) Taking the discontinuities of both sides of Eq. (74) yields (after applying the relevant discontinuity formulae, similar to Eqs. (158), (159)):

$$\begin{aligned} \Delta F^{(N)} &= \Delta G^{(N-2,2)} + \Delta N^{(N)}\underline{\omega}^{-}G + N^{(N)+} * G \\ &+ \Delta\Lambda^{(N-1)} \cdot {}_{\circ}\mathring{F}\underline{\mathring{\omega}}(G - \mathring{G}_{\circ}) . \end{aligned} \tag{173}$$

In the latter, we took into account the fact that G is irreducible (i. e. $G^{+} = G^{-} = G$ or $\Delta G = 0$). By applying Eqs. (170), (171), (172), Eq. (173) then becomes (since $N^{(N)+} * G = F^{(N)+} * G$, in view of the argument given after formula (164)):

$$\Delta G^{(N-2,2)} = F^{(N)+} * [F^{-} - G - (N^{-}\underline{\omega}^{-}G) - \Lambda^{-} \cdot {}_{\circ}\mathring{F}\underline{\mathring{\omega}}(G - \mathring{G}_{\circ})] .$$

But in view of (the analytic continuation of) Eq. (58), the expression enclosed in the bracket at the r. h. s. of this equation is equal to zero, and therefore: $\Delta G^{(N-2,2)} = 0$.

For completeness, one should also consider the equations (45), namely:

$$F^{(N_1+N_2)} = G^{(N_1,N_2)} + F^{(N_1+2)}\underline{\omega}G^{(2,N_2)} ,$$

which define the kernels $G^{(N_1,N_2)}$ in arbitrary (N_1, N_2)-channels in terms of the kernels $G^{(2,N_2)}$ (obtained in the same way as $G^{(N_1,2)}$ by taking the transposed version of all the previous equations). The proof of the irreducibility of $G^{(N_1,N_2)}$ (i. e. $\Delta G^{(N_1,N_2)} = 0$) is not different from the one given in the regularized case (see [2*b*]).

We can now summarize these results together with those of § 7.2 in the following:

THEOREM 2. — *Properties* A_0, A, A′, *A*, *A′* *together with the Asymptotic Completeness Equations for the functions* $F^{(N)}$ *imply the following properties*:

i) The functions $\Lambda^{(N-1)}$ *(resp.* $N^{(N)}$*) are general* $(N-1)$*-point (resp.* N*-point) functions which satisfy the following discontinuity formulae (in the* $(N-2, 2)$ *channel)*:

$$\begin{aligned} (\forall N \geqslant 4) \qquad \Delta\Lambda^{(N-1)} &= F^{(N)+} * \Lambda^{-} \qquad (\Lambda = \Lambda^{(3)}) \\ \Delta N^{(N)} &= F^{(N)+} * N^{-} \qquad (N = N^{(4)}) . \end{aligned}$$

ii) Under the additional conditions of theorem 1, all the kernels $G^{(N_1,N_2)}$ *(with* $G^{(2,2)} = G$*) defined in terms of the functions* $F^{(N)}$ *through the renormalized two-particle structure equations are general* N*-point functions which satisfy the two-particle irreducibility conditions*:

$$\Delta G^{(N_1,N_2)}(k;\, z^{(N_1)}, z^{(N_2)}) = 0 \qquad \textit{for} \qquad k^2 < (4\mu)^2 .$$

8. COMPLEMENTS ON THE CONSTRUCTION OF THE FOUR-POINT FUNCTION F AND OF THE « NORMAL GREEN FUNCTIONS » Λ AND *N* IN TERMS OF THE BETHE-SALPETER KERNEL G

In section 3 the problem of the construction of F in terms of G and $H^{(2)}$ was considered in the sense of formal series, with the results stated in propositions 2 and 3.

In the general approach of the Bethe-Salpeter equation, this problem can be treated, as first proposed by K. Symanzik in [4*b*] (App. B) on the basis of the incremental equations (85) (together with the corresponding transposed equations) and (96). Our assumptions on $H^{(2)}$ and G are those of properties A_0 and B.

Formula (89) (and the corresponding transposed formula) yield:

$$\mathring{F} = \mathring{F}_{\circ}(1 + 1\underline{\mathring{\omega}}\mathring{L}) + \mathring{L} \tag{174}$$

$$\mathring{F}_{\circ} = {}_{\circ}\mathring{F}_{\circ}(1 + \mathring{L}^t{}_{\circ}\underline{\mathring{\omega}}1) + \mathring{L}^t{}_{\circ}, \tag{175}$$

$${}_{\circ}F = G_{\circ} = g, \tag{24}$$

where the resolvents $\breve{L}$ and L^t are given by:

$$(1 + \underline{\mathring{\omega}}\mathring{L}) = [1 - \underline{\mathring{\omega}}(\mathring{G} - \mathring{G}_{\circ})]^{-1} \tag{176}$$

$$(1 + \mathring{L}^t\omega) = [1 - (\mathring{G} - {}_{\circ}\mathring{G})\underline{\mathring{\omega}}]^{-1}, \tag{177}$$

$\breve{L}$ and $\breve{G} - \mathring{G}_{\circ}$ (resp. $\breve{L}^t$ and $\mathring{G} - {}_{\circ}\breve{G}$) being regular from the left (resp. right) side.

Now, Eq. (96) yields F in terms of $\mathring{F}$ and $G - \mathring{G}$:

$$F = [1 - \mathring{F}(\underline{\omega} - \underline{\mathring{\omega}}) - (1 + \mathring{F}\underline{\mathring{\omega}})(G - \mathring{G})\underline{\omega}]^{-1}\,[\mathring{F} + (1 + \mathring{F}\underline{\mathring{\omega}})(G - \mathring{G})] \tag{178}$$

In the latter, the first factor is supposed to be well-defined and regular from the right-side, as it is the case for the kernel: $\mathring{F}(\omega - \mathring{\omega}) + (1 + \mathring{F}\underline{\mathring{\omega}})(G - \mathring{G})\omega$ (see our comment at the top of section 6).

We also note that the « normal » Green functions N, Λ, N^t, Λ^t can be expressed ([10]) in terms of F and of the increments of G, without any limiting procedure involved (in contrast with the definitions (59), (56) of Λ and N in terms of F); in fact, such expressions have been given for Λ^t and N^t respectively in formula (100) and (106), which we rewrite here (in view of Eq. (93), written at $k=0$):

$$\Lambda^t = (1 + 1\underline{\overset{\circ}{\omega}}\overset{\circ}{\mathrm{L}}).(1 + \underline{\overset{\circ}{\omega}}W) \tag{179}$$

$$.\underline{\omega}N^t = .\ [(1 + \underline{\overset{\circ}{\omega}}\overset{\circ}{\mathrm{L}}).(1 + \underline{\overset{\circ}{\omega}}W) - 1] \tag{180}$$

with
$$W = (\mathrm{G} - \overset{\circ}{\mathrm{G}})(1 + \underline{\omega}\mathrm{F}) + \frac{\omega - \overset{\circ}{\omega}}{\overset{\circ}{\omega}}\mathrm{F}. \tag{97}$$

In view of properties A_0 and B, the kernels W and $\overset{\circ}{\mathrm{L}}$ satisfy the left-sided regularity property, and this implies that the r. h. s. of Eqs. (179) and (180) are well-defined: in the approach of this section, we can thus introduce Λ^t and N^t through these formulae, N^t being automatically regular from the left-side (as W and $\overset{\circ}{\mathrm{L}}$). Formulae (179) and (180) then imply that $\Lambda^t = 1 + 1\underline{\omega}N^t$ and (by inverting the computation of § 6.2, between formulae (98) and (105)) that: $\mathrm{F} = N^t + \frac{\overset{\circ}{\omega}}{\omega}\overset{\circ}{\mathrm{F}}_{\circ}.\Lambda^t$. On the other hand, formulae (174) ... (180) and (97) show that N^t and Λ^t, as well as F, are expressible as pure functionals of G and $\mathrm{H}^{(2)}$.

We now show that if the four-point function G satisfies the irreducibility property (i. e. $\Delta\mathrm{G} = 0$, for $k^2 < (4\mu)^2$), then the function F, constructed by formulae (174)-(178), satisfies the asymptotic completeness relation (5). Let us take the discontinuities of both sides of Eq. (96). By making use of formulae (158), (159) and of the assumption $\Delta\mathrm{G} = 0$ (and by noticing that the quantities $\overset{\circ}{\mathrm{F}} = \mathrm{F}(o; z, z')$ and $\overset{\circ}{\omega} = \omega(o, z)$ have no discontinuities, we obtain:

$$\Delta\mathrm{F} - \overset{\circ}{\mathrm{F}}(\underline{\omega}^+ - \underline{\overset{\circ}{\omega}})\Delta\mathrm{F} - \overset{\circ}{\mathrm{F}} * \mathrm{F}^- = (1 + \overset{\circ}{\mathrm{F}}\underline{\overset{\circ}{\omega}})(\mathrm{G} - \overset{\circ}{\mathrm{G}})\underline{\omega}^+\Delta\mathrm{F} + (1 + \overset{\circ}{\mathrm{F}}\underline{\overset{\circ}{\omega}})(\mathrm{G} - \overset{\circ}{\mathrm{G}}) * \mathrm{F}^-$$

or:

$$[1 - \mathrm{F}(\underline{\omega}^+ - \underline{\overset{\circ}{\omega}}) - (1 + \overset{\circ}{\mathrm{F}}\underline{\overset{\circ}{\omega}})(\mathrm{G} - \overset{\circ}{\mathrm{G}})\underline{\omega}^+]\Delta\mathrm{F} = [\mathrm{F} + (1 + \mathrm{F}\underline{\omega})(\mathrm{G} - \mathrm{G})] * \mathrm{F}^- \tag{181}$$

Now, Eq. (178) entails, by analytic continuation:

$$[1 - \overset{\circ}{\mathrm{F}}(\underline{\omega}^+ - \underline{\overset{\circ}{\omega}}) - (1 + \overset{\circ}{\mathrm{F}}\underline{\overset{\circ}{\omega}})(\mathrm{G} - \overset{\circ}{\mathrm{G}})\underline{\omega}^+]\mathrm{F}^+ = \overset{\circ}{\mathrm{F}} + (1 + \overset{\circ}{\mathrm{F}}\underline{\overset{\circ}{\omega}})(\mathrm{G} - \overset{\circ}{\mathrm{G}}),$$

(all the integrations in this expression being taken on $\Gamma^+(k)$).

([10]) This has been done in [*4*], with the notation W for the present kernel ωN^t (see our remark at the end of § 6.2).

By applying the operation $* \mathrm{F}^-$ (from the right) on both sides of the latter, and by taking Eq. (181) into account, we then obtain:

$$[1-\mathring{\mathrm{F}}(\underline{\omega}^+-\underline{\mathring{\omega}})-(1+\mathring{\mathrm{F}}\underline{\mathring{\omega}})(\mathrm{G}-\mathring{\mathrm{G}})\underline{\omega}^+](\Delta\mathrm{F}-\mathrm{F}^+ * \mathrm{F}^-)=0. \tag{182}$$

The asymptotic completeness equation ($\Delta\mathrm{F} - \mathrm{F}^+ * \mathrm{F}^- = 0$) follows, provided the operator inside the bracket is invertible for all k in the region $k^2 < (4\mu)^2$.

In conclusion, we can state:

THEOREM 3. — *Being given the Bethe-Salpeter kernel* G, *as a general four-point function satisfying the asymptotic property* B *and the irreducibility condition* ($\Delta\mathrm{G} = 0$ *for* $k^2 < (4\mu)^2$), *and provided the two-point function* $\mathrm{H}^{(2)}$ *satisfies property* A_0, *the following results hold:*

i) The (amputated and connected) four-point function F *can be constructed in a unique way in terms of* G *and* $\mathrm{H}^{(2)}$, *with the normalization condition* $\mathrm{F}(o; o, o) = \mathrm{G}(o; o, o)$.

ii) The decomposition of F *in terms of the normal Green functions* Λ *and* N *can be constructed in terms of* G *and* $\mathrm{H}^{(2)}$*(via* F*), through formulae which avoid the limiting procedure.*

iii) The Green functions F, Λ *and* N *satisfy respectively the Asymptotic Completeness relations* (5′), (160), (161).

The construction of F *(in i) and the proof of iii) necessitate the use of an extra-assumption, namely the invertibility of the operator:*

$$[1 - \mathring{\mathrm{F}}(\underline{\omega} - \underline{\mathring{\omega}}) - (1 + \mathring{\mathrm{F}}\underline{\mathring{\omega}})(\mathrm{G}-\mathring{\mathrm{G}})\underline{\omega}] \qquad \text{on} \qquad \Gamma(k),$$

for all k in the relevant analyticity domain (containing the euclidean region and the physical points $(2\mu)^2 < k^2 < (4\mu)^2$).

Remark on the phenomenon of Landau ghosts.

The problem of invertibility mentioned at the end of theorem 3 is closely connected with the phenomenon of Landau ghosts (recalled in the introduction), and thereby with the problem of the existence of renormalizable field theories. As already observed, the axiomatic properties of G and $\mathrm{H}^{(2)}$, together with the asymptotic properties A_0 and B imply the analyticity and the right-sided regularity of the k-dependent kernel:

$$\mathscr{L} \equiv \mathscr{L}(k; z, z') = \mathring{\mathrm{F}}(\omega - \mathring{\omega}) + (1 + \mathring{\mathrm{F}}\underline{\mathring{\omega}})(\mathrm{G} - \mathring{\mathrm{G}})\omega.$$

Let us then treat $\mathscr{L}$ as a Fredholm kernel, dependent on the complex parameter k and acting on the space $\Gamma(k)$. It follows from [2] that $\mathscr{L}$ admits a resolvent kernel of the form $\dfrac{N(k; z, z')}{D(k)}$, where N and D are analytic functions defined in the domain $\hat{\mathrm{D}}$ (as sums of Fredholm series). From the expression (178) of F, we conclude that the axiomatic analyticity domain of F is preserved if and only if the Fredholm denominator $D(k)$ has no zero in the physical sheet of its domain. In fact, the existence of such a zero of $D(k)$ would imply a pole for F in its primitive domain; but this would express the impossibility for F and G to satisfy simultaneously the relevant axiomatic properties, and would therefore imply the inconsistency of the field theory considered, in the normal axiomatic framework.

We notice that if one tries for G a constant kernel $G = g$, one obtains (in view of Eqs. (174)...(177)) $\overset{\circ}{F} = g$, and since $G - \overset{\circ}{G} = 0$, the kernel $\mathscr{L}$ reduces to $g(\omega(k, z') - \omega(o, z'))$, whose resolvent is $\dfrac{g(\omega(k, z') - \omega(o, z'))}{1 - g\mathrm{I}(k)}$, with:

$$\mathrm{I}(k) = \int [\omega(k, z) - \omega(o, z)] d_4 z\,.$$

It can be seen (by an argument based on a subtracted dispersion relation and making use of the positivity of the discontinuity $\Delta\mathrm{I}$ of I on the set $k^2 > (2\mu)^2$) that, for k euclidean, and tending to infinity, $\mathrm{I}(k)$ behaves like $\gamma \ln(-k^2)$, with $\gamma > 0$. Therefore, in this case (as in the special case of the function $\mathrm{I}_0(k)$ corresponding to the choice:

$$\omega(k, z) = \left[\left(\frac{k}{2} + z\right)^2 - \mu^2\right]^{-1} x \left[\left(\left(\frac{k}{2} - z\right)^2 - \mu^2\right]^{-1}\right),$$

the function $D(k) = 1 - g\mathrm{I}(k)$ admits (for $g > 0$) a zero in euclidean space at $k^2 \approx - e^{\frac{1}{\gamma g}}$.

As far as the true kernel $\mathscr{L}$ is concerned, it is difficult to produce a rigorous argument, due to our insufficient knowledge of the properties of G. One can however argue that, in a family of theories indexed by the parameter $g = \mathrm{G}(o, o, o)$ the latter playing the role of a renormalized coupling constant, one may expect $|\mathrm{G} - \overset{\circ}{\mathrm{G}}|$ to be bounded at small g and large k by an expression of the form $g^2 \ln(-k^2)\underline{\mathrm{G}}(z, z')$, $\underline{\mathrm{G}}$ being a fixed Fredholm kernel. For g sufficiently small, the second term of $\mathscr{L}$ would then appear as a perturbation of the first one, and to the extent that $\overset{\circ}{F}$ behaves at infinity as a constant of the order and sign of g, the previous mathematical situation, yielding a zero of $D(k)$ in the euclidean region for $g > 0$, could represent a kind of generic obstruction, deeply connected with the sign of g. Properties of this type should be checked in the constructivist approach, in particular for the case of the massive Gross-Neveu model; it is indeed to be expected that for the « good sign » of g (namely, the one which makes the model constructible [7]) no Landau ghost will prevent the relevant kernel $\mathscr{L}$ to be invertible.

APPENDIX A

In [9], the convergence of the renormalized $\mathcal{G}$-convolution integrals was established by showing that the general Weinberg convergence criterion [17] was fulfilled by the integrand. In [10], we propose an alternative method ([11]) in which explicit bounds on the integrand are derived, the latter being valid uniformly in a certain region of complex space, thus allowing analytic continuation to be performed (starting from the euclidean region) into the two-particle physical region.

The following geometrical fact holds (see [10]):

PROPOSITION A-1. — *When the point (k, z, z') varies in an arbitrary compact subset* K *of a domain $(T_{M\eta} \times T_{\mu\eta} \times T_{\mu\eta})$ $(M < 4\mu)$, ramified around the threshold manifold $k^2 = (2\mu)^2$, the cycle of the integral* (22) *can be chosen to be $[\Gamma(k)]^{n-1}_{(\zeta_1,\dots,\zeta_{n-1})}$, where:*

$$\Gamma(k) = E_{(B)} + \Gamma_{(B)}(k);$$

B *denotes a positive constant (which may depend on the set* K *considered)* ; $E_{(B)}$ *is the part of the euclidean space* E *defined by the inequality*: $|\zeta|_E \geq B$, *and* $\Gamma_{(B)}(k)$ *is a chain contained in the bounded region*: $S_B = \{ \zeta \in \mathbb{C}^4; |\zeta|_E \leq B, \zeta \in T_{\mu\eta} \}$.

The complex space of variables $\zeta_1, \dots, \zeta_{n-1}$ is decomposed in the following sectorial regions (depending on the number B, introduced in proposition 1):

$$\sigma^{(B)}_{i_1\dots i_r} = \{ (\zeta_1, \dots, \zeta_{n-1}); \quad |\zeta_{i_1}|_E \geq \dots \geq |\zeta_{i_r}|_E \geq B; \quad \forall j, \quad j \neq i_1, \dots, i_r, \quad |\zeta_j|_E \leq B \}, \tag{A.1}$$

where, $(i_1, \dots, i_r)$ denotes any sequence of r elements of $\{1, 2, \dots, n-1\}$, $(1 \leq r \leq n-1)$, and

$$\sigma^{(B)}_0 = \{ (\zeta_1, \dots, \zeta_{n-1}); \quad \forall j, \quad |\zeta_j|_E \leq B \}. \tag{A.2}$$

Let $E^{(B)}_{i_1\dots i_r}$ be the euclidean subset of the projection of $\sigma^{(B)}_{i_1\dots i_r}$ onto $(\zeta_{i_1} \dots \zeta_{i_r})$-space. The basic property of the renormalized integral $\mathcal{R}_{\mathcal{G}_n}$ is the following

PROPOSITION A.2. — *With each sequence $i_1 \dots i_r$, there exists an associated decomposition of the integrand $\mathcal{R}_{\mathcal{G}_n}$*:

$$\mathcal{R}_{\mathcal{G}_n} = \sum_{U \in \mathcal{U}_{i_1\dots i_r}} X^U_{i_1\dots i_r}. \tag{A.3}$$

such that each term $X^U_{i_1\dots i_r}$ is uniformly integrable in the euclidean sector $E^{(B)}_{i_1\dots i_r}$, when the remaining variables $\zeta_j, j \neq i_1, \dots, i_r$, stay in S_B with (k, z, z') staying in the compact K; *the terms of this decomposition are labelled by the forests* U *of $\mathcal{G}_n$ which are « complete with respect to the set of planes » generated by the variables $\zeta_{i_1}, (\zeta_{i_1}, \zeta_{i_2}), \dots, (\zeta_{i_1} \dots \zeta_{i_r})$, and coincide with those defined in [9, proposition 3.1].*

In view of proposition A.1, the restriction of the integration contour $[\Gamma(k)]^{n-1}_{(\zeta_1\dots\zeta_n)}$ of (22) to any sectorial region $\sigma^{(B)}_{i_1\dots i_r}$ is the corresponding product cycle:

$$E^{(B)}_{i_1\dots i_r} \times [\Gamma_{(B)}(k)]^{n-r-1}_{\{\zeta_j; j \neq i_1,\dots,i_r\}}. \tag{A.4}$$

([11]) Although only applied here to the class of graphs $\mathcal{G}_n$ and in the case of renormalization parts with degree 0, this method should be valid in the general case.

and since $\Gamma_{(B)}(k)$ is contained in S_B, proposition A.2 entails the integrability of $\mathscr{R}_{\gamma_n}$ on this cycle. Summing over all sectorial regions $\sigma^{(B)}_{i_1\dots i_r}$ (including the bounded set $\sigma^{(B)}_0$) yields the integral (22); the analyticity of the latter with respect to (k, z, z') in the domain $T_{M\eta} \times T_{\mu\eta} \times T_{\mu\eta}$, ramified around $k^2 = (2\mu)^2$ with a two-sheeted structure (namely proposition 3, property *i*)), is a consequence of the uniform integrability property, contained in proposition A.2, and of the geometrical properties of $\Gamma_{(B)}(k)$ (described in [*2*]) when k turns around the threshold manifold $k^2 = (2\mu)^2$.

We now give a few indications on the proof of the proposition A.2, in the case when the functions C_j of property B_0 correspond to an *isotropic* behaviour of the derivatives of G.

The Weinberg criterion is replaced here by the following simple property which can easily be checked:

LEMMA. — *Let $\mathscr{C}_{\lambda_1\dots\lambda_r}$ be the class of continuous functions which admit bounds of the following type in the euclidean sector $E^{(B)}_{1,2,\dots,r}$:*

$$| f(\zeta_1, \dots, \zeta_r) | \leq C \prod_{1 \leq i \leq r} (1 + \| \zeta_i \|^{\lambda_i - 4 + \varepsilon}), \tag{A.5}$$

where $\lambda_1, \dots, \lambda_r$ are given integers, and ε is a sufficiently small positive number.

A sufficient condition for the functions of this class to be integrable over the set $E^{(B)}_{1,2,\dots,r}$ is that the following set of inequalities holds:

$$1 \leq q \leq r; \qquad \lambda_1 + \lambda_2 + \dots + \lambda_q \leq -1. \tag{A.6}$$

For applying this result to the proof of proposition A.2, the main part of the argument consists in checking that each complete forest term $X^U_{i_1\dots i_r}$ of the decomposition (A.3) admits a bound of the type (A.5) in the sector $E^{(B)}_{i_1\dots i_r}$ of the corresponding variables $\zeta_{i_1} \dots \zeta_{i_r}$, the constant C being uniform, when the variables $\zeta_j, j \neq i_1, \dots, i_r$ vary in S_B, with (k, z, z') staying in a compact K.

As described in [*9*, section3], each complete forest term is an ordered product, in which two types of operations are associated with the subgraphs of the forest: *i*) for a certain category of subgraphs, one takes a Taylor expansion at the renormalization point (e. g. zero): in the present case, it amounts to put at zero the external variables k, ζ_i, ζ_j of the subgraph considered (since each renormalization part has the degree zero). *ii*) For all the other subgraphs, the Taylor rest is taken instead.

The effect of the properties A_0 and B for the derivatives (namely formulae (19) and (20)) is the lowering of the degree at infinity by one unit in all the internal variables of the subgraphs of category *ii*) (the Taylor rest being written as an integral of a suitable derivative): in the subgraphs of category *i*), the degree at infinity is simply maintained. The most technical part of the argument is the proof that this mechanism of degree lowering precisely produces inequalities of the type (A.6) for the relevant exponents $\lambda_{i_1}, \dots, \lambda_{i_r}$: the order properties of the variables of the subgraphs of category *ii*) (considered as elements of the sequence $\zeta_{i_1}, \dots, \zeta_{i_r}$) implied by the definition of complete forests are essentially responsible for this fact. The same argument also allows one to keep control of the degrees at infinity in the external variables z, z', these degrees being maintained at zero after the integration (property *ii*) of proposition 3).

APPENDIX B

We give a proof of the discontinuity formula (25). From the assumption of irreducibility of G (namely $\Delta G(k; z, z') = 0$ for k real, $k^2 < (4\mu)^2$), it follows that G and all the functions $\mathcal{R}_{\mathcal{G}_n}[G \dots G; \omega \dots \omega]$ are analytic in a set of the form:

$$\mathcal{C} = \{ (k, z, z') : k \text{ real}, k^2 < (4\mu)^2, z \in T_{\mu\eta}, z' \in T_{\mu\eta} \}$$

(all internal variables $\zeta_1, \dots, \zeta_{n-1}$ varying also in $T_{\mu\eta}$).

At any point (k, z, z') in $\mathcal{C}$, we can then write:

$$\Delta F_n(k; z, z') = F_n^+(k; z, z') - F_n^-(k; z, z') \tag{B.1}$$

with

$$F_n^\varepsilon(k; z, z') = \int_{\Gamma^\varepsilon(k)} d\zeta_1 \dots \int_{\Gamma^\varepsilon(k)} d\zeta_{n-1} \mathcal{R}_{\mathcal{G}_n}[G \dots G; \omega \dots \omega](k; z, z'; \zeta_1 \dots \zeta_{n-1}) \tag{B.2}$$

$$\varepsilon = + \quad \text{or} \quad - .$$

In view of the previous remark, the integrand $\mathcal{R}_{\mathcal{G}_n}$ of (B.2) has the same determination in both formulae corresponding to $\varepsilon = +$ and $\varepsilon = -$ (namely, the limits $\mathcal{R}_{\mathcal{G}_n}^+$, $\mathcal{R}_{\mathcal{G}_n}^-$ from both sides $\operatorname{Im} k^{(0)} > 0$ and $\operatorname{Im} k^{(0)} < 0$ are equal).

The definition of the limiting contours $\Gamma^+(k)$, $\Gamma^-(k)$ (from the respective sides $\operatorname{Im} k^{(0)} > 0$, $\operatorname{Im} k^{(0)} < 0$ can be found in [2], as well as the following basic relation:

$$\Gamma^+(k) - \Gamma^-(k) = \tilde{e}(k); \tag{B.3}$$

$\tilde{e}(k)$ is a compact four-dimensional cycle surrounding the mass shell sphere

$$e(k) = \left\{ \zeta ; \left(\frac{k}{2} + \zeta\right)^2 = \left(\frac{k}{2} - \zeta\right)^2 = \mu^2 \right\}$$

(more precisely, the « Leray's coboundary of $e(k)$ ») and allowing the residue theorem to be applied on the latter, when a double pole factor $\omega(k, \zeta)$ is present.

In view of (B.3), formulae (B.1), (B.2) yield formally:

$$\Delta F_n(k; z, z') = \sum_{1 \le p \le n-1} \int_{[\Gamma^+(k)]^{p-1}} d\zeta_1 \dots d\zeta_{p-1} \int_{\tilde{e}(k)} d\zeta_p \int_{[\Gamma^-(k)]^{n-p-1}} d\zeta_{p+1} \dots d\zeta_{n-1} \mathcal{R}_{\mathcal{G}_n}(k; z, z'; \zeta_1 \dots \zeta_{n-1}) \tag{B.4}$$

The derivation of Eq. (25) then reduces to proving the following two facts (in view of the remark below):

i) $\mathcal{R}_{\mathcal{G}_n}$ is integrable on each multiple contour of the form $[\Gamma^+(k)]^p_{(\zeta_1 \dots \zeta_p)} \times [\Gamma^-(k)]^{n-p-1}_{(\zeta_{p+1} \dots \zeta_{n-1})}$ $(1 \le p \le n-1)$; this will entail that each term at the r. h. s. of (B.4) is well-defined.

ii) In the p^{th} term at the r. h. s. of (B.4), the integrand $\mathcal{R}_{\mathcal{G}_n}$ can be replaced by the product:

$$\mathcal{R}_{\mathcal{G}_p}(k; z, \zeta_p; \zeta_1 \dots \zeta_{p-1}) \omega(k, \zeta_p) \mathcal{R}_{\mathcal{G}_{n-p}}(k; \zeta_p, z'; \zeta_{p+1} \dots \zeta_{n-1}) \tag{B.5}$$

Remark. — The integral of the latter on the contour $[\Gamma^+(k)]^{p-1} \times \tilde{e}(k) \times [\Gamma^-(k)]^{n-p-1}$ yields:

$$\int_{\tilde{e}(k)} F_p^+(k; z, \zeta_p) \omega(k, \zeta_p) F_{n-p}^-(k; \zeta_p, z') d\zeta_p ,$$

which, in view of the residue theorem (on $e(k)$) is equal to: $F_p^+ * F_{n-p}^-$, namely to the p^{th} term at the r. h. s. of Eq. (25).

Proof of i). — In view of proposition (A.1), the contours $\Gamma^+(k)$, $\Gamma^-(k)$ can be decomposed as follows

$$\Gamma^\varepsilon(k) = \mathrm{E}_{(\mathrm{B})} + \Gamma^\varepsilon_{(\mathrm{B})}(k), \tag{B.6}$$

with
$$\Gamma^\varepsilon_{(\mathrm{B})}(k) \subset \mathrm{S_B}, \qquad \varepsilon = + \quad \text{or} \quad - . \tag{B.6'}$$

It follows that the restriction of the contour $[\Gamma^+(k)]^p_{(\zeta_1\ldots\zeta_p)} \times [\Gamma^-(k)]^{n-p-1}_{(\zeta_{p+1}\ldots\zeta_{n-1})}$ to any given sectorial region $\sigma^{(\mathrm{B})}_{i_1\ldots i_r}$ is equal to:

$$\mathrm{E}^{(\mathrm{B})}_{i_1\ldots i_r} \times \prod_{j\notin\{i_1,\ldots,i_r\}} [\Gamma^{\varepsilon_j}_{(\mathrm{B})}]_{(\zeta_j)}, \tag{B.7}$$

where $\varepsilon_j = +$ if $1 \leq j \leq p$, and $\varepsilon_j = -$ if $p + 1 \leq j \leq n - 1$; in view of the inclusion relation (B.6)′, the integrability of $\mathcal{R}_{\mathcal{G}_n}$ on the cycle described in (B.7) follows from proposition A.2. Summing over all sectorial regions $\sigma^{(\mathrm{B})}_{i_1\ldots i_r}$ yields the integral of $\mathcal{R}_{\mathcal{G}_n}$ on

$$[\Gamma^+(k)]^p_{(\zeta_1,\ldots,\zeta_p)} \times [\Gamma^-(k)]^{n-p-1}_{(\zeta_{p+1},\ldots,\zeta_{n-1})}.$$

Proof of *ii*): in view of (B.6), formula (B.3) yields: $\tilde{e}(k) = \Gamma^+_{(\mathrm{B})}(k) - \Gamma^-_{(\mathrm{B})}(k)$, so that $\tilde{e}(k)$ is contained in $\mathrm{S_B}$. It then follows that the restriction of the contour

$$[\Gamma^+(k)]^{p-1}_{(\zeta_1,\ldots,\zeta_{p-1})} \times [\tilde{e}(k)]_{(\zeta_p)} \times [\Gamma^-(k)]^{n-p-1}_{(\zeta_{p+1}\ldots\zeta_{n-1})}$$

to a given sectorial region $\sigma^{(\mathrm{B})}_{i_1\ldots i_r}$ is non-vanishing if and only if $p \notin \{ i_1, \ldots, i_r \}$, in which case one can write:

$$\sigma^{(\mathrm{B})}_{i_1\ldots i_r} \subset \sigma^{(\mathrm{B})}_{j_1\ldots j_{r_1}} \times \mathrm{B}_{(\zeta_p)} \times \sigma^{(\mathrm{B})}_{l_1\ldots l_{r_2}}, \tag{B.8}$$

where $\sigma^{(\mathrm{B})}_{j_1\ldots j_{r_1}}$ and $\sigma^{(\mathrm{B})}_{l_1\ldots l_{r_2}}$ are sectorial regions of $(\zeta_1, \ldots, \zeta_{p-1})$-space and $(\zeta_{p+1}, \ldots, \zeta_{n-1})$-space, defined by the restrictions $(j_1, \ldots, j_{r_1})$ and $(l_1, \ldots, l_{r_2})$ of the sequence $(i_1, \ldots, i_r)$ to the respective sets $\{ 1, \ldots, p - 1 \}$ and $\{ p + 1, \ldots, n - 1 \}$.

On the other hand, the decomposition of the integrand $\mathcal{R}_{\mathcal{G}_n}$ in complete forest terms $\mathrm{X}^{\mathrm{U}}_{\{i_1\ldots i_r\}}$ associated with a given region $\sigma^{(\mathrm{B})}_{i_1\ldots i_r}$ (see App. A) exhibits terms whose integral on $[\tilde{e}(k)]_{(\zeta_p)}$ vanishes: these are all the terms in which the double pole factor $\omega(k, \zeta_p)$ is replaced by the purely holomorphic factor $\omega(o, \zeta_p)$, whose residue on $e(k)$ is equal to zero; these terms correspond to all forests U whose at least one subgraph contains the loop l_p. Now all the remaining forests are of the form $\mathrm{U} = \mathrm{U}_1 \times \mathrm{U}_2$, where U_1 is a forest of $\mathcal{G}^{(1)}_p$ and U_2 is a forest of $\mathcal{G}^{(p+1)}_{n-p}$; moreover, by taking into account the inclusion (B.8) for $\sigma^{(\mathrm{B})}_{i_1\ldots i_r}$, one checks that the forests U_1, U_2 thus obtained are all the complete forests with respect to the corresponding sectors $\sigma_{j_1\ldots j_{r_1}}$ and $\sigma_{l_1\ldots l_{r_2}}$. So, by summing the terms of the decomposition of $\mathcal{R}_{\mathcal{G}_n}$ over this subset of « factorized » forests, one obtains the product (B.5). Since this argument holds for any sectorial region $\sigma^{(\mathrm{B})}_{i_1\ldots i_r}$ (such that $p \notin \{ i_1 \ldots i_r \}$), the statement of *ii*) is established.

APPENDIX C

We give the proof of proposition 5. Eq. (36) is equivalent to the following set of formulae (written for convenience in operator form, with the notations introduced in section 5, after property A, and for $k_0 = 0$):

$$\forall n \geq 2, \quad F_n - \mathring{F}_n = (G - \mathring{G})\underline{\omega} F_{n-1} + \sum_{1 \leq p \leq n-1} \mathring{F}_p(\underline{\omega} - \underline{\mathring{\omega}})F_{n-p}$$
$$+ \sum_{1 \leq p \leq n-2} \mathring{F}_p \underline{\mathring{\omega}}(G - \mathring{G})\underline{\omega} F_{n-p-1} + \mathring{F}_{n-1}\underline{\mathring{\omega}}(G - \mathring{G}) \quad \text{(C.1)}$$

By definition, we have:

$$F_n - \mathring{F}_n = \int (\mathscr{R}_{\mathscr{G}_n} - \mathring{\mathscr{R}}_{\mathscr{G}_n}) d\zeta_1 \dots d\zeta_{n-1} \quad \text{(C.2)}$$

and we can write:

$$\mathscr{R}_{\mathscr{G}_n} - \mathring{\mathscr{R}}_{\mathscr{G}_n} = \sum_{U \in \mathscr{U}} \mathscr{R}_{\mathscr{G}_n}^{(U)} - \mathring{\mathscr{R}}_{\mathscr{G}_n}^{(U)} \quad \text{(C.3)}$$

where $\mathscr{U}$ is the set of all forests of $\mathscr{G}_n$ and $\mathscr{R}_{\mathscr{G}_n}^{(U)}$ is the contribution to $\mathscr{R}_{\mathscr{G}_n}$ of a special forest U in $\mathscr{U}$. Each term $\mathscr{R}_{\mathscr{G}_n}^{(U)}$ has the structure of a monomial with respect to G and ω, which we write:

$$\mathscr{R}_{\mathscr{G}_n}^{(U)} = G^{(U,1)}\omega^{(U,1)}G^{(U,2)}\omega^{(U,2)} \dots \omega^{(U,n-1)}G^{(U,n)};$$

the notation $G^{(U,p)}$ (resp. $\omega^{(U,p)}$) means that the forest U prescribes a certain value for the factor G (resp. ω) which corresponds to the vertex v_p (resp. the loop l_p): for instance, if $p > 1$, $G^{(U,p)}$ can take either one of the following values: $G(k; \zeta_{p-1}, \zeta_p)$, $G(o; \zeta_{p-1}, \zeta_p)$, $G(o; o, \zeta_p)$, $G(o; \zeta_{p-1}, o)$, and $\omega^{(U,p)}$ is either $\omega(k, \zeta_p)$ or $\omega(0, \zeta_p)$).

We can then write (for each U):

$$\mathscr{R}_{\mathscr{G}_n}^{(U)} - \mathring{\mathscr{R}}_{\mathscr{G}_n}^{(U)} = (G^{(U,1)} - \mathring{G}^{(U,1)})\omega^{(U,1)}G^{(U,2)} \dots G^{(U,n)}$$
$$+ \sum_{1 \leq p \leq n-1} \mathring{G}^{(U,1)} \dots \mathring{G}^{(U,p)}(\omega^{(U,p)} - \mathring{\omega}^{(U,p)})G^{(U,p+1)} \dots G^{(U,n)}$$
$$+ \sum_{1 \leq p \leq n-2} \mathring{G}^{(U,1)} \dots \mathring{\omega}^{(U,p)}(G^{(U,p+1)} - \mathring{G}^{(U,p+1)})\omega^{(U,p+1)} \dots G^{(U,n)}$$
$$+ \mathring{G}^{(U,1)} \dots \mathring{G}^{(U,n-1)}\mathring{\omega}^{(U,n-1)}(G^{(U,n)} - \mathring{G}^{(U,n)}). \quad \text{(C.4)}$$

This formula already exhibits, for each individual term of (C.3), the structure of the r. h. s. of (C.1); the summation over the forests U remains to be studied (according to (C.2) (C.3)). Let us consider an arbitrary individual term in the r. h. s. of (C.4); it contains a unique incremental factor whose form is either $\omega^{(U,p)} - \mathring{\omega}^{(U,p)} (1 \leq p \leq n-1)$ or $G^{(U,p)} - \mathring{G}^{(U,p)}$ $(1 \leq p \leq n)$. Concerning the first case, we remark that the only forests U for which $\omega^{(U,p)} - \mathring{\omega}^{(U,p)}$ does not vanish are those whose *no* subgraph contains the corresponding loop l_p; in fact, if there is such a subgraph in the forest U, one has: $\omega^{(U,p)} = \mathring{\omega}^{(U,p)} = \omega(o, \zeta_p)$.

For this individual term of (C.4), the summation over $\mathcal{U}$ therefore reduces to the forests which are defined by couples (U'_p, U''_{n-p}), where U'_p (resp. U''_{n-p}) is an arbitrary forest of the subgraph $\mathcal{G}_p^{(1)}$ (resp. $\mathcal{G}_{n-p}^{(p+1)}$) which is on the left (resp. on the right) of the loop l_p in $\mathcal{G}_n$; for all these forests, one has $\omega^{(U,p)} - \mathring{\omega}^{(U,p)} = \omega(k, \zeta_p) - \omega(o, \zeta_p)$ and the result of the summation is thus equal to: $\mathcal{R}_{\mathcal{G}_p}(\omega - \mathring{\omega})\mathcal{R}_{\mathcal{G}_{n-p}}$. Integration in the sets of variables $(\zeta_1, \ldots, \zeta_{p-1})$, $(\zeta_{p+1}, \ldots, \zeta_{n-1})$ is thus feasible and yields: $F_p(k_0; z, \zeta_p)[\omega(k, \zeta_p) - \omega(k_0, \zeta_p)]F_{n-p}(k; \zeta_p, z')$. But since F_p and F_{n-p} are of degree zero at infinity in ζ, $(\omega(k, \zeta_p) - \omega(o, \zeta_p))d\zeta$ being of degree -1 (see formula (51)), the integration over ζ_p can be performed and yields the term:

$$F_p(\underline{\omega} - \underline{\omega})F_{n-p} \qquad \text{of the r. h. s. of (C.1).}$$

A similar argument holds for any term of (C.4) containing a factor $G^{(U,p)} - \mathring{G}^{(U,p)}$, since (again) such a factor vanishes as soon as the forest U includes a subgraph containing the vertex v_p. Each term at the r. h. s. of (C.1) is thus obtained after summation over the forests U.

Q. E. D.

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(Manuscrit reçu le 2 décembre 1985)