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The pressure of the two dimensional Coulomb gas at low and intermediate temperatures

by

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ABSTRACT. — The properties of the Mayer series of the pressure are investigated. For $\beta e^2 \equiv \alpha^2 \geqslant 8\pi$ it is proven that the series is asymptotic. For $\alpha^2 < 8\pi$ it has been previously proven (1) that only a finite number of terms of the series are finite; therefore the Mayer series is meaningless, nevertheless, its partial sum made up of the first finite terms (whose number increases as $\alpha^2 \to 8\pi$) is asymptotic to the pressure.

RÉSUMÉ. — On étudie les propriétés de la série de Mayer pour la pression. On montre que la série est asymptotique pour $\alpha^2 \geqq 8\pi$. Pour $\alpha^2 < 8\pi$

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on a précédemment démontré (1) qu'un nombre fini seulement de termes de la série sont finis, si bien que la série de Mayer n'a aucun sens ; néanmoins, la somme partielle des termes finis (dont le nombre augmente indéfiniment quand $\alpha^2 \to 8\pi$) est asymptotique à la pression.

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1. INTRODUCTION

Consider a two dimensional Coulomb gas made up of classical spinless particles with charge $\pm |e| \equiv \pm e$, with an ultraviolet cutoff (for instance a short range repulsive potential preventing collapse between particles of different charges), with activity λ and with inverse temperature $\beta = e^{-2}\alpha^2$. Call $p_c(\lambda, \alpha^2)$ its pressure in the infinite volume limit and write, formally, its Mayer series

$$p_c(\lambda, \alpha^2) = \sum_{K=1}^{\infty} a_{c,k}(\alpha^2)\lambda^k \tag{1.1}$$

then the following results have been proven in a previous work [1]:

a) If $\alpha^2 \geqslant 8\pi$ the coefficients $a_{c,k}(\alpha^2)$ of the Mayer series are all finite and those with k odd are zero.

b) If $\alpha^2 \in (\alpha_n^2, 8\pi)$ where $\alpha_n^2 = 8\pi\left(1 - \frac{1}{n}\right)$ the coefficients $a_{c,k}(\alpha^2)$ with $k \leqslant n$ are finite and those with k odd are zero.

The « physical » interpretation of this result, as discussed at length in [1], is that the Coulomb gas for $\alpha^2 > 4\pi$ can be interpreted as a gas consisting of multipoles with at most a fixed number of particles (which increases with α^2) ; when $\alpha^2 \geqslant 8\pi$ the multipoles can be made up of any number of particles.

Here we want to give a rigorous meaning to the Mayer series of the pressure for $\alpha^2 \geqslant 8\pi$ and to the finite partial sums of it when $\alpha^2 < 8\pi$.

The results we will prove are the following ones:

c) If $\alpha^2 \geqslant 8\pi$, $\forall$M integer > 0, λ small enough

$$\left| p_c(\lambda, \alpha^2) - \sum_{k=1}^{M} a_{c,k}(\alpha^2)\lambda^k \right| \leqslant (\text{const.})\lambda^{M+\tau} \tag{1.2}$$

where $\tau > 0$ and (const.) is an appropriate constant.

d) A sequence of thresholds $\{ \bar{\alpha}_n^2 \}$ exists satisfying

$$\alpha_n^2 \leqslant \bar{\alpha}_n^2 < 8\pi \tag{1.3}$$

such that if $\alpha^2 \in (\overline{\alpha}_n^2, 8\pi)$, $\forall M \leqslant n$, λ small enough

$$\left| p_c(\lambda, \alpha^2) - \sum_{k=1}^{M} a_{c,k}(\alpha^2)\lambda^k \right| \leqslant (\text{const.})\lambda^{M+\tau} \tag{1.4}$$

Results (1.2) and (1.4) substantiate the description of the Coulomb gas in the inverse temperature interval $e^{-2}[4\pi, 8\pi]$ given in [*1*] and prove the conjecture, made therein, that the pressure is a function of λ more and more regular (for λ small) as α^2 increases.

The following sections are devoted to the proof of (1.2) and (1.4).

2. THE SINE-GORDON FORMALISM AND THE STATEMENT OF THE PROBLEM

We shall use extensively the Sine-Gordon formalism and the field theory results proved in [*2*] and [*3*] in this paper. We start by giving some definitions.

The partition function for the neutral Coulomb gas, in a finite volume I, with inverse temperature β and activity λ is, [*1*],

$$Z_{c,(N)}^{Q=0}(I, \beta, \lambda) = \lim_{R\to\infty} Z^{(-R,N)}(I, \beta, \lambda) \tag{2.1}$$

where

$$Z^{(-R,N)}(I, \beta, \lambda) = \int P(d\Psi^{(-R,N)})e^{\lambda\int_I \cos\alpha\Psi_x^{(-R,N)}d^2x} \tag{2.2}$$

$P(d\Psi^{(-R,N)})$ is the measure of the gaussian field $\Psi^{(-R,N)}$ with covariance

$$C^{(-R,N)}(x, y) = \frac{1}{(2\pi)^2}\int_{R^2} d^2k e^{ik(x-y)}\left(\frac{1}{k^2+\gamma^{-2R}} - \frac{2}{k^2+\gamma^{2N}}\right) \tag{2.3}$$

γ^{2N} is the ultraviolet cutoff which is kept fixed and will be chosen equal to γ^2 (N = 1), in the following, $\gamma > 1$.

We define a new gaussian field on a different scale

$$\varphi_\xi^{[\leqslant R+N-1]} \equiv \Psi_{\gamma^R\xi}^{(-R,N)} \tag{2.4}$$

The covariance of $\varphi_\xi^{[\leqslant R]}$, (N = 1), is

$$C^{[\leqslant R]}(\xi, \eta) = C^{(-R,1)}(\gamma^R\xi, \gamma^R\eta) = \frac{1}{(2\pi)^2}\int d^2p e^{ip(\xi-\eta)}\left(\frac{1}{p^2+1} - \frac{1}{p^2+\gamma^{2(R+1)}}\right) \tag{2.5}$$

$$Z^{(-R,1)}(I, \beta, \lambda) = \int P(d\varphi^{[\leqslant R]})e^{\lambda(R)\int_{\Lambda(R)} :\cos\alpha\varphi_\xi^{[\leqslant R]}: d^2\xi} = Z_{\text{Yukawa}}^{(R)}(\Lambda(R), \beta, \lambda(R)) \tag{2.6}$$

where

$$\begin{cases} \Lambda(\mathrm{R}) = \gamma^{-\mathrm{R}}\mathrm{I} \ (|\Lambda(\mathrm{R})| = \gamma^{-2\mathrm{R}}|\mathrm{I}|) \\ \lambda(\mathrm{R}) = \lambda\gamma^{2\mathrm{R}} e^{-\frac{\alpha^2}{2}\mathrm{C}^{[\leq \mathrm{R}]}_{(0\ 0)}} = \lambda\gamma^{\left(2-\frac{\alpha^2}{4\pi}\right)\mathrm{R}} \end{cases} \tag{2.7}$$

and $Z^{(\mathrm{R})}_{\mathrm{Yukawa}}(\Lambda, \beta, \lambda)$ is the partition function of a Yukawa gas with an ultraviolet cutoff $\gamma^{2\mathrm{R}}$, volume Λ and activity λ.

Definitions (2.1) and (2.6) give

$$Z^{\mathrm{Q}=0}_{c,(1)}(\mathrm{I}, \beta, \lambda) = \lim_{\mathrm{R}\to\infty} Z^{(\mathrm{R})}_{\mathrm{Yukawa}}(\Lambda(\mathrm{R}), \beta, \lambda(\mathrm{R})) \tag{2.8}$$

which describes the fact that the infrared properties of the neutral Coulomb gas are connected to the ultraviolet properties of its « associated » Yukawa gas. This property, called « duality » in [*1*], is connected with the fact that the Coulomb potential in two dimensions diverges in the same way at zero and at infinity.

The pressure has the following expression

$$\begin{aligned} p_c(\lambda, \alpha^2) &= \lim_{\mathrm{I}\nearrow\mathrm{R}^2} \frac{1}{|\mathrm{I}|} \log Z^{\mathrm{Q}=0}_{c,(1)}(\mathrm{I}, \beta, \lambda) = \\ &= \lim_{\mathrm{I}\nearrow\mathrm{R}^2} \lim_{\mathrm{R}\to\infty} \frac{\log Z^{(\mathrm{R})}_{\mathrm{Yukawa}}(\Lambda(\mathrm{R}), \beta, \lambda(\mathrm{R}))}{|\mathrm{I}|} \end{aligned} \tag{2.9}$$

The properties of the Mayer expansion of $p_c(\lambda, \alpha^2)$ are therefore connected to the properties of the cumulant expansion of $\log Z^{(\mathrm{R})}_{\mathrm{Yukawa}}(\Lambda(\mathrm{R}), \beta, \lambda(\mathrm{R}))$. Formally

$$\log Z^{(\mathrm{R})}_{\mathrm{Yukawa}}(\Lambda(\mathrm{R}), \beta, \lambda(\mathrm{R})) = \sum_{k=1}^{\infty} \frac{1}{k!}\varepsilon^{\mathrm{T}}(\mathrm{V}^{(\mathrm{R})}_{0,\Lambda}\,;\, k) \tag{2.10}$$

where

$$\mathrm{V}^{(\mathrm{R})}_{0,\Lambda} \equiv \lambda(\mathrm{R}) \int_{\Lambda(\mathrm{R})} :\cos\alpha\varphi^{[\leq \mathrm{R}]}_{\xi}:\, d^2\xi \tag{2.11}$$

and $\varepsilon^{\mathrm{T}}(.\,;k)$ is the truncated expectation of order k, with respect to the gaussian measure $\mathrm{P}(d\varphi^{[\leq \mathrm{R}]})$

$$\varepsilon^{\mathrm{T}}(x; k) = \frac{\partial^k}{\partial\tau^k} \log \int \mathrm{P}(d\varphi^{[\leq \mathrm{R}]}) e^{\tau x}\bigg|_{\tau=0} \tag{2.12}$$

In [*1*] it was proven that the coefficients of the Mayer series which are given by the following expression

$$a_{c,k}(\alpha^2) = \lambda^{-k} \lim_{\mathrm{I}\nearrow\mathrm{R}^2} \lim_{\mathrm{R}\to\infty} \frac{1}{|\mathrm{I}|}\frac{1}{k!}\varepsilon^{\mathrm{T}}(\mathrm{V}^{(\mathrm{R})}_{0,\Lambda}\,;\,k) \equiv \lim_{\mathrm{I}\nearrow\mathrm{R}^2} \lim_{\mathrm{R}\to\infty} \frac{\mathrm{C}^{(\mathrm{R})}_k(\alpha^2)}{|\mathrm{I}|} \tag{2.13}$$

have some finite upper bounds for their moduli; proving eqs. (1.2) and (1.4) amounts to proving the following inequalities

I) $\alpha^2 \geqslant 8\pi$, $\forall$M integer > 0, λ small enough

$$e^{\sum_{k=1}^{M} C_k^{(R)}(\alpha^2)\lambda^k - O(\lambda^{M+\tau})|I|} \leqslant Z^{(R)}_{\text{Yukawa}} \leqslant e^{\sum_{k=1}^{M} C_k^{(R)}(\alpha^2)\lambda^k + O(\lambda^{M+\tau})|I|} \tag{2.14}$$

where $\tau > 0$, $O(\lambda^{M+\tau})$ is R and $|I|$ independent and $\lambda^k C_k^{(R)}(\alpha^2)$ is defined by

$$\lambda^k C_k^{(R)}(\alpha^2) = \frac{1}{k\,!}\,\varepsilon^T(V_{0,\Lambda}^{(R)}\,;\,k) \tag{2.15}$$

II) $\alpha^2 \in (\bar{\alpha}_n^2, 8\pi)$, $\forall M \leqslant n$, λ small enough; again the inequality (2.14) must be proven.

From eq. (2.14) the results (1.2) and (1.4) follow remembering that in [*1*] the following bounds for $C_k^{(R)}(\alpha^2)$ have been proven:

$$\lim_{R\to\infty} |C_k^{(R)}(\alpha^2)| \leqslant (\text{const.})\,|I| \tag{2.16}$$

for any k if $\alpha^2 \geqslant 8\pi$ (if k is odd, (const.) = 0) and $\forall k \leqslant n$ (if k is odd, (const.) = 0) if $\alpha^2 \in (\bar{\alpha}_n^2, 8\pi)$, where (const.) is R and $|I|$ independent.

If we define the following new interaction (choosing $M = 2M'$)

$$V_\Lambda^{(R)} = V_{0,\Lambda}^{(R)} - \sum_{k=1}^{2M'} C_k^{(R)}(\alpha^2)\lambda^k \tag{2.17}$$

and the new partition function

$$\tilde{Z}_y^{(R)} \equiv \int P(d\varphi^{[\leqslant R]})e^{V_\Lambda^{(R)}} \tag{2.18}$$

the inequalities (2.14) become

$$e^{-O(\lambda^{2M+\tau})|I|} \leqslant \tilde{Z}_y^{(R)} \leqslant e^{O(\lambda^{2M+\tau})|I|} \tag{2.19}$$

where we called M′ again M.

Inequalities (2.19) will be proven in the next sections. Before entering into the technical details we make the following remark:

Remark. — Proving inequalities (2.19) is just proving the ultraviolet stability for the two dimensional field theory with interaction $V_\Lambda^{(R)}$. This problem is formally the same as that of studying the massive Sine-Gordon field theory for $\alpha^2 \geqslant 4\pi$ (2) (6). In that case, which corresponds from a statistical mechanics point of view to the study of the short distance properties of a Yukawa gas, it has been proven that this field theory is superrenormalizable in the interval $[4\pi, 8\pi]$ by subtracting some constant counterterms from the original interaction $V_{0,\Lambda}^{(R)}$ which are divergent in the $R \to \infty$ limit. In that case the counterterms are the even truncated expectations with respect to the measure $P(d\varphi^{[\leqslant R]})$. It is therefore important to stress the main differences between the problem we are facing now and that discussed in [*2*] and [*5*]. They can be summarized in the following way:

i) In the true Yukawa gas (massive Sine-Gordon field theory) we are investigating an ultraviolet problem and the counterterms, needed for the theory to exist, diverge when we remove the cutoff. In the present case we are, on the other hand, studying the infrared problem for a Coulomb gas with a fixed ultraviolet cutoff. Therefore the counterterms $C_k^{(R)}$, in spite of having the same analytical structure as in the previous case, remain finite (zero if k is odd) in the limit $R \to \infty$ due to the reduction of this infrared problem to the ultraviolet one of a peculiar Yukawa gas with a dependance on R of the activity and of the gas volume (eq. (2.7)).

ii) The second major difference is that the temperature region where we want to prove inequalities (2.19) only partially overlaps with the one in [*2*] [*5*] and [*6*] where the ultraviolet problem was studied.

iii) Finally our goal is not to prove the existence of $\tilde{Z}_y^{(R)}$ in the limit $R \to \infty$ which is trivially true, but to prove inequalities (2.19) for *any* integer M if $\alpha^2 \geqslant 8\pi$ and for *any* $M \leqslant n$ if $\alpha^2 \in (\bar{\alpha}_n^2, 8\pi)$.

This will require general control of the terms of the cumulant expansions at any order. Although the differences pointed out in *i*), *ii*), *iii*) forbid us to simply translate the previous results to this case; we will, nevertheless, try to follow as close as possible the strategy used there.

3. THE PROPERTIES OF THE GAUSSIAN FIELD

The covariance (2.5) of the field $\varphi_\xi^{[\leqslant R]}$ implies that the following regularity properties hold with probability one: given a sample $\varphi^{[\leqslant q]}$ there exists a constant B such that

$$| \varphi_\xi^{[\leqslant q]} | \leqslant B \log \gamma^q = B' q \tag{3.1}$$

$$| \varphi_\xi^{[\leqslant q]} - \varphi_\eta^{[\leqslant q]} | \leqslant B(\gamma^q | \xi - \eta |)^{1-\varepsilon} \tag{3.2}$$

with $\varepsilon > 0$.

$\varepsilon > 0$ is due to the regularization used which is such that with probability one $\varphi^{[\leqslant q]}$ is Holder continuous but not differentiable. One could also introduce a stronger regularization obtaining a field differentiable with probability one, using, for instance an « iterated Pauli-Villars » regularization as was done in [*4*], but it will turn out that to prove our result we will not need it.

The field $\varphi^{[\leqslant R]}$ with covariance (2.5) can be written as a sum of R + 1 independent fields

$$\varphi_\xi^{[\leqslant R]} = \sum_{k=0}^{R} \varphi_\xi^{(k)} \tag{3.3}$$

where $\varphi_\xi^{(k)}$ has covariance

$$C_{(\xi,\eta)}^{(k)} = \frac{1}{(2\pi)^2}\int d^2p e^{ip.(\xi-\eta)}\left(\frac{1}{p^2+\gamma^{2k}} - \frac{1}{p^2+\gamma^{2(k+1)}}\right) \tag{3.4}$$

and they are identically distributed in the following sense

$$\varphi_\xi^{(k)} = \varphi_{\gamma^k \xi}^{(0)} \tag{3.5}$$

We start, now, to discuss the lower and upper bounds (2.19). The strategy is to reduce both estimates to the proof of a well-defined Lemma of the same type as Lemma 1 of [*2*]. This will be discussed in section 8 with the same techniques as in [*2*]. The reduction to this lemma for both the lower and upper bounds is the more difficult part of the work and is the content of the next three sections.

4. THE LOWER BOUND

The strategy consists in evaluating

$$\tilde{Z}_y^{(R)} = \int P(d\varphi^{[\leqslant R]})e^{V_\Lambda^{(R)}} \tag{2.18}$$

by integrating over the fields of « definite frequency » $\varphi^{(h)}$ one after the other, estimating after each integration the effective potential that has been produced and proving that it has the right properties so that an iterative procedure can be applied.

Fixing an arbitrary positive integer M, we define

$$\begin{cases} \tilde{V}_\Lambda^{(h)} \equiv \left[\sum_k^{2M}{}_{\!1} \frac{1}{k!}\varepsilon_{[h+1]}^T(\tilde{V}_\Lambda^{(h+1)};k)\right]_{(2M)} \\ \tilde{V}_\Lambda^{(R)} \equiv V_\Lambda^{(R)} \end{cases} \tag{4.1}$$

$[\ \ ]_{(2M)}$ indicates that we consider only the terms of order λ^s with $s \leqslant 2M$ and $\varepsilon_{[p]}^T(.\,;k)$ is the truncated expectation of order k with respect to the gaussian measure $P(d\varphi^{[p]})$. The proof of the (l. h. s.) inequality of eq. (2.19) follows if we prove for any h an inequality of the following type

$$\chi(\varphi^{[\leqslant h]})\int P(d\varphi^{(h+1)})\chi(\varphi^{(h+1)})e^{\tilde{V}_\Lambda^{(h+1)}} \geqslant \chi(\varphi^{[\leqslant h-1]})\chi(\varphi^{(h)})e^{\tilde{V}_\Lambda^{(h)}}e^{R_\Lambda^{(h)}} \tag{4.2}$$

where $R_\Lambda^{(h)}$ is the remainder produced by the cumulant expansion truncated to a certain order in λ (2M in this case) and the χ are characteristic functions we are going to specify in a moment.

Relations (4.2) are useful if the remainders $R_\Lambda^{(h)}$ satisfy

$$\left| \lim_{R\to\infty} \sum_{h=-1}^{R-1} R_\Lambda^{(h)} \right| \leqslant O(\lambda^{2M+\tau}) \, | \, I \, | \tag{4.3}$$

To give a precise meaning to the inequality (4.2) we define the following events in the space of fields $\varphi^{[\leqslant h]}$ and $\varphi^{(h)}$

$$\begin{aligned} E_\Delta^B &= \left\{ \varphi^{[\leqslant h]} \mid \sup_{\xi,\eta\in\Delta} \frac{|\varphi_\xi^{[\leqslant h]} - \varphi_\eta^{[\leqslant h]}|}{(\gamma^h |\xi-\eta|)^{1-\varepsilon}} \leqslant B(1+\gamma^h d(\Delta,\Lambda)) \right\} \\ E_\Delta^b &= \left\{ \varphi^{(h)} \mid \sup_{\xi,\eta\in\Delta} \frac{|\varphi_\xi^{(h)} - \varphi_\eta^{(h)}|}{(\gamma^h |\xi-\eta|)^{1-\varepsilon}} \leqslant b(1+\gamma^h d(\Delta,\Lambda)) \right\} \end{aligned} \tag{4.4}$$

where Δ is a tessera of linear size $\gamma^{-h}(\Delta \in Q_h)$, $d(\Delta, \Lambda)$ is the distance between Δ and the region $\Lambda \equiv \Lambda(R)$; B and b are two positive constants which have to satisfy the constraints needed to prove the following inequality

$$\chi_{\Delta'}^B(\varphi^{[\leqslant h-1]})\chi_\Delta^B(\varphi^{[\leqslant h]}) \geqslant \chi_{\Delta'}^B(\varphi^{[\leqslant h-1]})\chi_\Delta^b(\varphi^{(h)}) \tag{4.5}$$

where $\Delta' \in Q_{h-1}$ has linear size $\gamma^{-(h-1)}$, $\Delta' \supset \Delta \in Q_h$ and

$$\begin{aligned} \chi_\Delta^B(\varphi^{[\leqslant h]}) &\equiv \chi(\varphi^{[\leqslant h]} \in E_\Delta^B) \\ \chi_\Delta^b(\varphi^{(h)}) &\equiv \chi(\varphi^{(h)} \in E_\Delta^b) \end{aligned} \tag{4.6}$$

The first constraints that b and B have to satisfy are expressed in the following lemma

LEMMA 1. — Inequality (4.5) is true if

$$b < B\left(1 - \frac{1}{\gamma^{1-\varepsilon}}\right). \tag{4.7}$$

Proof. — It follows immediately from definitions (4.4).

If b and B satisfy the inequality (4.7) we have, $\forall h$

$$\prod_{0}^{h}{}_{s} \chi_{Q_s}^B(\varphi^{[\leqslant s]}) \geqslant \prod_{0}^{h-1}{}_{s} \chi_{Q_s}^B(\varphi^{[\leqslant s]})\chi^b(\varphi^{(h)}) \tag{4.8}$$

where

$$\begin{aligned} \chi_{Q_s}^B(\varphi^{[\leqslant s]}) &\equiv \prod_{\Delta\in Q_s} \chi_\Delta^B(\varphi^{[\leqslant s]}) \\ \chi_{Q_s}^b(\varphi^{(s)}) &\equiv \prod_{\Delta\in Q_s} \chi_\Delta^b(\varphi^{(s)}) \end{aligned} \tag{4.9}$$

Using these relations we can write

$$\tilde{Z}_y^{(R)} \geqslant \int P(d\varphi^{[\leqslant R]}) \prod_{0}^{R} {}_s\, \chi^{B}_{Q_s}(\varphi^{[\leqslant s]}) e^{\tilde{V}_\Lambda^{(R)}}$$

$$\geqslant \int P(d\varphi^{[\leqslant R-1]}) \prod_{0}^{R-1} {}_s\, \chi^{B}_{Q_s}(\varphi^{[\leqslant s]}) \left\{ \int P(d\varphi^{(R)}) \chi^{b}_{Q_R}(\varphi^{(R)}) e^{\tilde{V}_\Lambda^{(R)}} \right\} \tag{4.10}$$

and for the general term

$$\int P(d\varphi^{[\leqslant h]}) \prod_{0}^{h} {}_s\, \chi^{B}_{Q_s}(\varphi^{[\leqslant s]}) e^{\tilde{V}_\Lambda^{(h)}}$$

$$\geqslant \int P(d\varphi^{[\leqslant h-1]}) \prod_{0}^{h-1} {}_s\, \chi^{B}_{Q_s}(\varphi^{[\leqslant s]}) \left\{ \int P(d\varphi^{(h)}) \chi^{b}_{Q_h}(\varphi^{(h)}) e^{\tilde{V}_\Lambda^{(h)}} \right\} \tag{4.11}$$

Inequality (4.2) can now be written exactly for any $h \leqslant R$

$$\chi^{B}_{Q_{h-1}}(\varphi^{[\leqslant h-1]}) \int P(d\varphi^{(h)}) \chi^{b}_{Q_h}(\varphi^{(h)}) e^{\tilde{V}_\Lambda^{(h)}} \geqslant \chi^{B}_{Q_{h-1}}(\varphi^{[\leqslant h-1]}) e^{\tilde{V}_\Lambda^{(h-1)}} e^{R_\Lambda^{(h-1)}} \tag{4.12}$$

and the $R_\Lambda^{(h-1)}$'s have to satisfy the inequality (4.3).

Remark. — The proof of the lower bound is therefore reduced to the proof of inequalities (4.12) and (4.3). This will be discussed in sections 7 and 8. Although the detailed estimates of the $R_\Lambda^{(h)}$'s will be performed later on, it should be clear that for them we need some detailed estimates on the effective potential $\tilde{V}_\Lambda^{(h)}$, $\forall h$. Since these estimates, along with many others, are necessary for the proof of the upper bound we collect and discuss them in the next section.

Remark. — In the sketch of the strategy for the lower bound we have assumed that our procedure of integrating frequency by frequency has to be carried out from $h = R$ to $h = 0$. This is not strictly necessary; in fact, the strategy mimicking the ultraviolet proof for the Yukawa gas can be performed until we get a frequency ρ for which $\gamma^{-2\rho} = O(\gamma^{-2R} \,|\, I \,|)$. That is a finite number of times which goes to infinity as $I \nearrow R^2$. Then, when we are at frequency ρ we can perform a global estimate, provided $\alpha^2 > 4\pi$. This will be discussed in more detail in section 7.

5. GENERAL PROPERTIES OF THE EFFECTIVE POTENTIAL

We use for the general definition of $\tilde{V}_\Lambda^{(h)}$ the tree formalism extensively developed in [4] and [1]. Specifically with the notations of [1] we write

$$\begin{aligned}\tilde{V}_\Lambda^{(h)} = & \sum_{1}^{2M}{}_k \sum_{\substack{k(\gamma)=h \\ v(\gamma)=k}}{}_\gamma \frac{1}{n(\gamma)} \sum_{\underline{\sigma}} \int_{\Lambda^k} d\xi_1 \ldots d\xi_k \overline{V}(\gamma, \varphi^{[\leqslant h]}, \underline{\sigma}) \\ & - \sum_{1}^{2M} \sum_{\substack{k(\gamma)=h \\ v(\gamma)=k}}{}^*_\gamma \frac{1}{n(\gamma)} \sum_{\underline{\sigma}} \int_k d\xi_1 \ldots d\xi_k \overline{V}(\gamma, \underline{\sigma})\end{aligned} \tag{5.1}$$

where $k(\gamma)$ is the « root » of the tree, $n(\gamma)$ is the usual combinatorial factor, γ is a tree with definite frequencies at each bifurcation, $v(\gamma)$ is the number of final lines and $\underline{\sigma} = (\sigma_1, \ldots, \sigma_k)$ ($\sigma_i = \pm 1$) are the charges of the final lines. $\sum_{\substack{k(\gamma)=h \\ v(\gamma)=k}}{}_\gamma$ runs over all different trees with k final lines and root h.

Σ_γ^* differs from Σ_γ as, calling l the frequency of the lowest bifurcation of γ, (hereafter indicated by $h(\gamma)$) the sum over l runs from 0 to h instead of from $h+1$ to R as in Σ_γ. The second term in (5.1) corresponds to that part of the counterterms (see (4)) which has not yet been utilized going from level R to level h.

We now decompose Σ_γ in the following way: we fix the shape s of the tree, then we fix the absolute value of the charge at each vertex (bifurcation) v of the tree $Q_v \geqslant 0$. We call $\{ Q_v \}_s$ a compatible vertex charge configuration for a tree γ with shape s (a vertex can also be thought as a cluster of charges whose average size depends on the frequency h_v of the vertex itself (see (1)). Therefore we can write

$$\sum_{\substack{k(\gamma)=h \\ v(\gamma)=k}}{}_\gamma \sum_{\underline{\sigma}} = \sum_{v(s)=k}{}_s \sum_{\{Q_v\}_s} \sum_{\substack{s(\gamma)=s \\ k(\gamma)=h}}{}_\gamma \sum_{\{Q_v\}_s}{}_{\underline{\sigma}} \tag{5.2}$$

where $s(\gamma) = s$ means that the tree γ must have the shape s and $\sum_{\{Q_v\}_s}{}_{\underline{\sigma}}$ implies that the absolute value of the global charge of the vertex v must be Q_v.

Finally we decompose $\sum_{\{Q_v\}_s}^{\sigma}$ in the following way: we call a $\underline{\sigma}=(\sigma_1, \ldots, \sigma_k)$ satisfying the constraints imposed by $\{ Q_v \}_s$ an admissible configuration. We associate to each vertex v a label $\mu_v = +1$ if $Q_v > 0$ and $\mu_v = \pm 1$ if $Q_v = 0$; then we fix an admissible configuration for a given $\{ Q_v \}_s$: $\underline{\bar\sigma} = (\bar\sigma_1, \ldots, \bar\sigma_k)$ and define

$$\sigma_i = \bar\sigma_i \prod_{v \ni i} \mu_v \tag{5.3}$$

where $v \ni i$ means that the i-th final line of γ is inside the cluster associated to the vertex $v: \gamma_v$. It is clear that the sum over all the admissible configurations of a tree γ with fixed s and $\{ Q_v \}_s$ can be decomposed as a sum over a suitable family $\mathcal{S}$ of admissible configurations $\underline{\bar\sigma}$ times a sum over the configurations which can be obtained from a fixed $\underline{\bar\sigma}$ just summing over the $\underline{\mu}$'s and dividing by an appropriate factor $C(\underline{\bar\sigma})$ to take into account possible double countings.

Therefore

$$\begin{aligned}\sum_{\substack{k(\gamma)=h\\ v(\gamma)=k}}^{\gamma} \sum^{\underline{\sigma}} &= \left\{ \sum_{v(s)=k}^{s} \sum_{\{Q_v\}_s} \sum_{\{Q_v\}_s}^{\underline{\bar\sigma}} \frac{1}{c(\underline{\bar\sigma})} \frac{1}{n(s)} \right\} \sum_{\substack{s(\gamma)=s\\ k(\gamma)=h}}^{\gamma} \sum_{(\underline{\bar\sigma}\text{ fixed})}^{\underline{\sigma}} \\ &\equiv \sum_{(s,\{Q_v\}_s,\underline{\bar\sigma})} \sum_{\substack{s(\gamma)=s\\ k(\gamma)=h}}^{\gamma} \sum_{(\underline{\bar\sigma}\text{ fixed})}^{\underline{\sigma}}\end{aligned} \tag{5.4}$$

where $n(\gamma) = n(s)$ only if when we sum over the frequencies we do not impose any constraints between frequencies of different branches. Now we can write

$$\tilde{V}_\Lambda^{(h)} = \sum_{1}^{2M} {}_k \sum_{(s,\{Q_v\}_s,\underline{\bar\sigma})} \sum_{\substack{s(\gamma)=s\\ k(\gamma)=h}}^{\gamma} \tilde{V}(\gamma, \{ Q_v \}_s, \underline{\bar\sigma}) - [\text{counterterms}; h] \tag{5.5}$$

where

$$\tilde{V}(\gamma, \{ Q_v \}_s, \underline{\bar\sigma}) = \int_{\Lambda^k} d\xi_1 \ldots d\xi_k \sum_{(\underline{\bar\sigma}\text{ fixed})}^{\underline{\sigma}} \bar{V}(\gamma, \varphi^{[\leq h]}, \underline{\sigma}) \tag{5.6}$$

Remarks. — *i*) This decomposition is such that each term of the sum Σ satisfies the estimates we need. To prove them we need to use the essential cancellations provided by the $\sum_{(\underline{\bar\sigma}\text{ fixed})}^{\underline{\sigma}}$.

ii) We do not write explicitely the second term of (5.1) because these parts of the counterterms do not play any role at level h. A piece of them will be extracted and used when we go to the next level $h-1$.

We are still free to change the names of the final lines, changing the names of the coordinates and therefore to require that $\underline{\sigma}$ always appears as $\underline{\sigma} = (\bar{\sigma}_1, \ldots, \bar{\sigma}_{2k}) = (+, -, \ldots, +, -)$ if the charge $Q(\gamma)$ is $= 0$ and

$$\bar{\sigma} = (\bar{\bar{\sigma}}_1, \ldots, \bar{\bar{\sigma}}_{2k}, \bar{\bar{\sigma}}_{2k+1}, \ldots, \bar{\bar{\sigma}}_{2k+p}) = (+, -, \ldots, +, -, \pm, \pm, \ldots, \pm) \quad (5.7)$$

if $Q(\gamma) = \pm p$ being $Q(\gamma) \equiv Q$ the total charge of γ.

This can be done ordering hierarchically the bifurcations in the following way:

DEFINITION. — We say that a vertex is of order l if there are at least two final lines that, before merging in this vertex, meet l vertices preceding it (obviously going from top to bottom).

We start considering all the vertices of order zero and we arrange the names of the coordinates in such a way that the charges of the $\underline{\sigma}$ configuration associated to that vertex are $(\bar{\sigma}_1, \ldots, \bar{\sigma}_{2l}) = (+, -, \ldots, +, -)$ if the vertex is neutral; we do the same for the non-neutral vertices but in that case we label only the neutral part (for example if in a vertex w of order zero three lines merge with charges $(+, -, +)$ we label the first two only). We go on by considering the final lines of the order one vertices which have not yet been labelled and we proceed as before, order by order. With this choice $\underline{\sigma}$ appears as in (5.7) and in each vertex v the lines with opposite charges always have adjacent indices. Having given all these definitions, we can state the following theorems

If $Q(\gamma_0) = Q = 0$ and $\gamma_0(\bar{\gamma}_0)$ has only one bifurcation, then

$$\tilde{V}(\gamma, \{ Q_v \}_s, \underline{\sigma}) = \left(\frac{\lambda(R)}{2}\right)^{2k} \int_{\Lambda^{2k}} d\xi_1 \ldots d\xi_{2k} : \cos \alpha\varphi(\gamma, \underline{\sigma}) - 1 : F_{\gamma_0}(\underline{\xi}, \underline{\sigma}; Q=0) \quad (5.8a)$$

If $Q(\bar{\gamma}_0) = Q \neq 0$. Then

$$\tilde{V}(\bar{\gamma}_0, \{ Q_v \}_s, \underline{\bar{\sigma}}) = \left(\frac{\lambda(R)}{2}\right)^{2k+p} \int_{\Lambda^{2k+p}} d\xi_1 \ldots d\xi_{2k+p} : e^{i\alpha\varphi(\bar{\gamma}_0, \underline{\bar{\sigma}})} - 1 : F_{\gamma_0}(\underline{\xi}, \bar{\sigma}; Q \neq 0) \quad (5.8b)$$

where, in general,

$$\varphi(\gamma, \underline{\sigma}) = \sum_{i=1}^{2k} \bar{\sigma}_i \varphi_{\xi_i}^{[\leq h]} = \sum_{l=1}^{k} \Delta\varphi_{2l-1,2l}^{[\leq h]}$$

$$\varphi(\bar{\gamma}, \underline{\bar{\sigma}}) = \sum_{i=1}^{2k} \bar{\bar{\sigma}}_i \varphi_{\xi_i}^{[\leq h]} + \sum_{J=1}^{p} \bar{\bar{\sigma}}_{2k+J} \varphi_{\xi_{2k+J}}^{[\leq h]} = \sum_{l=1}^{k} \Delta\varphi_{2l-1,2l+\operatorname{sgn} Q}^{[\leq h]} \sum_{s=1}^{p} \varphi_{\xi_{2k+J}}^{[\leq h]} \quad (5.9)$$

with the definition

$$\Delta\varphi^{[\leqslant h]}_{2l-1,2l} = \varphi^{[\leqslant h]}_{\xi_{2l-1}} - \varphi^{[\leqslant h]}_{\xi_{2l}} \tag{5.10}$$

We decompose: $:\cos\alpha\varphi(\gamma,\bar\sigma) - 1: = :\cos\alpha\left(\sum_{l}^{k}{}_{1}\Delta\varphi_{2l-1,2l}\right) - 1:$ into several parts

$$:\cos\alpha\left(\sum_{l\,1}^{k}\Delta\varphi_{2l-1,2l}\right) - 1: = \frac{1}{2}\Sigma_\sigma : e^{i\alpha\sigma\Delta\varphi_{1,2}}\ldots e^{i\alpha\sigma\Delta\varphi_{2k-1,2k}} - 1:$$

$$= \frac{1}{2}\sum_{\sigma} : \prod_{l\,1}^{h}(\cos\alpha\sigma\Delta\varphi_{2l-1,2l} + i\sin\alpha\sigma\Delta\varphi_{2l-1,2l}) - 1:$$

$$= \sum_{\substack{\mathscr{P}\\|\mathscr{P}|\,\text{even}}} : P_{\mathscr{P}}(\varphi^{[\leqslant h]}): \tag{5.11}$$

where $\mathscr{P}$ is a subset $\{l_1, \ldots, l_q\} \subset \{1, \ldots, k\}$, $0 \leqslant q \leqslant k$, $|\mathscr{P}| = q$ and

$$:P_{\mathscr{P}}(\varphi^{[\leqslant h]}): = (i)^{|\mathscr{P}|} : \left(\prod_{J\,1}^{q}\sin\alpha\Delta\varphi^{[\leqslant h]}_{2l_J-1,2l_J}\right)\prod_{s\notin P}\cos\alpha\Delta\varphi^{[\leqslant h]}_{2s-1,2s}:$$

$$:P_{\Phi}(\varphi^{[\leqslant h]}): = :\left(\prod_{l\,1}^{k}\cos\alpha\Delta\varphi^{[\leqslant h]}_{2l-1,2l}\right) - 1: \tag{5.12}$$

We now observe that $P_{\mathscr{P}}(\varphi^{[\leqslant h]})$ is *odd* under the transformation

$$\xi_{2l_J-1} \rightleftarrows \xi_{2l_J}, \qquad \forall l_J \in \mathscr{P}$$

and *even* under the transformation

$$\xi_{2s-1} \rightleftarrows \xi_{2s}, \qquad \forall s \notin \mathscr{P}$$

Therefore we can define the following operations:

$$\begin{aligned}
S_{2i-1,2i}f(\xi_1, \ldots, \xi_{2k}) &= \frac{1}{2}(f(\xi_1, \ldots, \xi_{2i-1}, \xi_{2i}, \ldots, \xi_{2k}) \\
&\qquad + f(\xi_1, \ldots, \xi_{2i}, \xi_{2i-1}, \ldots, \xi_{2k})) \\
A_{2i-1,2i}f(\xi_1, \ldots, \xi_{2k}) &= \frac{1}{2}(f(\xi_1, \ldots, \xi_{2i-1}, \xi_{2i}, \ldots, \xi_{2k}) \\
&\qquad - f(\xi_1, \ldots, \xi_{2i}, \xi_{2i-1}, \ldots, \xi_{2k}))
\end{aligned} \tag{5.13}$$

and also the operation:

$$O_{\mathscr{P}}f \equiv \prod_{i\in\mathscr{P}} A_{2i-1,2i}\prod_{J\notin\mathscr{P}} S_{2J-1,2J}f \tag{5.14}$$

Of course

$$\sum_{\mathscr{P}} O_{\mathscr{P}} f = f \tag{5.15}$$

and, moreover, if f satisfies, for any l

$$f(\xi_1, \xi_2, \ldots, \xi_{2l-1}, \xi_{2l}, \ldots, \xi_{2k-1}, \xi_{2k}) = f(\xi_2, \xi_1, \ldots, \xi_{2l}, \xi_{2l-1}, \ldots, \xi_{2k}, \xi_{2k-1}) \tag{5.16}$$

$$\sum_{|\mathscr{P}|\,\text{even}} O_{\mathscr{P}} f = f.$$

THEOREM 1. — The following decomposition holds:

$$\tilde{V}(\gamma, \{ Q_v \}_s, \underline{\sigma}) = \left(\frac{\lambda(R)}{2}\right)^{2k} \sum_{|\mathscr{P}|\,\text{even}} \int_{\Lambda^{2k}} d\underline{\xi} : P_{\mathscr{P}}(\varphi^{[\leqslant h]}) : F_\gamma(\underline{\xi}, \underline{\sigma}; \mathscr{P}) \tag{5.17}$$

For non-neutral trees we proceed in a slightly different way; we divide the coordinates $\underline{\xi}$ of the final lines of $\bar{\gamma}$ in two groups: we call $\underline{\bar{\xi}}$ the $2\bar{k}$ which merge at some neutral bifurcation of $\bar{\gamma}$ and $\underline{\zeta}$ the $\bar{p}$ remaining ones.

Then we decompose $: e^{i\alpha\varphi(\bar{\gamma}, \underline{\bar{\bar{\sigma}}})} :$ in the following way:

$$: e^{i\alpha\varphi(\bar{\gamma}, \underline{\bar{\bar{\sigma}}})} : = : e^{i\alpha \sum_{l=1}^{\bar{k}} \Delta\varphi^{[\leqslant h]}_{2l-1,2l}} e^{i\alpha\varphi(\mathscr{L}(\bar{\gamma}))} : = \Sigma_N : P_N(\varphi^{[\leqslant h]}) e^{i\alpha\varphi(\mathscr{L}(\bar{\gamma}))} : \tag{5.18}$$

where $\varphi(\mathscr{L}(\bar{\gamma})) = \sum_{s=1}^{\bar{p}} \varphi^{[\leqslant h]}_{\zeta_s} \bar{\bar{\sigma}}_{2\bar{k}+s}$ and N is a subset of $\{ 1, \ldots, \bar{k} \}$.

Therefore

$$\tilde{V}(\bar{\gamma}, \{ Q_v \}_s, \underline{\bar{\bar{\sigma}}}) = \left(\frac{\lambda(R)}{2}\right)^{2\bar{k}+\bar{p}} \Sigma_N \int_{\Lambda^{2\bar{k}+\bar{p}}} d\underline{\bar{\xi}} d\underline{\zeta} : P_N(\varphi^{[\leqslant h]}) e^{i\alpha\varphi(\mathscr{L}(\bar{\gamma}))} : \cdot (F_{\bar{\gamma}}(\underline{\bar{\xi}}, \underline{\zeta}, \underline{\bar{\bar{\sigma}}}; N) \tag{5.19}$$

where O_N operates only the $\underline{\bar{\xi}}$ coordinates.

The decompositions (5.17) and (5.19) are useful as we have a recursive expression for $O_{\mathscr{P}}(F_\gamma(\underline{\zeta}, \underline{\sigma}; Q = 0))$ and $O_N(F_{\bar{\gamma}}(\underline{\bar{\xi}}, \underline{\zeta}, \underline{\bar{\bar{\sigma}}}; Q \neq 0))$.

This is the content of the next theorem, which together with (5.8 a) and (5.8 b) proves Theorem 1 as well.

THEOREM 2. — The following relations hold

$$F_\gamma(\underline{\xi}, \underline{\sigma}; \mathscr{P}) \doteq (\text{const.}) \sum_{\substack{\mathscr{P}_1,\dots,\mathscr{P}_{s_1}\\ |\mathscr{P}_i|\text{ even}}} \sum_{N_1,\dots,N_{s_2}} \left[O_{\mathscr{P}'\Delta(\mathscr{P}_1\cup\dots\cup N_{s_2})}(W_{(\gamma)}(\underline{\xi}, \underline{\sigma})) \cdot \prod_{i=1}^{s_1} F_{\gamma_i}(\underline{\xi}^{(i)}, \underline{\sigma}^{(i)}; \mathscr{P}_i) \prod_{J=1}^{s_2} F_{\bar\gamma_J}(\underline{\bar\xi}^{(J)}, \underline{\zeta}^{(J)}, \underline{\bar{\bar\sigma}}^{(J)}; N_J)\right] \quad (5.20)$$

$$F_{\bar\gamma}(\underline{\bar\xi}, \underline{\zeta}, \underline{\bar\sigma}; N) = (\text{const.}) \sum_{\substack{\mathscr{P}_1,\dots,\mathscr{P}_{s_1}\\ |\mathscr{P}_i|\text{ even}}} \sum_{N_1,\dots,N_{s_2}} \left[O_{N\Delta(\mathscr{P}_1\cup\dots\cup N_{s_2})}(W_{(\bar\gamma)}(\underline{\bar\xi}, \underline{\zeta}, \underline{\bar\sigma})) \cdot \prod_{i=1}^{s_1} F_{\gamma_i}(\underline{\xi}^{(i)}, \underline{\sigma}^{(i)}; \mathscr{P}_i) \prod_{J=1}^{s_2} F_{\bar\gamma_J}(\underline{\bar\xi}^{(J)}, \underline{\zeta}^{(J)}, \underline{\bar{\bar\sigma}}^{(J)}; N_J)\right]$$

where we have assumed that at the lowest bifurcation $h(\gamma)\,(h(\bar\gamma))$ s_1 neutral trees $\gamma_1, \dots, \gamma_{s_1}$ and s_2 non neutral trees $\bar\gamma_1, \dots, \bar\gamma_{s_2}$ merge. $\mathscr{P}\Delta\mathscr{G}$ is the symmetric difference $\mathscr{P}\backslash\mathscr{G} \cup \mathscr{G}\backslash\mathscr{P}$ and $O_{\mathscr{P}\Delta\mathscr{G}} f = \tilde f$ tells us that the function $\tilde f$ must be odd under the exchange $\xi_{2l-1} \leftrightarrows \xi_{2l}$ for each $l \in \mathscr{P}\Delta\mathscr{G}$.

The function W is associated to the truncated expectation at the lowest frequency; its explicit expression is given in the proof of the theorem. The set $\mathscr{P}' \subset \mathscr{P}$ and the operation $\tilde O_{\mathscr{P}'\Delta(\mathscr{P}_1\cup\dots\cup N_{s_2})}$ will be defined in the course of the proof.

Proof. — We first consider $Q = 0$ case; starting from the lowest bifurcation, $h(\gamma) = h+1$ the tree γ looks like $(k(\gamma) = h)$

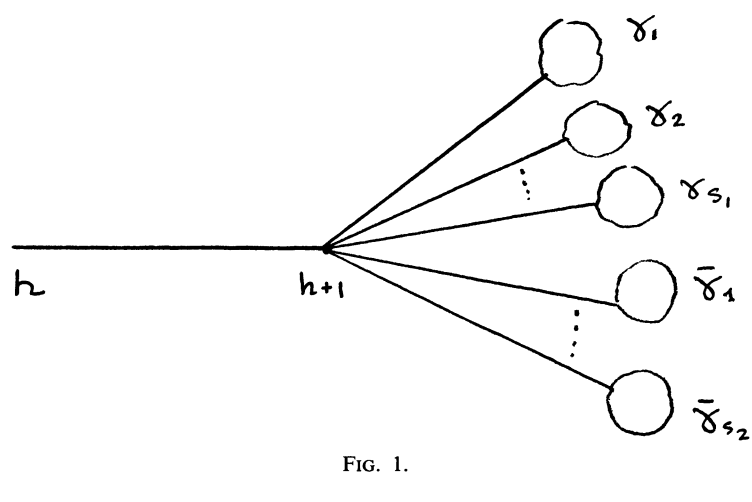

FIG. 1.

Given the subtrees $\gamma_1, \ldots, \gamma_{s_1}$ and $\overline{\gamma}_1, \ldots, \overline{\gamma}_{s_2}$ we assume expressions (5.17) and (5.19) for $\tilde{V}(\gamma_i, \{ Q_v \}_{s_i}, \underline{\sigma}^{(i)})$ and $\tilde{V}(\overline{\gamma}_s, \{ Q_v \}_{s_J}, \underline{\overline{\sigma}}^{(J)})$.

$$\tilde{V}(\gamma, \{ Q_v \}_s, \underline{\sigma})$$

$$= \sum_{\mu_0}^{(-1,+1)} \varepsilon^T_{[h+1]}(\tilde{V}(\gamma_1, \underline{\sigma}^{(1)}), \ldots, \tilde{V}(\gamma_{s_1}, \underline{\sigma}^{(s_1)}), \tilde{V}(\overline{\gamma}_1, \mu_0\underline{\overline{\sigma}}^{(1)}), \ldots, \tilde{V}(\overline{\gamma}_{s_2}, \mu_0\underline{\overline{\sigma}}^{(s_2)}))$$

$$= \sum_{\substack{\mathscr{P}_1, \ldots, \mathscr{P}_{s_1} \\ |\mathscr{P}_i| \text{ even}}} \sum_{N_1, \ldots, N_{s_2}} \left(\frac{\lambda(R)}{2}\right)^{2k} \int d\underline{\xi}^{(1)} \ldots d\underline{\xi}^{(s_1)} \int d\underline{\overline{\xi}}^{(1)} \ldots d\underline{\overline{\xi}}^{(s_2)} d\underline{\zeta}^{(1)} \ldots d\underline{\zeta}^{(s_2)} .$$

$$\Sigma_{\mu_0}\varepsilon^T_{[h+1]}(: P_{\mathscr{P}_1}(\varphi^{[\leqslant h+1]}) :, \ldots, : P_{\mathscr{P}_{s_1}}(\varphi^{[\leqslant h+1]}) :,$$

$$: P_{N_1}(\varphi^{[\leqslant h+1]})e^{i\alpha\mu_0\varphi(\mathscr{L}_1)} :, \ldots, : P_{N_{s_2}}(\varphi^{[\leqslant h+1]})e^{i\alpha\mu_0\varphi(\mathscr{L}_{s_2})} :)$$

$$\left[\prod_{i=1}^{s_1} F_{\gamma_i}(\underline{\xi}^{(i)}, \underline{\sigma}^{(i)} ; \mathscr{P}_i) \prod_{J=1}^{s_2} F_{\overline{\gamma}_J}(\underline{\overline{\xi}}^{(J)}, \underline{\zeta}^{(J)} ; N_J)\right] \quad (5.21)$$

where $\mathscr{L}_J = \mathscr{L}(\overline{\gamma}_J)$ (see eq. (5.18)).

We remark that $F_{\overline{\gamma}_J}$ does not depend on μ_0 as it is left invariant under the simultaneous change of sign of all the charges of $\overline{\gamma}_J$.

Moreover from definition (5.12), we can write

$$P_{\mathscr{P}}(\varphi^{[\leqslant h+1]}) = \frac{1}{2^k} \sum_{\underline{\tau}} \tau_{\mathscr{P}} : e^{i\alpha\underline{\tau}.\underline{\Delta\varphi}^{[\leqslant h+1]}} : = \frac{1}{2^k} \sum_{\underline{\tau}} \tau_{\mathscr{P}} : e^{i\alpha\underline{\tau}\mu_0.\underline{\Delta\varphi}^{[\leqslant h+1]}} : \quad (5.22)$$

where $\underline{\tau} = (\tau_1, \ldots, \tau_k)$, $\tau_{\mathscr{P}} = \tau_{l_1}.\tau_{l_2} \ldots \tau_{l_q}$
where $\mathscr{P} = \{ l_1, \ldots, l_q \}$

$$\underline{\tau}.\underline{\Delta\varphi}^{[\leqslant h+1]} = \sum_{l=1}^{k} \tau_l \Delta\varphi^{[\leqslant h+1]}_{2l-1,2l} \quad (5.23)$$

and

$$P_N(\varphi^{[\leqslant h+1]}) = \frac{1}{2^{\overline{k}}} \sum_{\underline{\overline{\tau}}} \overline{\tau}_N : e^{i\alpha\underline{\overline{\tau}}\mu_0\underline{\Delta\varphi}^{[\leqslant h+1]}} : \quad (5.24)$$

The (-1) present in P_Φ has been neglected since it has no effect in the truncated expectations.

Remembering the relation

$$: e^{i\alpha\varphi^{[\leqslant h+1]}} : = : e^{i\alpha\varphi^{[\leqslant h]}} : : e^{i\alpha\varphi^{[h+1]}} : \quad (5.25)$$

and using (5.22) and (5.24) to compute $\Sigma_{\mu_0}\varepsilon^T_{[h+1]}(\ \ )$ of (5.21) we get

$$[\Sigma_{\mu_0}\varepsilon^T_{[h+1]}(\ \ )] = \frac{1}{2^{c(k,\overline{k})}} \Sigma_{\underline{\tau}} : \cos \alpha\left(\underline{\tau}.\underline{\Delta\varphi}^{[\leqslant h]} + \sum_{J=1}^{s_2} \varphi^{[\leqslant h]}(\mathscr{L}_J)\right) :$$

$$. \ (\tau_{\mathscr{P}_1} \ldots \tau_{\mathscr{P}_{s_1}} \tau_{N_1} \ldots \tau_{N_{s_2}}) W_{(\gamma)}(\underline{\xi}, \underline{\tau}) \quad (5.26)$$

where

$$\underline{\tau} = \underline{\tau}^{(1)} \oplus \ldots \oplus \underline{\tau}^{(s_1)} \oplus \underline{\overline{\tau}}^{(1)} \oplus \ldots \oplus \underline{\overline{\tau}}^{(s_2)} \tag{5.27}$$

$$c(k, \overline{k}) = 2 \sum_{1}^{s_1} k_i + \sum_{1}^{s_2} \overline{k}_J \tag{5.28}$$

and

$$\begin{aligned} W_{(\gamma)}(\underline{\xi}, \underline{\tau}) = & e^{U^{[\leqslant h]}(\gamma_1, \ldots, \gamma_{s_1}, \overline{\gamma}_1, \ldots, \overline{\gamma}_{s_2}; \underline{\tau}^{(1)}, \ldots, \underline{\tau}^{(s_1)}, \underline{\overline{\tau}}^{(1)}, \ldots, \underline{\overline{\tau}}^{(s_2)})} \\ & \cdot \mathcal{E}^{T}_{[h+1]}(:e^{i\alpha\underline{\tau}^{(1)}.\underline{\Delta}\varphi^{(h+1)}}:, \ldots, :e^{i\alpha\underline{\tau}^{(s_1)}.\underline{\Delta}\varphi^{(h+1)}}:, :e^{i\alpha(\underline{\overline{\tau}}^{(1)}.\underline{\Delta}\varphi^{(h+1)} + \varphi^{(h+1)}(\mathcal{L}_1))}:, \ldots, \\ & :e^{i\alpha(\underline{\overline{\tau}}^{(s_2)}.\underline{\Delta}\varphi^{(h+1)} + \varphi^{(h+1)}(\mathcal{L}_{s_2}))}:) \end{aligned} \tag{5.29}$$

where $U^{[\leqslant h]}(\gamma_1, \ldots, \overline{\gamma}_{s_2}; \underline{\tau}^{(1)}, \ldots, \underline{\overline{\tau}}^{(s_2)})$ is the interaction energy between the clusters $\gamma_1, \ldots, \gamma_{s_1}, \overline{\gamma}_1, \ldots, \overline{\gamma}_{s_2}$ and $\underline{\tau}^{(1)}$ the vector defining the charges of γ_i with coordinates $\xi^{(i)}_{2l-1}$. Its explicit expression is given in the Appendix A.

Finally, we once more decompose $:\cos\alpha\left(\underline{\tau}.\underline{\Delta}\varphi^{[\leqslant h]} + \sum_{1}^{s_2}{}_{J}\, \varphi^{[\leqslant h]}(\mathcal{L}_J)\right):$ following eq. (5.11) obtaining

$$\begin{aligned} :\cos\alpha\left(\underline{\tau}.\underline{\Delta}\varphi^{[\leqslant h]} + \sum_{1}^{s_2}{}_{J}\, \varphi^{[\leqslant h]}(\mathcal{L}_J)\right): &= :\cos\alpha(\underline{\tau}.\underline{\Delta}\varphi^{[\leqslant h]} + \underline{1}.\underline{\Delta}\varphi^{[\leqslant h]}): \\ &= \sum_{\substack{\mathcal{P} \\ |\mathcal{P}|\ \mathrm{even}}} \tau_{\mathcal{P}'} :P_{\mathcal{P}}(\varphi^{[\leqslant h]}): + 1 \end{aligned} \tag{5.30}$$

The + 1 will be cancelled by the corresponding part of the counterterm, and $\mathcal{P} \in \{1, \ldots, k\}$,

$$\mathcal{P}' = \mathcal{P} \cap (\text{subset of } 1, \ldots, k \text{ where } \tau_i \text{ can take the values } +1 \text{ and } -1) \tag{5.31}$$

We have (where $c(k)$ is a constant depending only on k)

$$[\Sigma_{\mu_0}\mathcal{E}^{T}_{[h+1]}(\quad)] = c(k) \sum_{\substack{\mathcal{P} \\ |\mathcal{P}|\ \mathrm{even}}} :P_{\mathcal{P}}(\varphi^{[\leqslant h]}): \sum_{\tau} (\tau_{\mathcal{P}'}, \tau_{\mathcal{P}_1} \ldots \tau_{N_{s_1}}) W_{(\gamma)}(\underline{\xi}, \underline{\tau}) \tag{5.32}$$

(5.20) is proven noting from the explicit expression of $W_{(\gamma)}(\underline{\xi}, \underline{\tau})$ in (5.29) that changing the sign of $\tau^{(i)}_l$ is equivalent to performing the transformation

$$\xi^{(i)}_{2l-1} \longleftrightarrow \xi^{(i)}_{2l}$$

in the function $W_{(\gamma)}(\underline{\xi}, \underline{\sigma})$.

Therefore

$$\begin{aligned} &\Sigma_{\underline{\tau}}(\tau_{\mathcal{P}'}\tau_{\mathcal{P}_1} \ldots \tau_{\mathcal{P}_{s_1}}\tau_{N_1} \ldots \tau_{N_{s_2}}) W_{(\gamma)}(\underline{\xi}, \underline{\tau}) \sim \\ &O_{\mathcal{P}'\Delta(\mathcal{P}_1 \cup \ldots \cup \mathcal{P}_{s_1} \cup N_1 \cup \ldots \cup N_{s_2})}(W_{(\gamma)}(\underline{\xi}, \underline{\sigma})) \end{aligned}$$

and

$$F_\gamma(\underline{\xi}, \underline{\sigma}; \mathcal{P}) = \sum_{\substack{\mathcal{P}_1, \dots, \mathcal{P}_{s_1} \\ |\mathcal{P}_i| \text{ even}}} \sum_{N_1, \dots, N_{s_2}} O_{\mathcal{P}'\Delta(\mathcal{P}_1 \cup \dots \cup N_{s_2})}(W_{(\gamma)}(\underline{\xi}, \underline{\sigma})) \cdot \prod_{i}^{s_1} F_{\gamma_i}(\underline{\xi}^{(i)}, \underline{\sigma}^{(i)}; \mathcal{P}_i) \prod_{J}^{s_2} F_{\bar{\gamma}_J}(\underline{\bar{\xi}}^{(J)}, \underline{\zeta}^{(J)}, \underline{\bar{\sigma}}^{(J)}; N_J) \tag{5.33}$$

Remark. — $O_{\mathcal{P}'\Delta(\mathcal{P}_1 \cup \dots \cup N_{s_2})}$ is a symmetrization or antisymmetrization operator only with respect to those variables in the $\Delta\varphi_{2l-1,2l}$'s which are multiplied by the τ's (See eq. (5.30) and eq. (5.33).)

Thus equation (5.20) is proven; the case $Q \neq 0$ is proven analogously. This result also proves theorem I if we take into account (5.8 *a*) and (5.8 *b*).

We are now in the position to give the main result of this section, namely some estimates on the coefficients of $\tilde{V}(\gamma, \{ Q_v \}_s, \underline{\sigma})$ and $\tilde{V}(\bar{\gamma}, \{ Q_v \}_{\bar{s}}, \underline{\bar{\sigma}})$, which are the extension of the estimates proved in theorem 1 of (1).

Let $: P_{\mathcal{P}}(\varphi^{[\leq h]}) :$ be given; with $\mathcal{P} = \{ l_1, \dots, l_q \}$, we define the following expression:

$$[\text{Zeroes}; \mathcal{P}] \equiv \left(\prod_{J}^{q} | \xi_{2l_J-1} - \xi_{2l_J} |^{1-\varepsilon} \right) \tag{5.34a}$$

then $: P_{\mathcal{P}}(\varphi^{[\leq h]}) : (\gamma^{h(1-\varepsilon)|\mathcal{P}|} [\text{Zeroes}; \mathcal{P}])^{-1}$ does not have any zero when nearby coordinates coincide. If we have a sample $\varphi^{[\leq h]}$ such that $\chi^{B}_{Q_n}(\varphi^{[\leq h]}) = 1$, this quotient is bounded by $c(h)B^{|\mathcal{P}|}$ in the region Λ.

For $\mathcal{P} = \Phi$ we define

$$[\text{Zeroes}; \Phi] = (d(\underline{\xi}))^{2(1-\varepsilon)} . \gamma^{2h(1-\varepsilon)} \tag{5.34b}$$

where $d(\underline{\xi})$ is the length of the shortest polygonal path connecting the points $\xi_1, \dots, \xi_{2k}$.

LEMMA 2. — Let $h(\gamma) = h$ be the lowest bifurcation frequency and $k(\gamma) = h_0$ the root of the tree γ then

$$\gamma^{h(1-\varepsilon)|\mathcal{P}|} [\text{Zeroes}; \mathcal{P}] \quad F_\gamma(\underline{\xi}, \underline{\sigma}; \mathcal{P})$$

and

$$\gamma^{h(1-\varepsilon)|N|} [\text{Zeroes}; N] \quad F_{\bar{\gamma}}(\underline{\bar{\xi}}, \underline{\zeta}, \underline{\bar{\sigma}}; N)$$

have at each neutral bifurcation $v > v_0$ a « zero » of second order where by « zero » we mean that in the estimates of these functions at each neutral bifurcation v different from the lowest one we have a factor $\gamma^{-2(h_v - h_{v'})(1-\varepsilon)}$ where v' is the bifurcation immediately following v (going from top to bottem) and h_v, $h_{v'}$ are the associated frequencies.

Proof. — The proof will be by induction; we assume it true for the trees with final bifurcation of order n and we prove it for a tree with final bifurcation of order $n+1$. Let γ be such a tree ($Q(\gamma)=0$), of the following general type

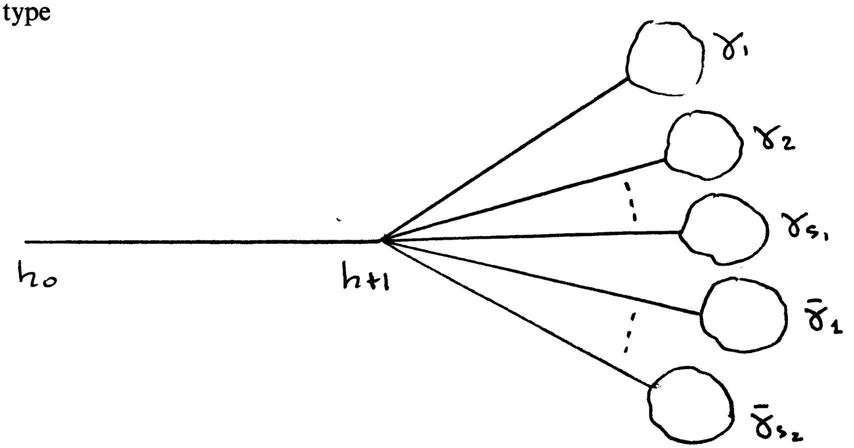

FIG. 2.

Let us consider a generic term $[(\mathscr{P}_1,\ldots,\mathscr{P}_{s_1},N_1,\ldots,N_{s_2})]$ of the sum (5.20) defining $O_{\mathscr{P}}(F_\gamma(\underline{\xi},\underline{\sigma};Q=0))$

$$(\gamma^{h|\mathscr{P}|})^{1-\varepsilon}[\text{Zeroes};\mathscr{P}]\,[(\mathscr{P}_1,\ldots,\mathscr{P}_{s_1},N_1,\ldots,N_{s_2})]$$

$$= \left\{ \left(\frac{\gamma^{h|\mathscr{P}|}}{\prod_1^{s_1}{}_i\,\gamma^{q_i|\mathscr{P}_i|}\prod_1^{s_2}{}_J\,\gamma^{\bar q_J|N_J|}} \right)^{1-\varepsilon} \frac{[\text{Zeroes};\mathscr{P}]}{\prod_1^{s_1}{}_i\,[\text{Zeroes};\mathscr{P}_i]\prod_1^{s_2}{}_J\,[\text{Zeroes};N_J]}\, O_{\mathscr{P}'\Delta(\mathscr{P}_1\cup\ldots\cup N_{s_2})}(W_{(\gamma)}(\underline{\xi};\underline{\sigma})) \right\}$$

$$\cdot \left[\prod_1^{s_1}{}_i\,\gamma^{q_i|\mathscr{P}_i|(1-\varepsilon)}\,[\text{Zeroes};\mathscr{P}_i] \prod_1^{s_2}{}_J\,\gamma^{\bar q_J|N_J|(1-\varepsilon)}\,[\text{Zeroes};N_J] \right.$$

$$\left. F_{\gamma_i}(\underline{\xi}^{(i)},\underline{\sigma}^{(i)};\mathscr{P}_i)F_{\gamma_J}(\underline{\bar\xi}^{(J)},\underline{\zeta}^{(J)},\underline{\bar\sigma}^{(J)};N_J)\right] \quad (5.35)$$

where

$$\begin{cases} q_i \equiv h(\gamma_i) & i:1,\ldots,S_1 \\ \bar q_J \equiv h(\bar\gamma_J) & J:1,\ldots,S_2 \end{cases} \quad (5.36)$$

We have to investigate the { } factor of (5.35).

$O_{\mathscr{P}\Delta(\ \ )}(W_{(\gamma)}(\ \ ))$ is symmetric under all the transformations $\xi_{2l-1} \rightleftarrows \xi_{2l}$, if $l \in \mathscr{P}' \cap (\mathscr{P}_1 \cup \ldots \cup N_{s_2})$; for $l \in \mathscr{P}' \Delta (\mathscr{P}_1 \cup \ldots \cup N_{s_2})$ it is antisymmetric and therefore has first order zeroes in $(\xi_{2l-1} - \zeta_{2l})^{1-\varepsilon}$.

Therefore

$$\mathrm{O}_{\mathscr{P}'\Delta(\quad)}(\mathrm{W}_{(\gamma)}(\quad))$$
$$= \gamma^{h(|\mathscr{P}'\backslash(\quad)|+|(\quad)\backslash\mathscr{P}'|)(1-\varepsilon)} [\text{Zeroes}; \mathscr{P}'\backslash(\quad)][\text{Zeroes}; (\quad)\backslash\mathscr{P}']\mathrm{G}(\underline{\xi}, \underline{\sigma}) \tag{5.37}$$

where $\mathrm{G}(\underline{\xi}, \underline{\sigma})$ does not have first-order zeroes anymore. We have

$$\{(5.35)\} = \frac{\gamma^{h(|\mathscr{P}'\backslash(\quad)|+|\mathscr{P}'\cap(\quad)|)(1-\varepsilon)}\gamma^{h(|\mathscr{P}'\backslash(\quad)|+|(\quad)\backslash\mathscr{P}'|)(1-\varepsilon)}}{\prod_{i=1}^{s_1} \gamma^{q_i|\mathscr{P}_i|(1-\varepsilon)} \prod_{J=1}^{s_2} \gamma^{\bar{q}_J|N_J|(1-\varepsilon)}} \cdot [\text{Zeroes}; \mathscr{P}'\backslash(\quad)]^2 \mathrm{G}(\underline{\xi}, \underline{\sigma})$$

$$\leqslant \prod_{i=1}^{s_1} \gamma^{(h-q_i)|\mathscr{P}_i|(1-\varepsilon)} \prod_{J=1}^{s_2} \gamma^{(h-\bar{q}_J)|N_J|(1-\varepsilon)} (\gamma^{h|\mathscr{P}'\backslash(\quad)|(1-\varepsilon)}[\text{Zeroes}; \mathscr{P}'\backslash(\quad)])^2 \mathrm{G}(\underline{\xi}, \underline{\sigma})$$

$$\leqslant \prod_{i=1}^{s_1} \gamma^{(h-q_i)|\mathscr{P}_i|(1-\varepsilon)} (\gamma^{h|\mathscr{P}'\backslash(\quad)|(1-\varepsilon)}[\text{Zeroes}; \mathscr{P}'\backslash(\quad)])^2 \mathrm{G}(\underline{\xi}, \underline{\sigma}). \tag{5.38}$$

If $\mathscr{P}_i \neq \Phi$, $|\mathscr{P}_i| \geqslant 2$ and we have produced s_1 second order « zeroes » associated to the neutral bifurcations $h(\gamma_1), \ldots, h(\gamma_{s_1})$.

If some $\mathscr{P}_i = \Phi$ there are two possibilities: either a zero associated to the bifurcation $h(\gamma_i)$ is in $(\gamma^{h|\mathscr{P}'\backslash(\quad)|(1-\varepsilon)}[\text{Zeroes}; \mathscr{P}'\backslash(\quad)])^2$ and is again of second order, or neither in $\mathscr{P}'$ nor in $(\mathscr{P}_1 \cup \ldots \cup N_{s_2})$ there are indices associated to the coordinates of γ_i in which case (remembering the definition of $\mathrm{W}_{(\gamma)}(\underline{\xi}, \underline{\sigma})$ and eq. (5.21)) there must be a zero of second order $(\gamma^h d(\underline{\xi}^{(i)}))^{2(1-\varepsilon)}$ in $\mathrm{G}(\underline{\xi}, \underline{\sigma})$.

This completes the proof of the Lemma in the $\mathrm{Q} = 0$ case (the argument for the non neutral tree is completely equivalent) provided we prove the inductive assumption for the trees with only one bifurcation. This is trivial as they do not have zeroes at all.

From this lemma and theorem 2 of [1] the following theorem is an easy corollary.

THEOREM 3. — Let $\Delta_i \in \mathrm{Q}_{h_0}$ ($h_0 = k(\gamma)$, $h = h(\gamma)$) then $\forall$ shapes s, $\forall \{ \mathrm{Q}_v \}_s$ and $\forall \underline{\sigma}$ we have, for $\mathrm{Q}(\gamma) = \mathrm{Q} = 0$

$$\sum_{\substack{\gamma \\ s(\gamma)=s \\ k(\gamma)=h_0 \\ v(\gamma)=2k}} \left(\frac{\lambda(\mathrm{R})}{2}\right)^{2k} \int_{\Delta_1 x \ldots x \Delta_{2k}} d\xi_1 \ldots d\xi_{2k} \gamma^{h_0|\mathscr{P}|(1-\varepsilon)}[\text{Zeroes}; \mathscr{P}] \,|\, \mathrm{F}_\gamma(\underline{\xi}, \underline{\sigma}; \mathscr{P}) \,|$$
$$\leqslant \begin{Bmatrix} \lambda^{2k}\gamma^{2k(\mathrm{R}-h_0)\varepsilon}, & \alpha^2 \geqslant 8\pi \\ \lambda^{2k}\gamma^{2k(\mathrm{R}-h_0)\left(2-\frac{\alpha^2}{4\pi}\right)}, & \alpha^2 < 8\pi \end{Bmatrix} e^{-\gamma^{h_0} d(\Delta_1, \ldots, \Delta_{2k})} \tag{5.39}$$

for $Q(\overline{\gamma}) = Q \neq 0$

$$\sum_{\substack{\overline{\gamma} \\ s(\overline{\gamma})=s \\ k(\overline{\gamma})=h_0 \\ v(\overline{\gamma})=n}} \left(\frac{\lambda(R)}{2}\right)^n \int_{\Delta_1 x \dots x \Delta n} d\xi_1 \dots d\xi_n \gamma^{h_0 |N|(1-\varepsilon)} [\text{Zeroes}; N] | F_{\overline{\gamma}}(\underline{\xi}, \underline{\overline{\overline{\sigma}}}; N) |$$

$$\leqslant \left\{ \begin{array}{ll} \lambda^n \gamma^{n(R-h_0)\varepsilon} \gamma^{-\frac{\alpha^2}{4\pi} h_0 Q^2} & \alpha^2 \geqslant 8\pi \\ \lambda^n \gamma^{n(R-h_0)\left(2-\frac{\alpha^2}{4\pi}\right)} \gamma^{-\frac{\alpha^2}{4\pi} h_0 Q^2} & \alpha^2 < 8\pi \end{array} \right\} e^{-\gamma h_0 d(\Delta_1, \dots, \Delta_{2k})} \qquad (5.40)$$

This theorem extends Theorem 2 of [*1*] using the estimates of Lemma 2. The proof is sketched in appendix A.

This result contains all the information we need on the general structure of $\tilde{V}_\Lambda^{(h)}$. In the next section we turn to the study of the upper bound, namely the right hand side of inequality (2.19).

6. THE UPPER BOUND

The proof of the upper bound is very involved; in this section our aim is to reduce it to the proof of an inequality similar to (4.12) but in the opposite direction. We think it is useful to first give a sketch of the main steps involved in this proof (see also [*2*]).

It turns out as will be proven in the next section, that the remainder produced integrating over the fields of definite frequency satisfy the right estimates which make it summable (see eq. (4.3)) if the field $\varphi^{[\leqslant h]}$ is Hölder continuous with a coefficient bounded by a constant B; this is easily achieved in the case of the lower bound just introducing appropriate characteristic functions constraining the field $\varphi^{[\leqslant h]}$ to be regular. On the other hand in the proof of the upper bound we cannot, *ab initio*, put any characteristic function in the integral defining $\tilde{Z}_Y^{(R)}$. Nevertheless, we have to exclude the regions where the fields are « rough ». For that purpose, following [*2*], we define an effective potential $\hat{V}_\Lambda^{(h)}(\mathscr{D}^{(h)})$, where for any choice of the field $\varphi^{[\leqslant h]}$ we subtract the regions (here indicated symbolically by $\mathscr{D}^{(h)}$, see later for a precise definition) where $\varphi^{[\leqslant h]}$ is « rough » (see eq. (3.23) of [*2*]). These regions have the drawback of being $\varphi^{[\leqslant h]}$-dependent and yet to be able to perform the integration at the level h with respect to the measure $P(d\varphi^{(h)})$, we should be free of this complicated dependance. Therefore we define a different effective potential $\hat{V}_\Lambda^{(h)}(\mathscr{D}^{(h-1)})$ which depends only on $\varphi^{[\leqslant h-1]}$ through its regions of integration and can therefore be integrated with respect to $P(d\varphi^{(h)})$.

The introduction of these new effective potentials is useful because between $\tilde{V}_\Lambda^{(h)}$, $\hat{V}_\Lambda^{(h)}(\mathcal{D}^{(h)})$ and $\hat{V}_\Lambda^{(h)}(\mathcal{D}^{(h-1)})$ we can prove the following relationships:

$$\begin{cases} \tilde{V}_\Lambda^{(R-1)} \leqslant \hat{V}_\Lambda^{(R-1)}(\mathcal{D}^{(R-1)}) \\ \hat{V}_\Lambda^{(h)}(\mathcal{D}^{(h)}) \leqslant \hat{V}_\Lambda^{(h)}(\mathcal{D}^{(h-1)}, \hat{R}^{(h)}) + [\text{other terms}] \end{cases} \tag{6.1}$$

where $\hat{R}^{(h)}$ are regions which will be defined later on. With [*other terms*] we mean terms produced at each step which have good estimates and can be safely added to the remainder, without spoiling its « summability » properties.

Once inequalities (6.1) are proven, the inequality for the upper bound is reduced to the following one:

$$\int P(d\varphi^{(h)})\chi^b_{(R^{(h)})^c}\overset{\circ}{\chi}{}^b_{R^{(h)}}e^{\hat{V}_\Lambda^{(h)}}(\mathcal{D}^{(h-1)}, \hat{R}^{(h)}) \leqslant e^{\left[\sum_1^{2M}{}_k \frac{1}{k!}\varepsilon^T_{[h]}(\hat{V}_\Lambda^{(h)}(\mathcal{D}^{(h-1)};k)\right]_{(2M)}}e^{\overline{R}_\Lambda^{(h-1)}} \tag{6.2}$$

(see eq. (3.32) of (2)) where $\overline{R}_\Lambda^{(h-1)}$ will be a remainder satisfying (4.3). Equations (6.1) and (6.2) allow us to apply an iterative mechanism as we can prove that

$$\left[\sum_1^{2M}{}_k \frac{1}{k!}\varepsilon^T_{[h]}(\hat{V}_\Lambda^{(h)}(\mathcal{D}^{(h-1)}); k)\right]_{(2M)} \leqslant \hat{V}_\Lambda^{(h-1)}(\mathcal{D}^{(h-1)}) + [\text{other terms}] \tag{6.3}$$

Remarks. — *i*) Inequality (6.2) is the counterpart for the upper bound, of inequality (4.12) needed for the lower bound. They will be proven in section 7. Their proof is a more or less straightforward extension of Lemma 1 of [*2*].

ii) In this case the hardest work will be the proofs of inequalities (6.1) and (6.3) which we can summarize symbolically as those inequalities which allow us to perform the following two steps:

STEP I).

$$\hat{V}_\Lambda^{(h)}(\mathcal{D}^{(h)}) \to \hat{V}_\Lambda^{(h)}(\mathcal{D}^{(h-1)}, \hat{R}^{(h)}) \tag{6.4}$$

STEP II).

$$\hat{V}_\Lambda^{(h)}(\mathcal{D}^{(h-1)}) \xrightarrow{\int P(d\varphi^{(h)})} \hat{V}_\Lambda^{(h-1)}(\mathcal{D}^{(h-1)})$$

The rest of this section will be devoted to making precise and to proving Steps I) and II).

The inequality (6.2) will be defined exactly at the end of this section.

6.1. Definition of $\hat{V}_\Lambda^{(h)}$

$\hat{V}_\Lambda^{(h)}(\mathcal{D}^{(h)})$ is defined subtracting from the regions of integration of $\tilde{V}_\Lambda^{(h)}$ those parts (field dependent!) where the fields are not Hölder continuous with a fixed bounded coefficient.

We define the following field dependent regions:

$$\mathrm{D}_l^{(h)}(\varphi^{[\leqslant h]}) \equiv \mathrm{D}_l^{(h)} \equiv \left\{ (\xi_{2l-1}, \xi_{2l}) \in \Lambda \times \Lambda \,\middle|\, \left| \sin \frac{\alpha}{2} \Delta\varphi^{[\leqslant h]}_{2l-1,2l} \right| \right.$$
$$\left. > \mathrm{B}(\gamma^h \,|\, \xi_{2l-1} - \xi_{2l} \,|)^{1-\varepsilon};\ \mathrm{B}(\gamma^h \,|\, \xi_{2l-1} - \xi_{2l} \,|)^{1-\varepsilon} \leqslant \delta < 1 \right\}$$
$$\mathrm{R}_l^{(h)}(\varphi^{(h)}) \equiv \mathrm{R}_l^{(h)} \equiv \left\{ \Delta \in \mathrm{Q}_h \,\middle|\, \exists \xi, \eta \in \Delta \text{ such that } \left| \sin \frac{\alpha}{2} \Delta\varphi^{(h)}_{\xi,\eta} \right| \right.$$
$$\left. > b(\gamma^h \,|\, \xi - \eta \,|)^{1-\varepsilon}(1 + \gamma^h d(\Delta, \Lambda)) \right\}$$
$$\hat{\mathrm{R}}^{(h)}(\varphi^{(h)}) \equiv \hat{\mathrm{R}}^{(h)} \equiv \{ \Delta \in \mathrm{Q}_h \,|\, \gamma^h d(\Delta, \mathrm{R}^{(h)}) \leqslant \mathscr{B} \} \tag{6.5}$$

where $\mathscr{B}$ will be defined in section 7. We have omitted in the regions D, R, $\hat{\mathrm{R}}$ the dependence on the various parameters.

The $\mathrm{D}_l^{(h)}$'s are the regions where the fields are « rough » and which have to be subtracted from $\tilde{\mathrm{V}}_\Lambda^{(h)}$.

In analogy with eq. (5.5) we write

$$\hat{\mathrm{V}}_\Lambda^{(h)}(\mathscr{D}^{(h)}) = \sum_{k=1}^{2\mathrm{M}} \sum_{(s,\{\mathrm{Q}_v\}_s,\underline{\sigma})} \sum_{\gamma \,\left\{\substack{s(\gamma)=s \\ k(\gamma)=h}\right.} \hat{\mathrm{V}}(\gamma, \{ \mathrm{Q}_v \}_s, \underline{\sigma}) \tag{6.6}$$

where

$$\hat{\mathrm{V}}(\gamma, \{ \mathrm{Q}_v \}_s, \underline{\sigma}) = \left(\frac{\lambda(\mathrm{R})}{2} \right)^{2k} \sum_{\substack{\mathscr{P} \\ |\mathscr{P}| \text{ even}}} \int_{\Lambda^{2k} \backslash \mathscr{D}^{(h)}(\mathscr{P})} d\underline{\xi} : \mathrm{P}_{\mathscr{P}}(\varphi^{[\leqslant h]}) \colon \mathrm{F}_\gamma(\underline{\xi}, \underline{\sigma}; \mathscr{P})$$
$$\hat{\mathrm{V}}(\bar{\gamma}, \{ \mathrm{Q}_v \}_{\bar{s}}, \underline{\bar{\sigma}}) = \left(\frac{\lambda(\mathrm{R})}{2} \right)^{2\bar{k}+\bar{p}} \Sigma_{\mathrm{N}} \int_{\Lambda^{2\bar{k}} \backslash \mathscr{D}^{(h)}(\mathrm{N}) \times \Lambda^{\bar{p}}} d\underline{\bar{\xi}} d\underline{\zeta} : \mathrm{P}_{\mathrm{N}}(\varphi^{[\leqslant h]}) e^{i\alpha\varphi(\mathscr{L}(\bar{\gamma}))} :$$
$$\mathrm{F}_{\bar{\gamma}}(\underline{\bar{\xi}}, \underline{\zeta}, \underline{\bar{\sigma}}; \mathrm{N}) \tag{6.7}$$

where, with $\mathrm{P} = \{ l_1, \ldots, l_q \}$

$$\Lambda^{2k} \backslash \mathscr{D}^{(h)}(\mathscr{P}) = \Lambda^2 \times \ldots \times \Lambda^2 \backslash \mathrm{D}_{l_1}^{(h)} \times \ldots \times \Lambda^2 \backslash \mathrm{D}_{l_q}^{(h)} \times \ldots \times \Lambda^2 \tag{6.8}$$

and exactly the same definition for $\mathscr{D}^{(h)}(\mathrm{N})$. From (6.8) it follows that

$$\mathscr{D}^{(h)}(\mathscr{P}) = \bigcup_{t=1}^{q} (\Lambda^2 \times \ldots \times \mathrm{D}_{l_t}^{(h)} \times \ldots \times \Lambda^2) \tag{6.9}$$

Remark. — In the case $\mathscr{P} = \Phi$, $: \mathrm{P}_\Phi : = : \prod_{\mathrm{J}=1}^{k} \cos \Delta\varphi^{[\leqslant h]}_{2\mathrm{J}-1,2\mathrm{J}} - 1 :$ is still decomposed in the following way

$$: \mathrm{P}_\Phi(\varphi^{[\leqslant h]}) : = \sum_{l=1}^{k} : \mathrm{P}_{\Phi_l}(\varphi^{[\leqslant h]}) : \tag{6.10}$$

where

$$: \mathrm{P}_{\Phi_l}(\varphi^{[\leqslant h]}) : = : (\cos \Delta\varphi^{[\leqslant h]}_{2l-1,2l} - 1) \prod_{\mathrm{J}=l+1}^{k} \cos \Delta\varphi^{[\leqslant h]}_{2\mathrm{J}-1,2\mathrm{J}} : \tag{6.11}$$

The definition of $\hat{V}_\Lambda^{(h)}(\mathscr{D}^{(h-1)})$ *is exactly the same as that of* $\hat{V}_\Lambda^{(h)}(\mathscr{D}^{(h)})$ *with* $h-1$ *instead of* h *in each* $D_{l_j}^{(h)}$.

The definition of $\hat{V}_\Lambda^{(h)}(\mathscr{D}^{(h-1)}, \hat{R}^{(h)})$ *is exactly the same as the previous one where in the region of integration each factor* $\Lambda^2 \backslash D_{l_j}^{(h)}$ *is substituted by* $(\Lambda \backslash \hat{R}^{(h)})^2 \backslash D_{l_j}^{(h-1)}$.

We are now in the position to describe the general strategy to perform Steps I) and II).

6.2. The general strategy for Step I) and Step II).

We start by considering Step I) and the order $\lambda^{n>2}$ part of $\hat{V}_\Lambda$.

Fixed an arbitrary level h we write the identity

$$\hat{V}^{(h)}_{\Lambda,O(\lambda>2)}(\mathscr{D}^{(h)}) = \hat{V}^{(h)}_{\Lambda,O(\lambda>2)}(\mathscr{D}^{(h-1)}, \hat{R}^{(h)}) + \left[\hat{V}^{(h)}_{\Lambda,O(\lambda>2)}(\mathscr{D}^{(h)}) - \hat{V}^{(h)}_{\Lambda,O(\lambda>2)}(\mathscr{D}^{(h-1)}, \hat{R}^{(h)}\right] \qquad (6.12)$$

then from the knowledge of the general structure it is easy to see that we can write

$$\left[\hat{V}^{(h)}_{\Lambda,O(\lambda>2)}(\mathscr{D}^{(h)}) - \hat{V}^{(h)}_{\Lambda,O(\lambda>2)}(\mathscr{D}^{(h-1)}, \hat{R}^{(h)})\right] = G^{(h)}_{1,O(\lambda>2)} + [\text{other terms}] \qquad (6.13a)$$

The dangerous part which we must be able to control is $G^{(h)}_{1,O(\lambda>2)}$ and will be made explicit in a moment. Similarly for the Step II) we can write (see eq. (6.3))

$$\left[\sum_{1}^{2M}{}_{k} \frac{1}{k!} \mathcal{E}^{T}_{[h]}(\hat{V}^{(h)}_{\Lambda}(\mathscr{D}^{(h-1)}; k)\right]_{(2M),O(\lambda>2)} = \hat{V}^{(h-1)}_{\Lambda,O(\lambda>2)}(\mathscr{D}^{(h-1)}) + \left[[\quad]_{(2M),O(\lambda>2)} - \hat{V}^{(h-1)}_{\Lambda,O(\lambda>2)}(\mathscr{D}^{(h-1)}\right]$$

and again we define

$$\left[[\quad]_{(2M),O(\lambda>2)} - \hat{V}^{(h-1)}_{\Lambda,O(\lambda>2)}(\mathscr{D}^{(h-1)}\right] \equiv G^{(h)}_{2,O(\lambda>2)} \qquad (6.13b)$$

As it will be clear looking at their explicit expressions, the terms $G^{(h)}_{1,O(\lambda>2)}$ and $G^{(h)}_{2,O(\lambda>2)}$ have a bad dependence on the fields and therefore cannot be integrated; neither we can just estimate them for each h and put them in the remainder as they would give rise to contributions in λ of order $\leqslant 2M$. We therefore have to eliminate them. To do this the following remark is crucial: *The analogous terms which appear at order* λ^2 *are negative.* Therefore one can hope that for λ small enough the $O(\lambda^2)$ terms dominate the $G^{(h)}_{i,O(\lambda>2)}$ parts in absolute value. Due to their negativity, inequalities (6.1) and (6.3) can be proven. This is exactly what happens for $h \geqslant h_0$ (where h_0 will be fixed later on) and can be summarized in the following way; We call $G^{(h)}_{O(\lambda^2)}$ the negative $O(\lambda^2)$ part which is present at the level h

(apart from $\hat{V}^{(h)}_{\Lambda,O(\lambda^2)}$), we split it into two well defined parts $G^{(h)}_{1,O(\lambda^2)}$ and $G^{(h)}_{2,O(\lambda^2)}$ and prove the following theorems:

THEOREM 4. — $\exists h_0$ and λ_0 (small enough) > 0, depending on α^2 and M, such that if $h \geqslant h_0$ and $\lambda \leqslant \lambda_0$ then

$$G^{(h)}_{1,O(\lambda^2)} + G^{(h)}_{1,O(\lambda>2)} \leqslant 0 \tag{6.14}$$

THEOREM 5. — $\exists h_0$ and λ_0 (small enough) > 0, depending on α^2 and M, such that if $h \geqslant h_0$ and $\lambda \leqslant \lambda_0$ then

$$G^{(h)}_{2,O(\lambda^2)} + G^{(h)}_{2,O(\lambda>2)} \leqslant ([\textit{negative part}] + [\textit{other terms}]) \leqslant [\textit{other terms}] \tag{6.15}$$

To prove these theorems we have to explicitely define the structure of$G^{(h)}_{i,O(\lambda^2)}$ and $G^{(h)}_{i,O(\lambda>2)}$.

Explicit expression of $G^{(h)}_{1:O(\lambda>2)}$.

We have the following lemma.

LEMMA 3. — If the inequality (4.7) : $b \leqslant B\left(1 - \frac{1}{\gamma^{1-\varepsilon}}\right)$ is satisfied then the following relation holds

$$D^{(h)}(\varphi^{[\leqslant h]})\backslash D^{(h-1)}(\varphi^{[\leqslant h-1]}) \subset (\hat{R}^{(h)} \times \hat{R}^{(h)})(\varphi^{(h)}) \tag{6.16}$$

Proof. — With obvious, inessential modifications the proof is the one for in Lemma 1 in [5].

From Lemma 3 we immediately have

$$\begin{aligned}\Lambda^2\backslash D^{(h)} = &\{ (\Lambda\backslash\hat{R}^{(h)})^2\backslash D^{(h-1)} \cup (\Lambda\backslash\hat{R}^{(h)})^2 \cap (D^{(h-1)}\backslash D^{(h)}) \} \\ &\cup [(\Lambda^2\backslash D^{(h)}) \cap (\hat{R}^{(h)} \times \hat{R}^{(h)}) \cup ((\Lambda\backslash\hat{R}^{(h)}) \times \hat{R}^{(h)} \cup \hat{R}^{(h)} \times (\Lambda\backslash\hat{R}^{(h)}))\backslash D^{(h-1)} \\ &\cup ((\Lambda\backslash\hat{R}^{(h)}) \times \hat{R}^{(h)} \cup \hat{R}^{(h)} \times (\Lambda\backslash\hat{R}^{(h)}) \cap (D^{(h-1)}\backslash D^{(h)})]\end{aligned} \tag{6.17}$$

and we can decompose the region $(\Lambda^2\backslash D^{(h)}_1 \times \ldots \times \Lambda^2\backslash D^{(h)}_n)$ in this way

$$\begin{aligned}(\Lambda^2\backslash D^{(h)}_1 \times \ldots \times \Lambda^2\backslash D^{(h)}_n) = &[(\Lambda\backslash\hat{R}^{(h)}_1)^2\backslash D^{(h-1)}_1 \times \ldots \times (\Lambda\backslash\hat{R}^{(h)}_n)^2\backslash D^{(h-1)}_n] \\ &\cup [(\Lambda\backslash\hat{R}^{(h)}_1)^2 \cap D^{(h-1)}_1\backslash D^{(h)}_1 \times \ldots \times (\Lambda\backslash\hat{R}^{(h)}_n)^2 \cap D^{(h-1)}_n\backslash D^{(h)}_n] \cup [\textit{other region}]\end{aligned} \tag{6.18}$$

$$\begin{aligned}[\textit{other region}] \subset \bigcup_{1}^{n}{}_{i} &[(\Lambda^2\backslash D^{(h)}_i) \cap (\hat{R}^{(h)}_i \times \hat{R}^{(h)}_i) \cup \\ &\cup ((\Lambda\backslash\hat{R}^{(h)}_i) \times \hat{R}^{(h)}_i \cup \hat{R}^{(h)}_i \times (\Lambda\backslash\hat{R}^{(h)}_i))\backslash D^{(h)}_i] \bigtimes_{1}^{n}{}_{J\neq i} (\Lambda^2\backslash D^{(h)}_J)\end{aligned} \tag{6.19}$$

Therefore from this definition we have

$$[(6.13)] = \sum_{3}^{2M}{}_{k} \sum_{(s,\{Q_v\}_s,\underline{\sigma})} \sum_{\substack{\gamma \\ s(\gamma)=s \\ k(\gamma)=h}} \Delta_1 \hat{V}(\gamma, \{ Q_v \}_s, \underline{\sigma}) \tag{6.20}$$

$$\Delta_1 \hat{V}(\gamma, \{ Q_v \}_s, \underline{\sigma})$$

$$= \left(\frac{\lambda(R)}{2}\right)^{2k} \sum_{\substack{\mathscr{P} \\ |\mathscr{P}|\,\text{even}}} \int_{[(\Lambda^{2k}\setminus\mathscr{R}(\mathscr{P}))\cap\mathscr{D}^{(h-1)}(\mathscr{P})\setminus\mathscr{D}^{(h)}(\mathscr{P})]} d\underline{\xi} : P_{\mathscr{P}}(\varphi^{[\leqslant h]}) : F_\gamma(\underline{\xi}, \underline{\sigma}; \mathscr{P})$$

$$+ \left(\frac{\lambda(R)}{2}\right)^{2k} \sum_{\substack{\mathscr{P} \\ |\mathscr{P}|\,\text{even}}} \int_{[\text{other region}]} d\underline{\xi} : P_{\mathscr{P}}(\varphi^{[\leqslant h]}) : F_\gamma(\underline{\xi}, \underline{\sigma}; \mathscr{P})$$

$$\equiv \Delta_{1,a} \hat{V}(\gamma, \{ Q_v \}_s, \underline{\sigma}) + \Delta_{1,b} \hat{V}(\gamma, \{ Q_v \}_s, \underline{\sigma}) \tag{6.21}$$

where for, fixed $\mathscr{P} \equiv \{ l_1, \ldots, l_q \}$,

$$[(\Lambda^{2k}\setminus\mathscr{R}(\mathscr{P}))\cap\mathscr{D}^{(h-1)}(\mathscr{P})\setminus\mathscr{D}^{(h)}(\mathscr{P})]$$

$$= \Lambda^2 \times \ldots \times (\Lambda\setminus \hat{R}_{l_1}^{(h)})^2 \cap D_{l_1}^{(h-1)}\setminus D_{l_1}^{(h)} \times \ldots \times (\Lambda\setminus \hat{R}_{l_q}^{(h)})^2 \cap D_{l_q}^{(h-1)}\setminus D_{l_q}^{(h)} \times \ldots \times \Lambda^2 \tag{6.22}$$

The definitions for the non neutral trees are exactly the same and we do not write them explicitely. Using (6.19) we define

$$G_{1,O(\lambda>2)}^{(h)} = \sum_{3}^{2M}{}_{k} \sum_{(s,\{Q_v\}_s,\underline{\sigma})} \sum_{\substack{\gamma \\ s(\gamma)=s \\ k(\gamma)=h}} \Delta_{1,a} \hat{V}(\gamma, \{ Q_v \}_s, \underline{\sigma}) \tag{6.23}$$

and we are left to prove that the same sum with $\Delta_{1,b}\hat{V}$ can be considered as [*other terms*] defined after eq. (6.1).

Explicit expression of $G_{2,O(\lambda>2)}^{(h)}$.
Again we define

$$[(6.13b)] \equiv G_{2,O(\lambda>2)}^{(h)} = \sum_{3}^{2M}{}_{k} \sum_{(s,\{Q_v\}_s,\underline{\sigma})} \sum_{\substack{\gamma \\ s(\gamma)=s \\ k(\gamma)=h \\ h(\gamma)=h+1}} \Delta_2 \hat{V}(\gamma, \{ Q_v \}_s, \underline{\sigma}) \tag{6.24}$$

and using the results of section 5 (in particular eq. (5.21)) we have

$$\Delta_2\hat{V}(\gamma, \{ Q_v \}_s, \underline{\sigma}) = -\left(\frac{\lambda(R)}{2}\right)^{2k} \sum_{\substack{\mathscr{P} \\ |\mathscr{P}| \text{ even}}} \left\{ \int_{\Lambda^{2k}\backslash\mathscr{D}^{(h)}(\mathscr{P})} d\underline{\xi} : P_{\mathscr{P}}(\varphi^{[\leq h]}) : F_\gamma(\underline{\xi}, \underline{\sigma}; \mathscr{P}) \right.$$

$$- \sum_{\substack{\mathscr{P}_1,\dots,\mathscr{P}_{s_1} \\ |\mathscr{P}_i| \text{ even}}} \sum_{N_1,\dots,N_{s_2}}$$

$$\int_{\substack{(\Lambda^{2k_1}\backslash\mathscr{D}^{(h)}(\mathscr{P}_1) \times \dots \times \Lambda^{2k_{s_1}}\backslash\mathscr{D}^{(h)}(\mathscr{P}_{s_1}) \\ \times\Lambda^{2\bar{k}_1}\backslash\mathscr{D}^{(h)}(N_1) \times \dots \times \Lambda^{2\bar{k}_{s_2}}\backslash\mathscr{D}^{(h)}(N_{s_2}) \\ \times\Lambda^{(\bar{p}_1 + \dots + \bar{p}_{s_2})})}} d\underline{\xi} : P_{\mathscr{P}}(\varphi^{[\leq h]}) : O_{\mathscr{P}'\Delta(\mathscr{P}_1\cup\dots\cup N_{s_2})}(W_{(\gamma)}(\underline{\xi}, \underline{\sigma}))$$

$$\left. \cdot \prod_{1}^{s_1}{}_i F_{\gamma_i}(\underline{\xi}^{(i)}, \underline{\sigma}^{(i)}; \mathscr{P}_i) \prod_{1}^{s_2}{}_J F_{\bar{\gamma}_J}(\underline{\bar{\xi}}^{(J)}, \underline{\zeta}^{(J)}, \underline{\bar{\sigma}}^{(J)}; N_J) \right\} \tag{6.25}$$

and a similar expression for the $Q \neq 0$ case.

Remark. — It is important to observe that in Σ_γ of (6.24) the frequency of the lowest bifurcation is fixed; this is crucial for the proof of Theorem 5 and follows from the definition of $G^{(h)}_{2,O(\lambda^{>2})}$ (see eq. (6.15)).

Explicit expression of $G^{(h)}_{O(\lambda^2)}$.

The explicit expression of $G^{(h)}_{O(\lambda^2)}$ is discussed in [*6*] where it is used as a tool to prove the ultraviolet stability of the massive Sine-Gordon theory for any $\alpha^2 < 8\pi$. Here we only state the final result applied to this case. Some remarks on it are collected in Appendix B.

THEOREM 6. — If b satisfies ineq. (4.7) with B large enough then at each level h of the iterative mechanism it is possible to isolate the following $O(\lambda^2)$ negative terms

$$G^{(h)}_{O(\lambda^2)} = G^{(h)}_{1,O(\lambda^2)} + G^{(h)}_{2,O(\lambda^2)}$$

where

$$G^{(h)}_{1,O(\lambda^2)} = -c_1\lambda^2(R) \int_{[(\Lambda\backslash\hat{R}^{(h)})^2\cap D^{(h-1)}\backslash D^{(h)}]} d\xi d\eta \sin^2\frac{\alpha}{2}\Delta\varphi^{[\leq h]}_{\xi,\eta} \sum_{q=h+1}^{R} e^{\alpha^2 c^{[<q]}}(e^{\alpha^2 c^{(q)}} - 1) \tag{6.26}$$

$$G^{(h)}_{2,O(\lambda^2)} = -A_1\lambda^2(R) \int_{D^{(h)}} d\xi d\eta \sin^2\frac{\alpha}{2}\Delta\varphi^{[\leq h]}_{\xi\eta} \sum_{q=h+1}^{R} \gamma^{-2(q-h)(1-\varepsilon)} e^{\alpha^2 c^{[<q]}}(e^{\alpha^2 c^{(q)}} - 1)$$

$$- A_2\lambda^2(R) \int_{D^{(h)}} d\xi d\eta\, [B(\gamma^h |\xi-\eta|)^{1-\varepsilon}]^2 \sum_{q=h+1}^{R} e^{\alpha^2 c^{[<q]}}(e^{\alpha^2 c^{(q)}} - 1) \tag{6.27}$$

where c_1, A_1, A_2 are constants > 0, $c^{[<q]} \equiv c^{[\leq q-1]}_{(\xi,\eta)}$ and $c^{(q)} \equiv c^{(q)}_{(\xi,\eta)}$.

Remark. — Once Theorems 4 and 5 and the following corollary (see the observations after eq. (6.23)) are proven, we have proven Steps I) and II), for $h \geqslant h_0$.

COROLLARY 1.

$$\left|\sum_{3}^{2M}{}_k \sum_{(s,\{Q_v\}_s,\bar{\sigma})} \sum_{\substack{\gamma \\ \left\{\begin{smallmatrix} s(\gamma)=s \\ k(\gamma)=h \end{smallmatrix}\right.}} \Delta_{1,b}\hat{V}(\gamma,\{Q_v\}_s,\bar{\sigma})\right| \leqslant [\text{other terms}] \quad (6.28)$$

We prove Theorem 5, Theorem 4 goes along the same lines and will be discussed later on.

Proof of theorem 5. — Remembering eq. (5.20), from eqs. (6.24), (6.25) it is clear that each $\Delta_2\hat{V}(\gamma,\{Q_v\}_s,\bar{\sigma})$ would be identically zero and so would be $G^{(h)}_{2,O(\lambda>2)}$ if the regions of integration of the different terms in (6.25) were equal. Therefore we only have to verify that the terms we add or subtract to make the regions of integration equal are controlled by $G^{(h)}_{2,O(\lambda^2)}$.

This will be done in two steps:

a) We eliminate the dangerous regions of $:P_{\mathscr{P}}:$ where the zeroes of $:P_{\mathscr{P}}:$ are not effective because of the lack of Hölder continuity.

b) We put back those regions $\mathscr{D}^{(h)}(\mathscr{P}_1),\ldots,\mathscr{D}^{(h)}(\mathscr{P}_{s_1}), \mathscr{D}^{(h)}(N_1),\ldots,\mathscr{D}^{(h)}(N_{s_2})$ which at this point are not needed anymore.

STEP *a*).

With arbitrarily fixed $\mathscr{P}_1,\ldots,\mathscr{P}_{s_1}, N_1,\ldots,N_{s_2}$ and $\mathscr{P}=\{l_1,\ldots,l_q\}$ we write

$$(\Lambda^{2k_1}\backslash\mathscr{D}^{(h)}(\mathscr{P}_1)\times\ldots\times\Lambda^{2\bar{k}s_2}\backslash\mathscr{D}^{(h)}(N_{s_2})\times\Lambda^{(\bar{p}_1+\ldots+\bar{p}_{s_2})})=J_1^2\times\ldots\times J_k^2\equiv\tilde{J} \quad (6.29)$$

where J_i^2 can be Λ^2 or $\Lambda^2\backslash D^{(h)}_{s_i}$ depending on i.

$$\tilde{J}\backslash\mathscr{D}^{(h)}(\mathscr{P})=(J_1^2\times\ldots\times J_{l_1}^2\backslash D^{(h)}_{2l_1-1,2l_1}\times\ldots\times J_{l_q}^2\backslash D^{(h)}_{2l_q-1,2l_q}\times\ldots\times J_k^2) \quad (6.30)$$

therefore

$$\tilde{J}\backslash(\tilde{J}\backslash\mathscr{D}^{(h)}(\mathscr{P}))=\bigcup_{1}^{q}{}_t(J_1^2\times\ldots\times J_{l_t-1}^2\times D^{(h)}_{l_t}\times J_{l_t+1}^2\times\ldots\times J_k^2) \quad (6.31)$$

Therefore Step *a*) is achieved if we prove that, given an arbitrary constant $\overline{c}$, it is possible to choose λ small enough (depending on $\overline{c}$) such that

$$G^{(h)}_{2,O(\lambda^2)} + \overline{c} \sum_{\substack{\gamma \\ s(\gamma)=s \\ k(\gamma)=h \\ h(\gamma)=h+1}} \left(\frac{\lambda(R)}{2}\right)^{2k} \int_{(J_1^2 \times \ldots \times J_{l_t-1}^2 \times D_{l_t}^{(h)} \times J_{l_t+1}^2 \times \ldots \times J_k^2)} d\underline{\zeta} : P_{\mathscr{P}}(\varphi^{[\leqslant h]}) :$$

$$\cdot\ O_{\mathscr{P}'\Delta(\ )}(W_{(\gamma)}(\underline{\xi}, \underline{\sigma})) \prod_{i}^{s_i} F_{\gamma_i}(\mathscr{P}_i) \prod_{J}^{s_2} F_{\overline{\gamma}_J}(N_J) \leqslant 0$$

$$\forall s, \{ Q_v \}_s, \underline{\sigma}, \qquad h \geqslant h_0, \qquad 1 < k \leqslant M \qquad (6.32)$$

Proof of inequality (6.32). — We observe, first of all, that the zeroes of $: P_{\mathscr{P}}(\varphi^{[\leqslant h]}) :$ associated to the indices $l \in \mathscr{P} \backslash (\mathscr{P}_1 \cup \ldots \cup N_{s_2})$ are not « effective » as the field is not forced to be Hölder continuous in the corresponding regions. Therefore we estimate $: P_{\mathscr{P}}(\varphi^{[\leqslant h]}) :$ in the following way: assume $l_t = 1$, then

$$| : P_{\mathscr{P}}(\varphi^{[\leqslant h]}) : |$$

$$\leqslant c(h) \left| \frac{\sin \alpha \Delta \varphi_{1,2}^{[\leqslant h]} : \overline{P}_{\mathscr{P} \cap (\ )}(\varphi^{[\leqslant h]})}{\gamma^{h(1-\varepsilon)|\mathscr{P} \cap (\ )|[\text{Zeroes}; \mathscr{P} \cap (\ )]}} \right| \gamma^{h(1-\varepsilon)|\mathscr{P} \cap (\ )|[\text{Zeroes}; \mathscr{P} \cap (\ )]}$$

$$\leqslant \overline{c}(h) \sin^2 \frac{\alpha}{2} \Delta \varphi_{12}^{[\leqslant h]} \left| \frac{: \overline{P}_{\mathscr{P} \cap (\ )}(\varphi^{[\leqslant h]}) :}{\gamma^{h(1-\varepsilon)|\mathscr{P} \cap (\ )|[\text{Zeroes}; \mathscr{P} \cap (\ )]}} \right| \frac{\gamma^{h(1-\varepsilon)|\mathscr{P} \cap (\ )|[\text{Zeroes}; \mathscr{P} \cap (\ )]}}{\gamma^{h(1-\varepsilon)} | \xi_1 - \xi_2 |^{1-\varepsilon}} \qquad (6.33)$$

$$\leqslant \overline{c}(h) \sin^2 \frac{\alpha}{2} \Delta \varphi_{12}^{[\leqslant h]} \frac{\gamma^{h(1-\varepsilon)|\mathscr{P} \cap (\ )|[\text{Zeroes}; \mathscr{P} \cap (\ )]}}{\gamma^{h(1-\varepsilon)} | \xi_1 - \xi_2 |^{1-\varepsilon}}$$

where $: \overline{P}_{\mathscr{P} \cap (\ )} :$ is that part of $: P_{\mathscr{P}} :$ which gives effective zeroes. The second term of the l. h. s. of (6.32) can be estimated by

$$\overline{c}\,(\text{const.})\, \lambda^2(R) \int_{D^{(h)}} d\xi_1 d\xi_2 \sin^2 \frac{\alpha}{2} \Delta \varphi_{1,2}^{[\leqslant h]}$$

$$\left\{ \sum_{\gamma} \int_{(J_1^2 \times \ldots \times \hat{J}_{l_t}^2 \times \ldots \times J_k^2)} d\underline{\xi} \backslash d\xi_1 d\xi_2 [\gamma^{h(1-\varepsilon)|\mathscr{P} \cap (\ )|[\text{Zeroes}; \mathscr{P} \cap (\ )]] \right.$$

$$\left. \frac{O_{\mathscr{P}'\Delta(\mathscr{P}_1 \cup \ldots \cup N_{s_2})}(W_{(\gamma)}(\underline{\xi}, \underline{\sigma}))}{\gamma^{h(1-\varepsilon)} | \xi_1 - \xi_2 |^{1-\varepsilon}} \prod_{1}^{s_1}{}_{i} F_{\gamma_i}(\mathscr{P}_i) \prod_{1}^{s_2}{}_{J} F_{\overline{\gamma}_J}(N_J) \right\} \qquad (6.34)$$

There are now different possibilities to investigate:

i) The final lines associated to ξ_1, ξ_2 merge into a neutral bifurcation before the lowest one.

ii) The only neutral bifurcation the final lines ξ_1, ξ_2 meet is the lowest one (we are assuming the tree to be neutral).

CASE *i*).

Here there are two different possible situations, either

a) (ξ_1, ξ_2) belong to a neutral subtree γ_i

or

b) (ξ_1, ξ_2) belong to a non neutral subtree $\bar{\gamma}_J$.

We start considering the case *i*), *a*): we rewrite $\{(6.34)\}$ in the following way

$$\{(6.34)\} = \sum_{\substack{\gamma_i \\ \left\{\begin{array}{l} s(\gamma_i)=s_i \\ k(\gamma_i)=h+1 \\ h(\gamma_i)=q_i>h+1 \end{array}\right.}} (\lambda(\mathrm{R}))^{2(k_i-1)}$$

$$\int_{(\mathrm{J}_1^{(i)2}\times\ldots\times\hat{\mathrm{J}}_{l_t}^{(i)2}\times\ldots\times\mathrm{J}_{k_i}^{(i)2})} d\underline{\xi}^{(i)}\backslash d\xi_1 d\xi_2 \left([\gamma^{h(1-\varepsilon)|\mathscr{P}\cap\mathscr{P}_i|}[\mathrm{Zeroes};\mathscr{P}\cap\mathscr{P}_i]]\right.$$

$$\cdot\,[\gamma^{h(1-\varepsilon)|\mathscr{P}_i\backslash\mathscr{P}|}[\mathrm{Zeroes};\mathscr{P}_i\backslash\mathscr{P}]]\mathrm{F}_{\gamma_i}(\underline{\xi}^{(i)},\underline{\sigma}^{(i)};\mathscr{P}_i)$$

$$\cdot\left\{\sum_{\substack{\gamma\backslash\gamma_i \\ \left\{\begin{array}{l} s(\gamma\backslash\gamma_i)=s\backslash s_i \\ k(\gamma)=h \\ k(\gamma)=h+1 \end{array}\right.}} (\lambda(\mathrm{R}))^{2(k-k_i)} \int_{(\mathrm{J}_1^2\times\ldots\times\widehat{(\mathrm{J}_1^{(i)2}\times\ldots\times\mathrm{J}_{k_i}^{(i)2})}\times\ldots\times\mathrm{J}_k^2)} d\underline{\xi}\backslash d\underline{\xi}^{(i)}\right.$$

$$\frac{\mathrm{O}_{\mathscr{P}'\Delta(\mathscr{P}_1\cup\ldots\cup\mathrm{N}_s)}(\mathrm{W}_{(\gamma)}(\underline{\xi},\underline{\sigma}))}{[\gamma^{h(1-\varepsilon)|(\mathscr{P}_1\cup\ldots\cup\mathrm{N}_{s_2})\backslash\mathscr{P}'|}[\mathrm{Zeroes};(\quad)\backslash\mathscr{P}']]\gamma^{h(1-\varepsilon)}|\xi_1-\xi_2|^{1-\varepsilon}}$$

$$\cdot\left[\prod_{\substack{1 \\ s\neq i}}^{s_1}[\gamma^{h(1-\varepsilon)|\mathscr{P}_s|}[\mathrm{Zeroes};\mathscr{P}_s]]\ \mathrm{F}_{\gamma_s}(\underline{\xi}^{(s)},\underline{\sigma}^{(s)};\mathscr{P}_s)\right.$$

$$\left.\left.\cdot\prod_{\substack{1 \\ \mathrm{J}}}^{s_2}[\gamma^{h(1-\varepsilon)|\mathrm{N}_\mathrm{J}|}[\mathrm{Zeroes};\mathrm{N}_\mathrm{J}]]\ \mathrm{F}_{\bar{\gamma}_\mathrm{J}}(\underline{\bar{\xi}}^{(\mathrm{J})},\underline{\zeta}^{(\mathrm{J})},\underline{\bar{\sigma}}^{(\mathrm{J})};\mathrm{N}_s)]\right]\right\} \tag{6.35}$$

where one has to remember that (ξ_1, ξ_2) are associated to an index l_t which does not belong to $\mathscr{P}_i$.

$\{(6.35)\}$ can now be estimated as in Theorem 3 of section 5 obtaining

$$\{(6.35)\} \leqslant \left\{\begin{array}{ll} \lambda^{2(k-k_i)}\gamma^{2(k-k_i)(\mathrm{R}-h)\varepsilon} & \alpha^2 \geqslant 8\pi \\ \lambda^{2(k-k_i)}\gamma^{2(k-k_i)(\mathrm{R}-h)\left(2-\frac{\alpha^2}{4\pi}\right)} & \alpha^2 < 8\pi \end{array}\right. \tag{6.36}$$

and the first part of (6.35) as

$$\sum_{\substack{\gamma_i \\ \{ s(\gamma_i)=s_i \\ h(\gamma_i)=q_i \geq h+1}} \gamma^{-(q_i-h)(1-\varepsilon)|\mathscr{P}_i|} \cdot \lambda(\mathrm{R})^{2(k_i-1)} \int_{(\mathrm{J}_1^{(i)2}\times\ldots\times \mathrm{J}_{k_i}^{(i)2})} d\underline{\xi}^{(i)}\backslash d\xi_1 d\xi_2 \, [\gamma^{q_i(1-\varepsilon)|\mathscr{P}_i|}[\text{Zeroes};\mathscr{P}_i]] \; \mathrm{F}_{\gamma_i}(\mathscr{P}_i)$$

$$\leqslant \sum_{q_i=h+2}^{\mathrm{R}} \gamma^{-(q_i-h)(1-\varepsilon)|\mathscr{P}_i|} \cdot e^{\alpha^2 c^{[<q_i]}_{(\xi_1,\xi_2)}}(e^{\alpha^2 c^{[<q_i]}_{(\xi_1,\xi_2)}}-1) \begin{cases} \lambda^{2(k_1-1)}\gamma^{2(k_i-1)(\mathrm{R}-h)\varepsilon} & \alpha^2 \geqslant 8\pi \\ \gamma^{2(k_i-1)}\gamma^{2(k_1-1)(\mathrm{R}-h)\left(2-\frac{\alpha^2}{4\pi}\right)} & \alpha^2 < 8\pi \end{cases} \tag{6.37}$$

which if $|\mathscr{P}_i|$ is even, globally gives

$$[(6.34)] \leqslant \bar{c}\,(\text{const.})\,\lambda^2(\mathrm{R}) \int_{\mathrm{D}^{(h)}} d\xi_1 d\xi_2 \sin^2\frac{\alpha}{2}\Delta\varphi^{[\leqslant h]}_{1,2} \sum_{q=h+1}^{\mathrm{R}} e^{\alpha^2 c^{[<q]}}(e^{\alpha^2 c^{(q)}}-1)\gamma^{-2(q-h)(1-\varepsilon)} \cdot \begin{cases} \lambda^{2(k-1)}\gamma^{2(k-1)(\mathrm{R}-h)\varepsilon} & \alpha^2 \geqslant 8\pi \\ \lambda^{2(k-1)}\gamma^{2(k-1)(\mathrm{R}-h)\left(2-\frac{\alpha^2}{4\pi}\right)} & \alpha^2 < 8\pi \end{cases} \tag{6.38}$$

which for λ small enough and $\mathrm{R}-h \leqslant \bar{h}$ finite is dominated by the first term of $\mathrm{G}^{(h)}_{2,\mathrm{O}(\lambda^2)}$ (eq. (6.27)).

Remark. — If $|\mathscr{P}_i| = 0$ ($\mathscr{P}_i = \Phi$), remembering the remark after eq. (6.9), it is easy to recognize that in $\mathrm{O}_{\mathscr{P}'\Delta(\mathscr{P}_1\cup\ldots\cup \mathrm{N}_{s_2})}(\mathrm{W}_{00})$ there is a second order zero which can again be estimated by $\gamma^{-2(q-h)(1-\varepsilon)}$ due to the $(\cos\Delta\varphi - 1)$ factor present in each term $:\mathrm{P}_{\Phi_l}:$ of $:\mathrm{P}_{\Phi}(\varphi^{[\leqslant h+1]}):$ producing the same estimates.

CASE *i*) *b*).

In this case we rewrite $\{(6.34)\}$ in the following way

$$\{(6.34)\} = \sum_{\substack{\bar{\gamma}_{\mathrm{J}} \\ \{ s(\bar{\gamma}_{\mathrm{J}})=s_{\mathrm{J}} \\ h(\bar{\gamma}_{\mathrm{J}})=\bar{q}_{\mathrm{J}} \geqslant h+1}} (\lambda(\mathrm{R}))^{n_{\mathrm{J}}-2} \int d\underline{\bar{\xi}}^{(\mathrm{J})} d\underline{\zeta}^{(\mathrm{J})}\backslash d\xi_1 d\xi_2 \, [\gamma^{h(1-\varepsilon)|\mathrm{N}_{\mathrm{J}}|}[\text{Zeroes};\mathrm{N}_{\mathrm{J}}]] \; \mathrm{F}_{\bar{\gamma}_{\mathrm{J}}}(\underline{\bar{\xi}}^{(\mathrm{J})}, \underline{\zeta}^{(\mathrm{J})}, \ldots)$$

$$\cdot \left\{ \sum_{\substack{\gamma\backslash\bar{\gamma}_{\mathrm{J}} \\ \{ s(\bar{\gamma}\backslash\gamma_{\mathrm{J}})=s\backslash s_{\mathrm{J}} \\ k(\gamma)=h \\ h(\gamma)=h+1}} (\lambda(\mathrm{R}))^{2k-n_{\mathrm{J}}} \int d\underline{\xi}\backslash d\underline{\bar{\xi}}^{(\mathrm{J})} d\underline{\zeta}^{(\mathrm{J})} \frac{\mathrm{O}_{\mathscr{P}'\Delta(\mathscr{P}_1\cup\ldots\cup \mathrm{N}_{s_2})}(\mathrm{W}_{(\gamma)}(\underline{\xi},\underline{\sigma}))}{\gamma^{h(1-\varepsilon)|(\quad)\backslash\mathscr{P}'|}[\text{Zeroes};(\quad)\backslash\mathscr{P}']\gamma^{h(1-\varepsilon)}|\xi_1-\xi_2|^{(1-\varepsilon)}} \right.$$

$$\times \prod_{i=1}^{s_1} \gamma^{h(1-\varepsilon)|\mathscr{P}_i|} [\text{Zeroes}; \mathscr{P}_i] \;\; \mathrm{F}_{\gamma_i}(\underline{\xi}^{(i)}, \underline{\sigma}^{(i)}; \mathscr{P}_i)$$

$$\cdot \prod_{\substack{s=1 \\ s \neq \mathrm{J}}}^{s_2} \gamma^{h(1-\varepsilon)|\mathrm{N}_s|} [\text{Zeroes}; \mathrm{N}_s] \;\; \mathrm{F}_{\overline{\gamma}_s}(\underline{\overline{\xi}}^{(s)}, \ldots; \mathrm{N}_s) \Big\} \tag{6.39}$$

The { } factor is again estimated using Theorem 3. The first part can be estimated considering now $\overline{\gamma}_{\mathrm{J}}$ as the whole tree and iterating the previous proof if the lines (1, 2) are in one of the neutral subtrees which merge in the lowest bifurcation of $\overline{\gamma}_{\mathrm{J}}$; otherwise one iterates the procedure as many times as needed to arrive at such a situation.

CASE *ii*).

In this case (ξ_1, ξ_2) merge together only in the lowest bifurcation (this case is trivial) or in a non neutral bifurcation and then the line going out from this bifurcation again merges in to a non neutral one and so on and so forth. Therefore, neutrality is restored only at the lowest bifurcation. This is an easier case; in fact there will be at least one line (of coordinate ξ_3) merging into the same non-neutral bifurcations of frequency q as (ξ_1, ξ_2) which therefore is at a distance $|\xi_3 - \xi_2| \sim \gamma^{-q}$ from (ξ_1, ξ_2). The integration over ξ_3 gives a factor γ^{-2q} but if ξ_3 is associated to ξ_4 in $\Delta\varphi_{3,4}$ only at the lowest bifurcation $h+1$ to get the usual estimates, it will be enough that ξ_3 be at distance $\gamma^{-(h+1)}$ from ξ_4 and that the integration over it gives a factor $\gamma^{-2(h+1)}$; that is, a factor $\sim \gamma^{-2(q-h)}$ has been gained. This argument can be easily generalized to all the possible situations of case *ii*). The non neutral case can be worked out in a similar way both for case *i*) and for case *ii*) and we do not report it here.

This completes the Step *a*) of Theorem 5.

STEP *b*).

Once that all the regions in $\mathscr{D}^{(h)}(\mathscr{P})$ have been removed we are left with the problem of putting back the regions in $\mathscr{D}^{(h)}(\mathscr{P}_1), \ldots, \mathscr{D}^{(h)}(\mathrm{N}_{s_2})$ which are not simultaneously in $\mathscr{D}^{(h)}(\mathscr{P})$. The absence of the regions $\mathscr{D}^{(h)}$ relative to the indices $l \in (\mathscr{P}_1 \cup \ldots \cup \mathrm{N}_{s_2}) \backslash \mathscr{P}'$ is not needed anymore as the zeroes associated to the couples (ξ_{2l-1}, ξ_{2l}) of coordinates do not depend on the properties of the field $\varphi^{[\leq h]}$ anymore but are present in $\mathrm{O}_{\mathscr{P}'\Delta(\mathscr{P}_1 \cup \ldots \cup \mathrm{N}_{s_2})}(\mathrm{W}_{(\gamma)}(\underline{\xi}, \underline{\sigma}))$.

Proceding as in Step *a*), it is clear that for any $l \in (\mathscr{P}_1 \cup \ldots \cup N_{s_2}) \backslash \mathscr{P}'$ we have to control with $G^{(h)}_{2,O(\lambda^2)}$ a term of the following type

$$\bar{c} \sum_{\substack{\gamma \\ s(\gamma)=s \\ k(\gamma)=h \\ h(\gamma)=h+1}} \left(\frac{\lambda(R)}{2}\right)^{2k} \int_{(\mathscr{L}_1^2 \times \ldots \times \mathscr{L}_{l-1}^2 \times D_l^{(h)} \times \mathscr{L}_{l+1}^2 \times \ldots \times \mathscr{L}_k^2)} d\underline{\xi} : P_{\mathscr{P}}(\varphi^{[\leq h]}) : O_{\mathscr{P}'\Delta(\ )}(W_{(\gamma)}(\underline{\xi}, \underline{\sigma})) \prod_1^{s_1}{}_i F_{\gamma_i}(\mathscr{P}_i) \prod_1^{s_2}{}_J F_{\bar{\gamma}_J}(N_J) \quad (6.40)$$

for $Q = 0$, where now the regions $\mathscr{L}_J^2$ are such that the zeroes of $: P_{\mathscr{P}}(\varphi^{[\leq h]}) :$ are all effective. We mimick the proof of Step *a*) and again call ξ_1, ξ_2 the coordinates associated to the index *l*, belonging to the subtree γ_i. Once again

$$[(6.39)] \leq \bar{c}\,(\text{const.})\,(\lambda(R))^2 \int_{D^{(h)}} d\xi_1 d\xi_2 (\gamma^h |\xi_1 - \xi_2|)^{2(1-\varepsilon)} \cdot \left\{ \sum_\gamma \int_{(\mathscr{L}_1^2 \times \ldots \times \mathscr{L}_l^2 \times \ldots \times \mathscr{L}_k^2)} d\underline{\xi} \backslash d\xi_1 d\xi_2 \right.$$
$$\left(\left| \frac{: P_{\mathscr{P}}(\varphi^{[\leq h]}) :}{[\gamma^{h(1-\varepsilon)|\mathscr{P}|}[\text{Zeroes}; \mathscr{P}]]} \frac{O_{\mathscr{P}'\Delta(\ )}(W_{(\gamma)}(\underline{\xi}, \underline{\sigma}))}{[\gamma^{h(1-\varepsilon)|\mathscr{P}_\Delta(\ )|}[\text{Zeroes}; \mathscr{P}\Delta(\ )]]} \right| [\gamma^{h(1-\varepsilon)|\mathscr{P}\backslash(\ )|}[\text{Zeroes}; \mathscr{P}\backslash(\ )]]^2 \right.$$
$$[\gamma^{h(1-\varepsilon)|(\mathscr{P}_1 \cup \ldots \cup \hat{\mathscr{P}}_i \cup \ldots \cup N_{s_2})|}[\text{Zeroes}; (\mathscr{P}_1 \cup \ldots \cup \hat{\mathscr{P}}_i \ldots \cup N_{s_2})]] \left. \prod_1^{s_1}{}_{l \neq i} F_{\gamma_l}(\mathscr{P}_l) \prod_1^{s_2}{}_J F_{\bar{\gamma}_J}(N_J) \right)$$
$$\left. \cdot \left(\frac{[\gamma^{h(1-\varepsilon)|\mathscr{P}_i|}[\text{Zeroes}; \mathscr{P}_i]]}{(\gamma^h |\xi_1 - \xi_2|)^{2(1-\varepsilon)}} \gamma^{-2(q-h)(1-\varepsilon)} F_{\gamma_i}(\mathscr{P}_i) \right) \gamma^{2(q-h)(1-\varepsilon)} \right\}. \quad (6.41)$$

At this point we proceed as before but now the last factor $\gamma^{2(q-h)(1-\varepsilon)}$ cancels the factor $\gamma^{-2(q-h)(1-\varepsilon)}$ of (6.38) and we get

$$[(6.39)] \leq \bar{c}\,(\text{const.})\,(\lambda(R))^2 \int_{D^{(h)}} d\xi_1 d\xi_2 (\gamma^h |\xi_1 - \xi_2|)^{2(1-\varepsilon)} \sum_{h+1}^{R}{}_q e^{\alpha^2 c^{[<q]}} (e^{\alpha^2 c^{(q)}} - 1) \cdot \begin{cases} \lambda^{2(k-1)} \gamma^{2(k-1)(R-h)\varepsilon} & (\alpha^2 \geq 8\pi) \\ \lambda^{2(k-1)} \gamma^{2(k-1)(R-h)\left(2 - \frac{\alpha^2}{4\pi}\right)} & (\alpha^2 < 8\pi) \end{cases} \quad (6.42)$$

which for λ small enough and $R - h \leq \bar{h}$ finite is dominated by the second term of $G^{(h)}_{2,O(\lambda^2)}$ (eq. (6.27)).

The other cases are treated in a similar way and we do not discuss them here. This completes Step *b*) and therefore the proof of Theorem 5.

Remarks. — 1) It is completely trivial to extract from $O_{\mathscr{P}_i}(F_{\gamma_i}(\ \ ))$ a factor $\gamma^{\frac{\alpha^2}{2\pi}q_i}e^{-\gamma^{q_i}|\xi_1-\xi_2|}$; to extract a factor $e^{\alpha^2 c[<q_i]}(e^{\alpha^2 c(q_i)}-1)$ requires a little more care.

2) If γ is a non neutral tree the last factors in (6.38) and (6.31) appear multiplicated by an extra factor $\gamma^{-\frac{\alpha^2}{4\pi}hQ^2}$ according to the estimates (5.40) of Theorem 3.

3) From (6.41) and (6.38) we need $h \geqslant R - \bar{h} \equiv h_0$ of Theorems 4 and 5.

To complete the iterative mechanism for the upper bound, we still need to prove Corollary 1, Theorem 4 and, more important, we have to show that we can perform the iteration also when $h < h_0$.

The proof of Theorem 4 is simpler than that of Theorem 5; the main difference is that now we also sum over the frequency of the last bifurcation of γ: $h(\gamma)$. We do not have to subtract nor add new regions of integration as all the zeros of the fields in $:P_{\mathscr{P}}(\varphi^{[\leqslant h]}):$ are effective in the present ones of $G^{(h)}_{1,O(\lambda>2)}$ (see eq. (6.21)). These indications are enough to easily carry through its proof.

To prove the iteration when $h < h_0$ requires the knowledge of the structure of the remainders $R^{(h)}_\Lambda$ and $\bar{R}^{(h)}_\Lambda$ produced at each step of the iteration. What we prove is that both $G^{(h)}_{1,O(\lambda>2)}$ and $G^{(h)}_{2,O(\lambda>2)}$ have estimates which allow us to add these terms to the remainder without spoiling its properties. Therefore we give some general properties of the remainder which will be proven in sections 7 and 8 and then prove the following Theorem:

THEOREM 7. — For $h < h_0$ and λ small enough we have

$$|G^{(h)}_{O(\lambda>2)}| \leqslant [\text{other terms}] \tag{6.43}$$

6.3. Some properties of the remainder.

As will be made explicit in the next two sections, the remainder $\bar{R}^{(h)}_\Lambda$ will be of two different types:

$$\bar{R}^{(h)}_\Lambda = R^{(h)}_{1,\Lambda} + R^{(h)}_{2,\Lambda} \tag{6.44}$$

where

$$|R^{(h)}_{1,\Lambda}| \leqslant \delta_1(B,\dots)O(\lambda^{2M+\tau})\gamma^{2h}|\Lambda(R)|$$

and

$$|R^{(h)}_{2,\Lambda}| \leqslant \delta_2(B,\dots)O(\lambda^2)\gamma^{2h}|\Lambda(R)\cap\hat{R}^{(h)}| \tag{6.45}$$

If $\delta_1(B,\dots)$ and $\delta_2(B,\dots)$ do not depend on h, in a « bad way » a remainder satisfying (6.45) can easily be controlled. For the $R^{(h)}_{2,\Lambda}$ part we just follow the procedure of [2] (see in particular inequality (5.15) of [2]) finally obtaining a contribution of type

$$\exp\ [(\text{cost})e^{\delta_2(B,\dots)O(\lambda^2)-c_1B^2}]\gamma^{2h}\,|\,\Lambda(R)\,| \tag{6.46}$$

with $c_1 > 0$, which can be bounded by $O(\lambda^{2M+\tau})\gamma^{2h}\,|\,\Lambda(R)\,|$ choosing λ small enough and B large enough. Then, in this case, the remainders at differents levels sum up giving

$$O(\lambda^{2M+\tau})\,(\text{const.})\sum_{h\,0}^{R}\gamma^{2h}\,|\,\Lambda(R)\,| = (\text{const.})O(\lambda^{2M+\tau})\sum_{h\,0}^{R}\gamma^{-2(R-h)}\,|\,I\,|$$

$$\leqslant (\text{const.})\,O(\lambda^{2M+\tau})\,|\,I\,| \tag{6.47}$$

(See inequality (4.3)).

Consider now $G^{(h)}_{2,O(\lambda>2)}$; if $h \geqslant R - \bar{h} \equiv h_0$ we have proven Theorem 5 which tells us that this contribution disappears. Let h be $< h_0$, then looking at the estimates (6.38) and (6.41) where $D^{(h)}$ is in fact $D^{(h)} \cap \Lambda^2(R)$, we can bound the integral increasing the region of integration to $\Lambda^2(R)$ obtaining

$$[(6.34)],\ [(6.39)] \leqslant \bar{c}\,(\text{const.})\,\lambda^{2k}\begin{cases}(\gamma^{2(k-1)(R-h)\varepsilon})\gamma^{-2(R-h)}\,|\,I\,| \\ (\gamma^{2(k-1)(R-h)\left(2-\frac{\alpha^2}{4\pi}\right)})\gamma^{-2(R-h)}\,|\,I\,|\end{cases}$$

$$= \bar{c}\,(\text{const.})\,\lambda^{2k}\begin{cases}\gamma^{2(k-1)(R-h)\varepsilon-(2-\delta_1)(R-h)}\gamma^{-\delta_1(R-h)}\,|\,I\,| & \alpha^2 \geqslant 8\pi \\ \gamma^{2(k-1)(R-h)\left(2-\frac{\alpha^2}{4\pi}\right)-(2-\delta_1)(R-h)}\gamma^{-\delta_1(R-h)}\,|\,I\,| & \alpha^2 < 8\pi\end{cases} \tag{6.48}$$

with $2 > \delta_1 > 0$. The factor $\gamma^{-\delta_1(R-h)}\,|\,I\,|$ makes these contributions summable if

$$[2(M-1)\varepsilon - (2-\delta_1)] \equiv \delta_{2,1} < 0$$

and if

$$\left[2(M-1)\left(2-\frac{\alpha^2}{4\pi}\right) - (2-\delta_1)\right] \equiv \delta_{2,2} < 0 \tag{6.49}$$

Inequalities (6.49) can always be satisfied for a fixed arbitrary M choosing ε sufficiently small as we are allowed to do if $\alpha^2 \geqslant 8\pi$ and if $\alpha^2 \in (\alpha^2_{2,M}, 8\pi)$ since in this interval α^2 satisfies

$$2M\left(2-\frac{\alpha^2}{4\pi}\right) - 2 < 0 \tag{6.50}$$

Thus we can choose $a\bar{\alpha}^2_{2M} > \alpha^2_{2M}$ such that for $\alpha^2 \in (\bar{\alpha}^2_{2M}, 8\pi)$, we have:

$$\left[2(M-1)\left(2-\frac{\alpha^2}{4\pi}\right) - (2-\delta_1)\right] < 0 \tag{6.51}$$

Once (6.49) is satisfied we have

$$[(6.34)],\ [(6.39)] \leqslant (\bar{c}\,(\text{const.})\,\lambda^4\gamma^{-\frac{\delta_1}{2}(R-h_0)})\gamma^{-\frac{\delta_1}{2}(R-h)}\,|\,I\,| \tag{6.52}$$

and to be allowed to put these terms in the remainder without spoiling Theorem 5 for $h \geqslant h_0$, the following inequalities must be simultaneously satisfied:

$$a) \qquad \lambda^4 \gamma^{-\frac{\delta_1}{2}(\mathrm{R}-h_0)} \leqslant \mathrm{O}(\lambda^{2\mathrm{M}+\tau})$$

$$b) \qquad \left\{ \begin{array}{l} \lambda^{2k}\gamma^{2k(\mathrm{R}-h_0)\varepsilon} \\ \lambda^{2k}\gamma^{2k(\mathrm{R}-h_0)\left(2-\frac{\alpha^2}{4\pi}\right)} \end{array} \right. \leqslant \mathrm{O}(\lambda^{\tilde{\varepsilon}}), \quad \tilde{\varepsilon} > 0 \quad (1 \leqslant k < \mathrm{M}) \qquad (6.53)$$

a) and *b)* can be simultaneously satisfied choosing $\mathrm{R} - h_0$ large enough, and $(2 - \alpha^2/4\pi)$ small enough. These relations also impose constraints on $\overline{\alpha}^2_{2\mathrm{M}}$ which are easy to work out.

Once *a)* and *b)* are satisfied Theorem 5 is still valid and for $h < h_0$ [(6.34)], [(6.39)] and therefore $\mathrm{G}^{(h)}_{2,\mathrm{O}(\lambda>2)}$ can be put in to the remainder.

This proves inequality (6.3) for any h (See section 7 for the discussion of what happens when h is much smaller as well as the remark after Lemma 1).

For the term $\mathrm{G}^{(h)}_{1,\mathrm{O}(\lambda>2)}$, if $h < h_0$, it is easy to recognize that

$$\forall s, \{ \mathrm{Q}_v \}_s, \underline{\overline{\sigma}} \sum_{\substack{\gamma \\ s(\gamma)=s \\ k(\gamma)=h \\ v(\gamma)=n}} | \Delta_{1,a}\hat{\mathrm{V}}(\gamma, \{ \mathrm{Q}_v \}_s, \underline{\overline{\sigma}}) |$$

$$\leqslant (\text{const.})\, \lambda^n \left\{ \begin{array}{c} \gamma^{n(\mathrm{R}-h)\varepsilon} \\ \gamma^{n(\mathrm{R}-h)\left(2-\frac{\alpha^2}{4\pi}\right)} \end{array} \right\} \gamma^{-\frac{\alpha^2}{4\pi}\mathrm{Q}^{2h}} \gamma^{-2(\mathrm{R}-h)} | \mathrm{I} | \qquad (6.54)$$

where the factor $\gamma^{-2(\mathrm{R}-h)}$ appears because all zeroes of $:\mathrm{P}_{\mathscr{P}}(\varphi^{[\leqslant h]}):$ are effective. Therefore, for $h < h_0$, the same estimate (6.52) with a different constant is satisfied by $\mathrm{G}^{(h)}_{1,\mathrm{O}(\lambda>2)}$ and Theorem 7 is proven.

We are left with the

Proof of corollary 1. — We observe that in $\Delta_{1,b}\hat{\mathrm{V}}$ the region of integration (called [other region] and defined in eq. (6.19)) is such that all zeroes of $:\mathrm{P}_{\mathscr{P}}:$ are effective and therefore with similar techniques used to prove Theorem 5 it is even easier to show that

$$| \Delta_{1,b}\hat{\mathrm{V}}(\gamma_1 \{ \mathrm{Q}_v \}_{s_1} \underline{\overline{\sigma}}) | \leqslant (\text{const.})\, \lambda^n \left\{ \begin{array}{c} \gamma^{n(\mathrm{R}-h)\varepsilon} \\ \gamma^{n(\mathrm{R}-h)\left(2-\frac{\alpha^2}{4\pi}\right)} \end{array} \right\} \gamma^{2h} | \Lambda(\mathrm{R}) \cap \hat{\mathrm{R}}^{(h)} | \qquad (6.55)$$

then if $h \geqslant h_0$, $\mathrm{R} - h$ is bounded by a constant and therefore it satisfies the estimate (6.44) of $\mathrm{R}^{(h)}_{2,\Lambda}$ and can be incorporated into the remainder. If $h < h_0$ we bound $\gamma^{2h} | \Lambda(\mathrm{R}) \cap \hat{\mathrm{R}}^{(h)} |$ by $\gamma^{-2(\mathrm{R}-h)} | \mathrm{I} |$ and proceed as after estimates (6.48).

This completes the iterative mechanism for the upper bound. In the next section it will be shown that this allows us to reduce the upper bound to a well defined inequality of the same type as inequality (4.12) for the lower bound.

Remark. — In the proof of the last inequality which will be given in the next section it will happen that the $\delta_1(B, \ldots)$ and $\delta_2(B, \ldots)$ have some extra dependance on h due to the factors $\gamma^{n(R-h)\varepsilon}$ or $\gamma^{n(R-h)\left(2-\frac{\alpha^2}{4\pi}\right)}$. Therefore, for these terms as well one has to proceed as discussed in this subsection.

7. REDUCTION TO THE FINAL INEQUALITY FOR THE UPPER BOUND

From inequalities (6.1) and the considerations of subsection 6.3 it follows

$$\begin{aligned}\tilde{Z}_y^{(R)} &= \int P(d\varphi^{[\leq R]})e^{\tilde{V}_\Lambda^{(R)}} = \int P(d\varphi^{[\leq R-1]}) \int P(d\varphi^{(R)})e^{\tilde{V}_\Lambda^{(R)}} \\ &\leq \int P(d\varphi^{[\leq R-1]})e^{\tilde{V}_\Lambda^{(R-1)}} e^{\delta_1(B,\ldots)\gamma^{2R}|\Lambda(R)|} \\ &\leq \int P(d\varphi^{[\leq R-1]})e^{\tilde{V}_\Lambda^{(R-1)}(\mathscr{D}^{(R-1)})} e^{\delta_1(B,\ldots)\gamma^{2R}|\Lambda(R)|} \qquad (7.1)\end{aligned}$$

where we used in particular the first inequality of (6.1) and the inequality

$$\int P(d\varphi^{(R)})e^{\tilde{V}_\Lambda^{(R)}} \leq e^{\tilde{V}_\Lambda^{(R-1)}} e^{\delta_1(B,\ldots)\gamma^{2R}|\Lambda(R)|} \qquad (7.2)$$

Then we apply the second inequality of (6.1) and get

$$\begin{aligned}\tilde{Z}_y^{(R)} &\leq e^{\delta_1(B,\ldots)\gamma^{2R}|\Lambda(R)|} \int P(d\varphi^{[\leq R-1]})e^{\tilde{V}_\Lambda^{(R-1)}(\mathscr{D}^{(R-1)})} \\ &\leq e^{\delta_1(B,\ldots)\gamma^{2R}|\Lambda(R)|} \int P(d\varphi^{[\leq R-1]}) \left\{ \int P(d\varphi^{(R-1)})e^{\tilde{V}_\Lambda^{(R-1)}(\mathscr{D}^{(R-2)},\hat{R}^{(R-1)})} \right. \\ &\left. \quad . e^{\delta_2(B,\ldots)O(\lambda^2)\gamma^{2(R-1)}|\Lambda(R)\cap\hat{R}^{(R-1)}|} \right\} \qquad (7.3)\end{aligned}$$

We now introduce the following events in the space of fields $\varphi^{(h)}$:

Let $\Delta \in Q_h$; define

$$E_\Delta^b = \left\{ \varphi^{(h)} \,\middle|\, \sup_{\xi,\eta\in\Delta} \frac{|\varphi_\xi^{(h)} - \varphi_\eta^{(h)}|}{(\gamma^h|\xi-\eta|)^{1-\varepsilon}} < b(1+\gamma^h d(\Delta, \Lambda)) \right\} \qquad (7.4)$$

and the characteristic functions

$$\begin{cases} \chi_\Delta^b \equiv \chi(E_\Delta^b) \\ \mathring{\chi}_\Delta^b \equiv 1 - \chi(E_\Delta^b) \end{cases} \qquad (7.5)$$

as well as

$$\begin{cases} \overset{\circ}{\chi}{}^{b}_{G_h} \equiv \prod_{\Delta \in G_h} \overset{\circ}{\chi}{}^{b}_{\Delta} \\ \chi^{b}_{G_h^c} \equiv \prod_{\Delta \in G_h^c} \chi^{b}_{\Delta} \end{cases} \tag{7.6}$$

where with G_n we indicate an arbitrary region consisting of tesserae $\in Q_n$ and, at the same time, the set of tesserae of the region.

We introduce the following decomposition of the identity:

$$1 = \Sigma_{G_h} \chi^b_{G_h^c} \overset{\circ}{\chi}{}^b_{G_h} \tag{7.7}$$

and write the r.h.s. of (7.3) for a general $h \leqslant R - 1$ in the following way

$$\begin{aligned} \{(7.3)\} = \Sigma_{G_h} \int P(d\varphi^{(h)}) \overset{\circ}{\chi}{}^b_{G_h} \chi^b_{G_h^c} e^{\tilde{V}_\Lambda^{(h)}(\mathscr{D}^{(h-1)}, \hat{G}^{(h)})} e^{\delta_2(B,\ldots) O(\lambda^2) \gamma^{2h} |\Lambda(R) \cap \hat{G}_h|} \\ \leqslant e^{[\sum_1^{2M} k \frac{1}{k!} \varepsilon^T_{[h]}(\hat{V}_\Lambda^{(h)}(\mathscr{D}^{(h-1)}); k)]_{(2M)}} e^{\delta_1(B,\ldots)(\gamma^{-\frac{\delta_1}{2}(R-h)} |I|) O(\lambda^{2M+\tau})} \\ \cdot \Sigma_{G_h} e^{\delta_2(B,\ldots) O(\lambda^2) \gamma^{2h} |\Lambda(R) \cap \hat{G}_h|} \left(\int P(d\varphi^{(h)}) \overset{\circ}{\chi}{}^b_{G_h} \right)^{1/2} \end{aligned} \tag{7.8}$$

where $\hat{G}_h$ is connected to G_h as $\hat{R}^{(h)}$ to $R^{(h)}$. Writing inequality (7.8) we have assumed the following inequality

$$\begin{aligned} \int P(d\varphi^{(h)}) \overset{\circ}{\chi}{}^b_{G_h} \chi^b_{G_h^c} e^{\hat{V}_\Lambda^{(h)}(\mathscr{D}^{(h-1)}, \hat{G}_h)} \leqslant e^{[\sum_1^{2M} k \frac{1}{k!} \varepsilon^T_{[h]}(\hat{V}_\Lambda^{(h)}(\mathscr{D}^{(h-1)}); k)]_{(2M)}} \\ \cdot e^{\delta_1(B,\ldots)(\gamma^{-\frac{\delta_1}{2}(R-h)} |I|) O(\lambda^{2M+\tau})} e^{\delta_2(B,\ldots) O(\lambda^2) \gamma^{2h} |\Lambda(R) \cap \hat{G}_h|} \cdot \left(\int P(d\varphi^{(h)}) \overset{\circ}{\chi}{}^b_{G_h} \right)^{1/2} \end{aligned} \tag{7.9}$$

Assuming inequality (7.9) of which (7.2) is a particularly simple case, one immediate gets the recursive relation. In fact, from equation (7.8) and inequality (6.3) we get, for $h \geqslant h_0$ also applying inequality (5.15) and Prop. 1 of [2]

$$\begin{aligned} \{(7.3)\} \leqslant e^{\hat{V}_\Lambda^{(h-1)}(\mathscr{D}^{(h-1)}) + \delta_1(B,\ldots)(\gamma^{-\frac{\delta_1}{2}(R-h)} |I|) O(\lambda^{2M+\tau})} \cdot \left(\Sigma_{G_h} e^{\delta_2(B,\ldots) O(\lambda^2) \gamma^{2h} |\Lambda(R) \cap \hat{G}_h|} \right. \\ \left. \cdot \left(\int P(d\varphi^{(h)}) \overset{\circ}{\chi}{}^b_{G_h} \right) \right) \leqslant e^{\hat{V}_\Lambda^{(h-1)}(\mathscr{D}^{(h-1)}) + \delta_1(B,\ldots)(\gamma^{-\frac{\delta_1}{2}(R-h)} |I|) O(\lambda^{2M+\tau})} \end{aligned} \tag{7.10}$$

where the constants incorporated in the terms $\delta_1(B, \ldots)$ and $\delta_2(B, \ldots)$ may change at different steps.

When $h < h_0$, following the discussion of subsection 6.3, some of the terms proportional to $\gamma^{2h} |\Lambda(R) \cap \hat{G}_h|$ are incorporated in the first part of the remainder.

The iterative procedure for the upper bound is therefore completed. The remainder produced at each level h is summable and $O(\lambda^{2M+\tau})$ as claimed. Thus we are only left with the proof of the inequality (4.12) for the lower bound and (7.9) for the upper bound. This is an easy task since they have already been proven with only slight differences in [2] and [3]. We will discuss them in the next sections pointing out some small improvements necessary to apply them to our case.

Remark. — In the computation of the effective potential $\hat{V}^{(h)}$ by the iterative mechanism, the estimates change a little when h is small enough so that a tessera $\Delta \in Q_h$ is larger than $\Lambda(R)$, namely when

$$\gamma^{-2h} \geqslant \gamma^{-2R} \, |I|$$

In fact the estimates are improven as one can easily see in the case $2M = 2$. This also implies that the iterative mechanism can be performed without any problem up to $h = 0$.

8. THE MAIN INEQUALITY

The last inequalities which remain to be proven are:

$$\chi^{B}_{Q_{h-1}}(\varphi^{[\leqslant h-1]}) \int P(d\varphi^{(h)}) \chi^{b}_{Q_h}(\varphi^{(h)}) e^{\tilde{V}^{(h)}} \geqslant \chi^{B}_{Q_{h-1}}(\varphi^{[\leqslant h-1]}) e^{\tilde{V}^{(h-1)}_{\Lambda}} e^{R^{(h-1)}_{\Lambda}} \tag{4.12}$$

and

$$\int P(d\varphi^{(h)}) \mathring{\chi}^{b}_{G_h} \chi^{b}_{G^c_h} e^{\hat{V}_{\Lambda}(\mathscr{D}^{(h-1)}, G_h)} \leqslant e^{\left[\sum_{1}^{2M} {}_k \frac{1}{k!} \mathcal{E}^{T}_{[h]}(\hat{V}^{(h)}_{\Lambda}(\mathscr{D}^{(h-1)}); k)\right]_{(2M)}} \cdot$$

$$\left\{ e^{\delta_1(B,\ldots)(\gamma^{-\frac{\delta_1}{2}(R-h)}|I|) O(\lambda^{2M+\tau})} \cdot e^{\delta_2(B,\ldots) O(\lambda^2) \gamma^{2h} |\Lambda(R) \cap \hat{G}_h|} \cdot \left(\int P(d\varphi^{(h)}) \mathring{\chi}^{b}_{G_h} \right)^{1/2} \right\} \tag{7.9}$$

$\forall 2M$, $\forall h \leqslant R$ if $\alpha^2 \geqslant 8\pi$ and $\forall 2M \leqslant 2n$ if $\overline{\alpha}^2_{2n} < \alpha^2 < 8\pi$.

Remark. — Some constraints which imply $\overline{\alpha}^2_{2n} > \alpha^2_{2n}$ have already been discussed in subsection 6.3; others will appear at the end of this section.

We prove the following Lemma

LEMMA 4. — $\forall h \leqslant R$, $\forall 2M$ if $\alpha^2 \geqslant 8\pi$ and $\forall 2M \leqslant 2n$ if $\overline{\alpha}^2_{2n} < \alpha^2 < 8\pi$, for λ small enough and $b = \sigma B > \sigma B_0$ satisfying the inequality (4.7), provided B_0 is sufficiently large, the inequalities (4.12) and (7.9) are true.

Remark. — The proof of this Lemma follows as close as possible the proof of Lemma 1 in [2] and of Lemma 1 in [3]. Here we repeat only the general lines of the proof stressing some points.

Proof. — It consists of two steps: first, one reduces the inequalities (4.12) and (7.9) using « the scale invariance » of the fields (see eq. (3.5)), to two inequalities which do not depend on h anymore. Secondly one proves two modified inequalities from which (4.12) and (7.9) follow.

STEP one.

We observe that in both inequalities (4.12) and (7.9) the integration is made respect to $\varphi^{(h)}$ and the extra dependance on $\varphi^{[\leqslant h-1]}$ is controlled in (4.12) by the characteristic function $\chi^{B}_{Q_{h-1}}(\varphi^{[\leqslant h-1]})$ and in (7.9) by the fact that in $\hat{V}_\Lambda(\mathscr{D}^{(h-1)}, \hat{G}_h)$ the fields $\varphi^{[\leqslant h-1]}$ are « smooth ». Therefore, the interactions $\tilde{V}^{(h)}_\Lambda$ and $\hat{V}_\Lambda(\mathscr{D}^{(h-1)}, \hat{G}_h)$ can be thought of as functions of $\varphi^{(h)}$ only and rewritten again as sum of terms like (5.17), (5.19) or (6.7) (see eqs. (5.5) and (6.6)) where now $:P_{\mathscr{P}}(\varphi^{[\leqslant h]}):$ and $:P_N(\varphi^{[\leqslant h]}):$ are substituted by $:P_{\mathscr{P}}(\varphi^{(h)}):$ or $:P_N(\varphi^{(h)}):$ and the remaining $\varphi^{[\leqslant h-1]}$-dependance is put in the new coefficients:

$$\tilde{V}(\gamma, \{ Q_v \}_s, \underline{\sigma}) = \Sigma_{\mathscr{P}} \int_{\Lambda^{2k}} d\underline{\xi} : P_{\mathscr{P}}(\varphi^{(h)}) : H_{\mathscr{P}}(\underline{\xi}, \varphi^{[\leqslant h-1]}; \gamma, \underline{\sigma}, Q=0)$$
$$+ \text{(terms not depending on } \varphi^{(h)})$$
$$\tilde{V}(\bar{\gamma}, \{ Q_v \}_s, \underline{\bar{\sigma}}) = \Sigma_N \int_{\Lambda^{2\bar{k}+\bar{p}}} d\underline{\bar{\xi}} d\underline{\zeta} : P_N(\varphi^{(h)}) e^{i\alpha\varphi^{(h)}_{\mathscr{Y}}(\bar{\gamma})} :$$
$$H_N(\underline{\bar{\xi}}, \underline{\zeta}, (\varphi^{[\leqslant h-1]}; \bar{\gamma}, \underline{\bar{\sigma}}, Q \neq 0) + \text{(terms not depending on } \varphi) \quad (8.1)$$

and similar expressions for $\hat{V}(\gamma, \{ Q_v \}_s, \underline{\sigma})$ and $\hat{V}(\bar{\gamma}, \{ Q_v \}_s, \underline{\bar{\sigma}})$. One immediately realizes that the coefficients $H_{\mathscr{P}}$ and H_N of (8.1) satisfy inequalities similar to those of Theorem 3 of section 5 with $H_{\mathscr{P}}$ and H_N substituting $O_{\mathscr{P}}(F_\gamma(\ \ ))$ and $O_N(F_{\bar{\gamma}}(\ \ ))$ and the right hand side of the inequalities multiplied by $B^{v(\gamma)/2}$.

Therefore, $\forall \underline{\sigma}, \{ Q_v \}_s, s$

$$\sum_{\substack{\gamma \\ s(\gamma)=s \\ k(\gamma)=h \\ v(\gamma)=n}} \int_{\Delta_1 \times \ldots \times \Delta_n} d\underline{\xi}\, [\text{Zeroes}; \mathscr{P}(N)] \,|\, H_{\mathscr{P}(N)}(\underline{\xi}, \varphi^{[\leqslant h-1]}; \gamma, \underline{\sigma}, Q = 0) \,|$$
$$\leqslant (\text{const.})\, \lambda^n \begin{Bmatrix} \gamma^{n(R-h)\varepsilon} & (\alpha^2 \geqslant 8\pi) \\ \gamma^{n(R-h)\left(2-\frac{\alpha^2}{4\pi}\right)} & (\alpha^2 < 8\pi) \end{Bmatrix} \gamma^{-\frac{\alpha^2}{4\pi} Q^{2h}} B^{\frac{v(\gamma)}{2}} e^{-\gamma^h d(\Delta_1 \ldots \Delta_n)} \quad (8.2)$$

which is the generalization of eqs. (3.28), (3.29) of (2).

Therefore we can write

$$\tilde{V}^{(h)}_\Lambda = \sum_{k=1}^{2M} \sum_{(s,\{Q_v\}_s,\underline{\sigma})} \sum_{\substack{\gamma \\ \begin{cases} s(\gamma)=s \\ k(\gamma)=h \end{cases}}} \tilde{V}(\gamma, \{ Q_v \}_s, \underline{\sigma}) \text{- [counterterms]} \quad (8.3)$$

and $\tilde{V}(\gamma, \{ Q_v \}_s, \underline{\sigma})$ is given by eq. (8.1). Similar expressions can be written for $\hat{V}_\Lambda^{(h)}(\mathscr{D}^{(h-1)}, \hat{G}_h)$ with the obvious modifications in the regions of integration (see section 6).

Remark. — The factor $B^{\frac{v(\gamma)}{2}}$ originates from the regularity of the fields present in the coefficients.

STEP two.

The second step of the proof consists in transforming all the inequalities (4.12) and (7.9) into two inequalities on a fixed scale ($h = 0$). This is achieved by defining a field Z in the following way

$$Z(\gamma^h \xi) \equiv \varphi^{(0)}_{\gamma^h \xi} = \varphi^{(h)}_{\xi} \tag{8.4}$$

$$x \equiv \gamma^h \xi \tag{8.5}$$

Z(x) is a gaussian random field with covariance

$$\langle Z(x)Z(y) \rangle = C(x, y) = \frac{1}{(2\pi)^2} \int d^2p e^{ip(x-y)} \left(\frac{1}{p^2+1} - \frac{1}{p^2+\gamma^2} \right) \tag{8.6}$$

We express $\tilde{V}_\Lambda^{(h)}$ and $\hat{V}_\Lambda^{(h)}$ in terms of Z(x) and we recall that $\Lambda(R) \to \tilde{\Lambda}$ with

$$|\tilde{\Lambda}| = \gamma^{-2(R-h)} |I| \tag{8.7}$$

and

$$\chi^b_{\tilde{\Delta}} = \left\{ Z(.) \,\middle|\, \sup_{\substack{x, y \\ x, y \in \tilde{\Delta}}} \frac{|Z(x) - Z(y)|}{|x-y|^{1-\varepsilon}} \leqslant b(1 + d(\tilde{\Delta}, \tilde{\Lambda})) \right\} \equiv \chi^b_{\tilde{\Delta}}(Z) \tag{8.8}$$

where $|\tilde{\Delta}| = 1$. Similarly,

$$\mathring{\chi}^b_{G_h}(\varphi^{(h)}) = \mathring{\chi}^b_{G}(Z), \qquad \chi^b_{G_h^c}(\varphi^{(h)}) = \chi^b_{G^c}(Z) \tag{8.9}$$

where G is the same set as in G_h, now made up of tesserae of side size 1.

We define a function $H_J(Z)$ in the following way:

$$H_J(Z) = \sum_{1}^{2M}{}_{k} \sum_{(s,\{Q_v\}_s,\underline{\sigma})} \tilde{H}(s, \underline{\sigma}, Q; Z) \tag{8.10}$$

where

$$\begin{aligned} &\tilde{H}(s, \underline{\sigma}, Q=0; Z) = \Sigma_{\mathscr{P}} \int_{J^{2k}} d\underline{x} : P_{\mathscr{P}}(Z) : H_{\mathscr{P}}(\underline{x}; s, \underline{\sigma}, Q=0) \\ &\tilde{H}(s, \bar{\underline{\sigma}}, Q \neq 0; Z) = \Sigma_N \int_{J^{2\bar{k}+\bar{p}}} d\underline{x} : P_N(Z) e^{i\alpha Z(\mathscr{L}(\bar{s}))} : H_N(\underline{x}; s, \underline{\sigma}, Q \neq 0) \end{aligned} \tag{8.11}$$

the coefficients satisfying the following bounds

$$\int_{\tilde\Delta_1 \times \ldots \times \tilde\Delta_k} dx_1 \ldots dx_k\,[\text{Zeroes}; \mathscr{P}(\mathrm{N})]\,|\,\mathrm{H}_{\mathscr{P}(\mathrm{N})}(x; s, \bar\sigma, \underline{\mathrm{Q}})\,| \leqslant \mathrm{G}_{(k)}\lambda^k e^{-d(\tilde\Delta_1,\ldots,\tilde\Delta_k)} \tag{8.12}$$

Remark. — If $\mathrm{H}_{\mathrm{J}}(\mathrm{Z})$ is $\tilde{\mathrm{V}}_{\Lambda}^{(h)}$ or $\hat{\mathrm{V}}_{\Lambda}^{(h)}$ then the constants $\mathrm{G}_{(k)}$ of inequality (8.12) are given by

$$\mathrm{G}_{(k)} = (\text{const.})\gamma^{-\frac{\alpha^2}{4\pi}h\mathrm{Q}^2} \left\{ \begin{matrix} \gamma^{k(\mathrm{R}-h)\varepsilon} & (\alpha^2 \geqslant 8\pi) \\ \gamma^{k(\mathrm{R}-h)\left(2-\frac{\alpha^2}{4\pi}\right)} & (\alpha^2 < 8\pi) \end{matrix} \right\} \mathrm{B}^k \tag{8.13}$$

where $\mathrm{J} = \tilde\Lambda$ or an analogous region.

Inequalities (4.12) and (7.9) are proven once we prove the following inequalities

$$\int \mathrm{P}(d\mathrm{Z})\chi^{\circ b}_{\Phi}\chi^{b}_{\Phi^c = \mathrm{Q}_0} e^{\mathrm{H}_\mathrm{J}(\mathrm{Z})} \geqslant e^{-\delta(\mathrm{G}_{(.)},\lambda)|\mathrm{J}|} e^{\left[\sum_1^{2\mathrm{M}} \lambda^k \frac{1}{k!}\varepsilon^{\mathrm{T}}(\mathrm{H}_\mathrm{J};k)\right]_{(2\mathrm{M})}}$$

$$\int \mathrm{P}(d\mathrm{Z})\chi^{\circ b}_{\mathrm{G}}\chi^{b}_{\mathrm{G}^c} e^{\mathrm{H}_{\mathrm{J}\setminus\mathrm{G}}(\mathrm{Z})} \leqslant \left(e^{\delta(\mathrm{G}_{(.)},\lambda)|\mathrm{J}| + \delta'(\mathrm{G}_{(.)},\lambda)|\mathrm{J}\cap\hat{\mathrm{G}}|}\right). e^{\left[\sum_1^{2\mathrm{M}} \lambda^k \frac{1}{k!}\varepsilon^{\mathrm{T}}(\mathrm{H}_\mathrm{J};k)\right]_{(2\mathrm{M})}} \tag{8.14}$$

and the structure of δ and δ' is such that when one substitutes for $\mathrm{G}_{(.)}$ the expression (8.13) and for $|\,\mathrm{J}\,|\,|\,\tilde\Lambda\,| = \gamma^{-2[\mathrm{R}-h)}\,|\,\mathrm{I}\,|$:

$$\begin{aligned} \delta(\mathrm{G}_{(.)},\lambda)\,|\,\mathrm{J}\,| &\to \delta_1(\mathrm{B};\alpha^2,\mathrm{M},\lambda,h)\,|\,\tilde\Lambda\,| \leqslant \bar\delta_1(\mathrm{B})\mathrm{O}(\lambda^{2\mathrm{M}+\tau})\gamma^{-\frac{\delta_1}{2}(\mathrm{R}-h)}\,|\,\mathrm{I}\,| \\ \delta'(\mathrm{G}_{(.)},\lambda)\,|\,\mathrm{J}\cap\hat{\mathrm{G}}\,| &\to \delta_2(\mathrm{B};\alpha^2,\mathrm{M},\lambda,h)\,|\,\tilde\Lambda\cap\hat{\mathrm{G}}\,| \leqslant (\bar\delta_2(\mathrm{B})\mathrm{O}(\lambda^2)) \\ &\qquad .\gamma^{2h}\,|\,\Lambda(\mathrm{R})\cap\hat{\mathrm{R}}^{(h)}\,| \end{aligned} \tag{8.15}$$

where $\bar\delta_1(\mathrm{B}) = c_1\mathrm{B}^{\rho_1}$, $\bar\delta_2(\mathrm{B}) = c_2\mathrm{B}^{\rho_2}$ with appropriate $\rho_1, \rho_2 > 0$ and with two constants $c_1, c_2 > 0$ independent of the parameters.

Inequalities (8.14) are exactly the inequalities (3.32) and (3.34) proven in [*2*]. However the proof has to be slightly improved to get an estimate of $\delta(\mathrm{G}_{(.)},\lambda)$, better than (3.33) [*2*], which satisfies (8.15). In fact the estimates (3.33) in [*2*] written in our present notations, are:

$$\delta_1(\mathrm{B};\alpha^2,\mathrm{M},\lambda,h) = (\text{const.})\;[(\bar{\mathrm{H}}_h\mathrm{B}^\rho e^{\rho\bar{\mathrm{H}}_h\mathrm{B}^\rho})^{2\mathrm{M}+1} + e^{-\rho'\mathrm{B}^2+\rho\bar{\mathrm{H}}_h\mathrm{B}^\rho}] \tag{8.16}$$

where $$\bar{\mathrm{H}}_h = \left\{ \begin{matrix} \gamma^{2\mathrm{M}(\mathrm{R}-h)\varepsilon} & (\alpha^2 \geqslant 8\pi) \\ \gamma^{2\mathrm{M}(\mathrm{R}-h)\left(2-\frac{\alpha^2}{4\pi}\right)} & (\alpha^2 < 8\pi) \end{matrix} \right. ;\quad \rho, \rho' > 0$$

The factors $e^{\rho\bar{\mathrm{H}}_h\mathrm{B}^\rho}$ due to their dependance in $(\mathrm{R} - h)$ would ruin the necessary summability of the remainder.

Nevertheless, going through the proof as discussed in detail in (3) and using the same ideas as in subsection 6.3, it is easy to get a better estimate for $\delta(\mathrm{G}_{(.)},\lambda)$ satisfying inequalities (8.15).

This will lead to a third constraint on $\overline{\alpha}_n^2$ and will be discussed in Appendix C.

Remark. — We do not put any particular emphasis on the exact value of the coefficients $\overline{\alpha}_n^2$ because we have not looked for their optimal values; from a « physical point of view » we would think that the true values are those of the coefficients α_n^2. Nevertheless it can easily be checked collecting all the constraints that a possible choice for $\overline{\alpha}_n^2$ is

$$\overline{\alpha}_n^2 = 8\pi\left(1 - \frac{1}{4n}\right) \tag{8.17}$$

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APPENDIX A

Proof of Theorem 3. — The proof is a straightforward extension of Lemma 2 and of Theorem 1 of [1] and is again an inductive one.

We define

$$C^{[\leq h_0]}_{\gamma,\gamma} = \sum_{i,J\in\gamma} \bar{\sigma}_i\bar{\sigma}_J C^{[\leq h_0]}(\xi_i, \xi_J) \tag{1.a}$$

where i and J run over the indices of all the final lines of the tree γ and

$$C^{[\leq h_0]}_{\gamma,\gamma}(0) = \sum_{i,J} \bar{\sigma}_i\bar{\sigma}_J C^{[\leq h_0]}_{(0,0)} = Q^2_\gamma(h_0+1)C^{(0)}_{(0,0)} \tag{2.a}$$

We rewrite eq. (5.29) in the following way:

$$W_{(\gamma)}(\underline{\xi}, \underline{\tau}) = \left(e^{U^{[\leq h]}(\gamma_1,\ldots,\bar{\gamma}_{s_2};\tau^{(1)},\ldots,\tau^{(s_2)}) - \frac{\alpha^2}{2}\left(\sum_i^{s_1} c^{[\leq h]}_{\gamma_i,\gamma_i}(0) + \sum_J^{s_2} c^{[\leq h]}_{\bar{\gamma}_J,\bar{\gamma}_J}(0)\right)}\right) . e^{\frac{\alpha^2}{2}\left(\sum_i^{s_1} c^{[\leq h]}_{\gamma_i,\gamma_i}(0) + \sum_J^{s_2} c^{[\leq h]}_{\gamma_J,\gamma_J}(0)\right)} . \mathcal{E}^{T}_{[h+1]}(\ ) \tag{3.a}$$

then we define

$$U_0^{[\leq h]}(\gamma_1,\ldots,\bar{\gamma}_{s_2}) \equiv U^{[\leq h]}_{(\gamma_1,\ldots,\bar{\gamma}_{s_2};\tau^{(1)},\ldots,\bar{\tau}^{(s_2)})} \quad \text{(all coordinates equal)} \tag{4.a}$$

i. e.

$$U_0^{[\leq h]}(\gamma_1,\ldots,\bar{\gamma}_{s_2}) = -\frac{\alpha^2}{2}\sum_{\substack{l,s\\ l\neq s}} c^{[\leq h]}_{\gamma_l,\gamma_s}(0) \tag{5.a}$$

where l, s both run over $(1, \ldots, s_2)$ and γ_l is a $\bar{\gamma}_J$. Therefore defining

$$\delta U^{[\leq h]} \equiv U^{[\leq h]}(\gamma_1,\ldots,\bar{\gamma}_{s_2};\underline{\tau}^{(1)},\ldots,\underline{\tau}^{(s_2)}) - U_0^{[\leq h]}(\gamma_1,\ldots,\bar{\gamma}_{s_2}) \tag{6.a}$$

$$W_{(\gamma)}(\underline{\xi}, \underline{\tau}) = \left(e^{\gamma U^{[\leq h]}(\gamma_1,\ldots,\bar{\gamma}_{s_2};\underline{\tau}^{(1)},\ldots,\bar{\tau}^{(s_2)})}e^{-\frac{\alpha^2}{2}c^{[\leq h]}_{\gamma,\gamma}(0)}\right) . \prod_i^{s_1} e^{\frac{\alpha^2}{2}c^{[\leq h]}_{\gamma_i,\gamma_i}(0)} \prod_J^{s_2} e^{\frac{\alpha^2}{2}c^{[\leq h]}_{\bar{\gamma}_J,\bar{\gamma}_J}(0)} \mathcal{E}^{T}_{[h+1]}(\ )$$

$$= e^{-\frac{\alpha^2}{2}c^{[\leq h]}_{\gamma,\gamma}(0)} \delta W_{(\gamma)}(\underline{\xi}, \underline{\tau}) \prod_i^{s_1} e^{\frac{\alpha^2}{2}c^{[\leq h]}_{\gamma_i,\gamma_i}(0)} \prod_J^{s_2} e^{\frac{\alpha^2}{2}c^{[\leq h]}_{\bar{\gamma}_J,\bar{\gamma}_J}(0)} \tag{7.a}$$

we have

$$e^{\frac{\alpha^2}{2}c^{[\leq h]}_{\gamma,\gamma}(0)} F_\gamma(\underline{\xi}, \underline{\bar{\sigma}}; \mathcal{P}) = \sum_{\substack{\mathcal{P}_1\ldots\mathcal{P}_{s_1}\\ |\mathcal{P}_i|\,\text{even}}} \sum_{N_1\ldots N_{s_2}} O_{\mathcal{P}'\Delta(\mathcal{P}_1\cup\ldots\cup N_{s_2})}(\delta W_{(\gamma)}(\underline{\xi}, \underline{\bar{\sigma}}))$$

$$. \prod_i^{s_1} e^{\frac{\alpha^2}{2}c^{[\leq h]}_{\gamma_i,\gamma_i}(0)} F_{\gamma_i}(\underline{\xi}^{(i)}, \underline{\bar{\sigma}}^{(i)}; \mathcal{P}_i) \prod_J^{s_2} e^{\frac{\alpha^2}{2}c^{[\leq h]}_{\bar{\gamma}_J,\bar{\gamma}_J}(0)} F_{\bar{\gamma}_J}(\underline{\bar{\xi}}^{(J)}, \underline{\zeta}^{(J)}, \underline{\bar{\bar{\sigma}}}^{(J)}; N_J)$$

$$. \left(e^{-\frac{\alpha^2}{2}(c^{[\leq h]}_{\gamma,\gamma}(0) - c^{[\leq h_0]}_{\gamma,\gamma}(0))}\right) \tag{8.a}$$

We now prove the following Lemma

LEMMA 1*a*. — The following estimates hold for a generic tree γ ($h(\gamma) = h$)

$$(\gamma^{h|\mathscr{F}|})^{1-\varepsilon}[\text{Zeroes};\mathscr{F}] \quad F_\gamma(\underline{\xi},\underline{\sigma};\mathscr{F}) \leqslant \text{const.}) e^{-\frac{\alpha^2}{2}c^{[\leqslant h]}_{\gamma,\gamma}(0)}$$

$$\cdot \prod_{v>v_0} e^{-\frac{\alpha^2}{2}(c^{[\leqslant h_v]}_{\gamma_v,\gamma_v}(0) - c^{[\leqslant h_{v'}]}_{\gamma_v,\gamma_v}(0))} \prod_{1}^{n}{}_i\, e^{\frac{\alpha^2}{2}c^{[\leqslant h_{v_i}]}(0)} \prod_{v\geqslant v_0} \gamma^{-2h_v(s_v-1)} \prod_{v>v_0,} \gamma^{-2(1-\varepsilon)(h_v-h_{v'})\delta Q_v,0}$$

$$\left(\prod_{v\geqslant v_0} \frac{[\text{exp. decay factor at } h_v]}{\gamma^{-2h_v(s_v-1)}}\right) \tag{9.a}$$

where $\mathscr{F}$ stands for both $\mathscr{P}$ and N. Following the notations of [1], we have

i) γ_v is the subtree whose lowest bifurcation is v.

ii) $\prod_{v\geqslant v_0} \gamma^{-2h_v(s_v-1)}$ are the volume factors due to the exponentially decay factors present at each bifurcation, s_v is the number of lines entering into v (from right to left).

iii) $\prod_{v>v_0} \gamma^{-2(1-\varepsilon)(h_v-h_{v'})\delta Q_v,0}$ are the « zeroes » (of second order) discussed in Lemma 2 of the Section 5.

iv) The exponential decay factor at the generic bifurcation v is a factor $\exp - k\gamma^{h_v}d^*(x_v)$ where $d^*(x_v)$ is the length of the shortest path connecting the clusters (bifurcations) $v^{(1)}, \ldots, v^{(s_v)}$ which come immediately before v (from right to left); $k > 0$.

v) v_i is the vertex where the i-th final line first merges.

Proof. — The proof follows immediately from (8.*a*) observing that Lemma 2 is still valid with $\delta W_{(\gamma)}(\underline{\xi}, \underline{\tau})$ instead of $W_{(\gamma)}(\underline{\xi}, \underline{\tau})$ and that in $\delta W_{(\gamma)}(\underline{\xi}, \underline{\sigma})$ (in $G(\underline{\xi}, \underline{\sigma})$ of eq. (5.37)) there is an exponentially decaying factor

$$\exp = k\gamma^{h(\gamma)}d^*(x_{h(\gamma)})$$

with $k > 0$.
Now observing that

$$c^{[\leqslant h]}_{\gamma,\gamma}(0) = Q^2_\gamma c^{[\leqslant h]}_{(0,0)} = Q^2_\gamma(h+1)c^{(0)}_{(0,0)} = Q^2_\gamma \frac{(h+1)}{2\pi}\log\gamma \tag{10.a}$$

and

$$\sum_{v\geqslant v_0}(s_v - 1) = n - 1 = n_{v_0} - 1 .$$

v_0 being the lowest bifurcation and n_v the number of final lines which eventually merge in the vertex v, it is simple to realize that, choosing $n_{v_0} = 2k$, $Q_\gamma = Q_{v_0} = 0$

$$\left(\frac{\lambda(R)}{2}\right)^{2k} \sum_{\substack{\gamma \\ s(\gamma)=s \\ k(\gamma)=h_0 \\ v(\gamma)=2k}} (\gamma^{2(h_0-h)})^{1-\varepsilon} \int_{\Delta_1\times\ldots\times\Delta_{2k}} d\xi_1\ldots d\xi_{2k}\gamma^{-2(h_0-h)(1-\varepsilon)}[\text{Zeroes};\mathscr{P}]|\,O_{\mathscr{P}}(F_\gamma(\underline{\xi},\underline{\sigma};Q=0)|$$

$$\leqslant (\text{const.})\left(\frac{\lambda(R)}{2}\right)^{2k} \sum_{\{h_v\}} (\gamma^{2(h_0-h)})^{1-\varepsilon}\left[\prod_{v>v_0} \gamma^{-\left[\left(2-\frac{\alpha^2}{4\pi}\right)(n_v-1)-\frac{\alpha^2}{4\pi}+\frac{\alpha^2}{4\pi}Q_v^2+2(1-\varepsilon)\delta Q_v,0\right](h_v-h_{v'})}\right]$$

$$\cdot\,\gamma^{-\left[\left(2-\frac{\alpha^2}{4\pi}\right)(n_{v_0}-1)-\frac{\alpha^2}{4\pi}+\frac{\alpha^2}{4\pi}Q^2_{v_0}\right]h_{v_0}}\left(\int_{\Delta_1\times\ldots\times\Delta_{2k}} d\xi_1\ldots d\xi_{2k}\prod_{v\geqslant v_0}\frac{e^{-k\gamma^{h_v}d^*(x_v)}}{\gamma^{-2h_v(s_v-1)}}\right) \tag{11.a}$$

and, as discussed in [1],

$$
\begin{aligned}
((11.a)) &\leqslant (\text{const.})\,|\Delta|\, e^{-k\gamma^{h_0} d(\Delta_1,\dots,\Delta_{2k})} \\
&\leqslant (\text{const.})\,\gamma^{-2h_0} e^{-k\gamma^{h_0} d(\Delta_1,\dots,\Delta_{2k})}
\end{aligned}
\tag{12.a}
$$

where $\Delta_i \in Q_{h_0}$.

Thus.

$$
\begin{aligned}
[\text{l. h. s. } (5.39)] \leqslant c_{2k}\left(\frac{\lambda(R)}{2}\right)^{2k} \sum_{\{h_v\}} (\gamma^{2(h_0-h)})^{1-\varepsilon}\,|\Delta| \\
\left[\prod_{v>v_0} \gamma^{-\left[\left(2-\frac{\alpha^2}{4\pi}\right)(n_v-1)-\frac{\alpha^2}{4\pi}+\frac{\alpha^2}{4\pi}Q_v^2+2(1-\varepsilon)\delta Q_v,0\right](h_v-h_{v'})}\right. \\
\left.\cdot\ \gamma^{-\left[\left(2-\frac{\alpha^2}{4\pi}\right)(n_{v_0}-1)-\frac{\alpha^2}{4\pi}+\frac{\alpha^2}{4\pi}Q_{v_0}^2\right]h}\right] e^{-k\gamma^{h_0} d(\Delta_1,\dots,\Delta_{2k})}
\end{aligned}
\tag{13.a}
$$

where c_{2k} is a constant depending only on the number of final lines and $\Sigma_{\{h_v\}}$ is the sum over all the possible frequencies at the bifurcations of γ.

$$
\begin{aligned}
[\text{l. h. s. } (5.39)] \leqslant \bar{c}_{2k}(\lambda(R))^{2k}\gamma^{-2h_0\varepsilon}. \sum_{\{h_v\}}\left[\prod_{v>v_0} \gamma^{\left[\left(\frac{\alpha^2}{4\pi}-2\right)(n_v-1)+\frac{\alpha^2}{4\pi}-2(1-\varepsilon)\delta Q_v,0-\frac{\alpha^2}{4\pi}Q_v^2\right](h_v-h_{v'})}\right. \\
\left.\gamma^{\left[\left(\frac{\alpha^2}{4\pi}-2\right)n_{v_0}+2\varepsilon-\frac{\alpha^2}{4\pi}Q_{v_0}^2\right]h}\right] e^{-k\gamma^{h_0} d(\Delta_1,\dots,\Delta_{2k})}.
\end{aligned}
\tag{14.a}
$$

Assuming $\alpha^2 \geqslant 8\pi$, $Q_{v_0} = 0$ then $2(1-\varepsilon)\delta_{Q_v,0} + \dfrac{\alpha^2}{4\pi}Q_v^2 \geqslant 2(1-\varepsilon)$ and [1]

$$
\begin{aligned}
[\text{l. h. s. } (5.39)] &\leqslant (\text{const.})(\lambda(R))^{2k}\gamma^{-2h_0\varepsilon} \sum_{\{h_v\}} \prod_{v_i}\left[\left(\gamma^{\left(\frac{\alpha^2}{4\pi}-2\right)n_{v_i}h_{v_i}+2\varepsilon h_{v_i}}\right)\right. \\
&\quad \cdot \prod_{v'>v_i}\left[\gamma^{-2\varepsilon(\#(v_i<v')-1)h_{v'}}\left[\prod_{v''>v'}\gamma^{-2\varepsilon(\#(v'<v'')-1)h_{v''}}\dots\gamma^{-2\varepsilon(\#(\tilde{v}_0<v_0)-1)h}\right]\dots\right]\Bigg] \\
&\quad \cdot\ e^{-k\gamma^{h_0}d(\Delta_1,\dots,\Delta_{2k})} \leqslant (\text{const.})\,(\lambda(R))^{2k}\gamma^{-2h_0\varepsilon}\Big(\gamma^{\left(\frac{\alpha^2}{4\pi}-2\right)n_{v_0}R}\gamma^{2\varepsilon\#(v_i)R}\gamma^{-2\varepsilon\#(v_i)h_0+2\varepsilon\#(v')h_0} \\
&\quad \gamma^{-2\varepsilon\#(v')h_0-2\varepsilon\#(v'')h_0}\dots\gamma^{2\varepsilon h_0}\Big) \\
e^{-k\gamma^{h_0}d(\Delta_1,\dots,\Delta_{2k})} &\leqslant (\text{const.})\,(\lambda(R))^{2k}\gamma^{\left(\frac{\alpha^2}{4\pi}-2\right)2kR}\gamma^{2k\varepsilon(R-h_0)}e^{-k\gamma^{h_0}d(\Delta_1,\dots,\Delta_{2k})} \\
&\leqslant (\text{const.})\,\lambda^{2k}\gamma^{2k(R-h_0)\varepsilon}e^{-k\gamma^{h_0}d(\Delta_1,\dots,\Delta_{2k})}
\end{aligned}
\tag{15.a}
$$

and $^{\#}(v)$ is the number of bifurcations of a fixed order.

If $\alpha^2 < 8\pi$, $\dfrac{\alpha^2}{4\pi} - 2(1-\varepsilon)\delta_{Q_v,0} - \dfrac{\alpha^2}{4\pi}Q_v^2 \leqslant 0$, since ε is arbitrary; $Q_{v_0} = 0$

$$
\begin{aligned}
[\text{l. h. s. } (5.39)] &\leqslant (\text{const.})\,(\lambda(R))^{2k}\gamma^{-2h_0\varepsilon}\sum_{\{h_v\}}\left[\prod_{v>v_0}\gamma^{\left[\left(\frac{\alpha^2}{4\pi}-2\right)(n_v-1)\right](h_v-h_{v'})}\gamma^{\left[\left(\frac{\alpha^2}{4\pi}-2\right)n_{v_0}+2\varepsilon\right]h}\right] \\
&\leqslant (\text{const.})\,(\lambda(R))^{2k}\gamma^{-2h_0\varepsilon}\gamma^{\left(\frac{\alpha^2}{4\pi}-2\right)2kh_0}\gamma^{2h_0\varepsilon} \leqslant (\text{const.})\,\lambda^{2k}\gamma^{2k(R-h_0)\left(2-\frac{\alpha^2}{4\pi}\right)}.
\end{aligned}
\tag{16.a}
$$

The proof in the $Q_{v_0} \neq 0$ case is completely equivalent.

---

(1) $\#(v < v')$ is the number of bifurcations which come immediately before v' (from right to left).

Explicit expression of $\mathrm{U}^{[\leqslant h]}_{(\gamma_1,\ldots,\overline{\gamma}_{s_2};\underline{\tau}^{(1)},\ldots,\underline{\overline{\tau}}^{(s_2)})}$ *(see eq. 5.29).*

$$\mathrm{U}^{[\leqslant h]}((\gamma_1,\ldots,\gamma_{s_1},\overline{\gamma}_1,\ldots,\overline{\gamma}_{s_2});\underline{\tau}^{(1)},\ldots,\underline{\tau}^{(s_1)},\underline{\overline{\tau}}^{(1)},\ldots,\underline{\overline{\tau}}^{(s_2)})$$

$$= -\frac{\alpha^2}{2}\left\{\sum_{\substack{i\in(1,\ldots,s_1)\\ l\in(1,\ldots,s_1)\\ i\neq l}} c^{[\leqslant h]}_{\gamma_i,\gamma_l}(\underline{\tau}^{(i)},\underline{\tau}^{(l)}) + \sum_{\substack{J\in(1,\ldots,s_1)\\ t\in(1,\ldots,s_2)\\ J\neq t}} c^{[\leqslant h]}_{\overline{\gamma}_J,\overline{\gamma}_t}(\underline{\overline{\tau}}^{(J)},\underline{\overline{\tau}}^{(t)}) + \sum_{\substack{i\in(1,\ldots,s_1)\\ J\in(1,\ldots,s_2)}} c^{[\leqslant h]}_{\gamma_i,\overline{\gamma}_J}(\underline{\tau}^{(i)},\underline{\overline{\tau}}^{(J)})\right\} \qquad (17.a)$$

where

$$c^{[\leqslant h]}_{\gamma_i,\gamma_J}(\underline{\tau}^{(i)},\underline{\tau}^{(l)}) = \sum_{\substack{\alpha\in(1,\ldots,k_i)\\ \beta\in(1,\ldots,k_J)}} \tau^{(1)}_\alpha\tau^{(J)}_\beta \langle \Delta\varphi^{[\leqslant h]}_{2\alpha-1,2\alpha}\Delta\varphi^{[\leqslant h]}_{2\beta-1,2\beta}\rangle$$

$$c^{[\leqslant h]}_{\gamma_i,\overline{\gamma}_J}(\underline{\tau}^{(i)},\underline{\overline{\tau}}^{(J)}) = \sum_{\substack{\alpha\in(1,\ldots,k_i)\\ \beta\in(1,\ldots,\overline{k}_J)}} \tau^{(1)}_\alpha\overline{\tau}^{(J)}_\xi \langle \Delta\varphi^{[\leqslant h]}_{2\alpha-1,2\alpha}\Delta\varphi^{[\leqslant h]}_{2\beta-1,2\beta}\rangle + \sum_{\substack{\alpha\in(1,\ldots,k_i)\\ \gamma\in(1,\ldots,\overline{p}_J)}} \tau^{(i)}_\alpha \langle \Delta\varphi^{[\leqslant h]}_{2\alpha-1,2\alpha}\varphi^{[\leqslant h]}_{2\overline{k}_J+\gamma}\rangle$$

$$c^{[\leqslant h]}_{\overline{\gamma}_J,\overline{\gamma}_t}(\underline{\overline{\tau}}^{(J)},\underline{\overline{\tau}}^{(t)}) = \sum_{\substack{\alpha\in(1,\ldots,\overline{k}_J)\\ \beta\in(1,\ldots,\overline{k}_t)}} \overline{\tau}^{(J)}_\alpha\overline{\tau}^{(t)}_\beta \langle \Delta\varphi^{[\leqslant h]}_{2\alpha-1,2\alpha}\Delta\varphi^{[\leqslant h]}_{2\beta-1,2\beta}\rangle + \sum_{\substack{\alpha\in(1,\ldots,\overline{k}_J)\\ \gamma\in(1,\ldots,\overline{p}_t)}} \overline{\tau}^{(J)}_\alpha \langle \Delta\varphi^{[\leqslant h]}_{2\alpha-1,2\alpha}\varphi^{[\leqslant h]}_{2\overline{k}_t+\gamma}\rangle$$

$$+ \sum_{\substack{\delta\in(1,\ldots,\overline{p}_J)\\ \beta\in(1,\ldots,\overline{k}_t)}} \overline{\tau}^{(t)}_\beta \langle \varphi^{[\leqslant h]}_{2\overline{k}_J+\delta}\Delta\varphi^{[\leqslant h]}_{2\beta-1,2\beta}\rangle + \sum_{\substack{\delta\in(1,\ldots,\overline{p}_J)\\ \gamma\in(1,\ldots,\overline{p}_t)}} \langle \varphi^{[\leqslant h]}_{2\overline{k}_J+\delta}\varphi^{[\leqslant h]}_{2\overline{k}_t+\gamma}\rangle \qquad (18.a)$$

APPENDIX B

Theorem 6 tells us that at each level h from the part of $\hat{V}^{(h)}$ of order λ^2 some pieces can be extracted ($G^{(h)}_{1,2;O(\lambda^2)}$ in the text) which allow us via Theorems 4 and 5 to perform Steps I) and II). To obtain their explicit expressions and to prove Theorem 6 we need to prove the recursive structure of $\hat{V}^{(h)}$ at second order. This is done in the following theorem proved in [6].

THEOREM 1*b*. — At each level k, provided b satisfies (4.7) with B large enough $\hat{V}^{(k)}|_{O(\lambda^2)}$ has the following expression

$$\hat{V}^{(k)}|_{O(\lambda^2)} = \hat{V}^{(k)}(\mathscr{D}^{(k-1)}, \hat{R}^{(k)})|_{O(\lambda^2)} + [\mathscr{G}^{(k)}_{1,O(\lambda^2)} + \mathscr{G}^{(k)}_{2,O(\lambda^2)}]$$
$$+ [\overline{\theta} V^{(k)}_{O(\lambda^2)}[(\Lambda \backslash \hat{R}^{(k)})^2 \cap D^{(k-1)} \backslash D^{(k)}] + U^{(k)}_{2,c}[D^{(k-1)}] + W^{(k)}_c[D^{(k)}] + \Delta^{(k)}_c[D^{(k)}]] \quad (1.b)$$

where

$$\left\{\begin{aligned} V^{(k)}_{O(\lambda^2)}[J] &= -\overline{c}\lambda^2 \sum_{q=k+1}^{R} \int_J d\xi d\eta e^{\alpha^2 c[<q]}(e^{\alpha^2 c(q)} - 1) \sin^2 \frac{\alpha}{2} \Delta\varphi^{[\leqslant k]}_{\xi\eta} \\ W^{[k]}_c[J] &= c\rho \sum_{q=0}^{k-1} p^{[(k+2)-q]} \sum_{\tilde{h}=k+1}^{R} \overline{Y}^{\tilde{h}}_k[J] \\ \Delta^{(k)}_c[J] &= \delta \sum_{q=0}^{k-1} p^{[(k+1)-q]} Y^{k+1}_k[J] \\ U^{(k)}_{2,c}[J] &= -c_4\lambda^2 \sum_{q=k+1}^{R} \int_J d\xi d\eta e^{\alpha^2 c[<q]}(e^{\alpha^2 c(q)} - 1)[\tilde{\tilde{B}}(\gamma^{k-1}|\xi - \eta|)^{1-\varepsilon}]^2 \end{aligned}\right. \quad (2.b)$$

$$\overline{Y}^{\tilde{h}}_k[J] = -\gamma^{-2(1-\varepsilon)[\tilde{h}-(k+2)]}\lambda^2 \int_J d\xi d\eta e^{\alpha^2 c[<\tilde{h}]}(e^{\alpha^2 c(\tilde{h})} - 1) \sin^2 \frac{\alpha}{2} \Delta\varphi^{[\leqslant k]}_{\xi\eta}$$
$$Y^{\tilde{h}}_k[J] = -\lambda^2 \int_J d\xi d\eta e^{\alpha^2 c[<\tilde{h}]}(e^{\alpha^2 c(\tilde{h})} - 1) \sin^2 \frac{\alpha}{2} \Delta\varphi^{[\leqslant k]}_{\xi\eta} \quad (3.b)$$

where $\overline{c}$ is an appropriate positive constant, $\overline{\theta} \in [0, 1]$, $\tilde{\tilde{B}} = B\sqrt{\rho}$ and between c_4', c, ρ, δ, p the following relations hold

a) $p \in (0, 1)$, $\delta > 0$, $c > 0$, c_4, appropriately small, > 0

b) $\rho p \gamma^{2(1-\varepsilon)} = 1$

c) $\dfrac{(c + \delta)p}{1 - p} < 1$

d) $\dfrac{b}{B} = \dfrac{\gamma^{1-\varepsilon} - 1/\sqrt{p}}{\gamma^{2(1-\varepsilon)}} < \dfrac{\gamma^{1-\varepsilon} - 1}{\gamma^{2(1-\varepsilon)}} < \dfrac{\gamma^{1-\varepsilon} - 1}{\gamma^{1-\varepsilon}}$.

Remarks. — *i*) The crucial point of the Theorem 1.*b* is that at each step $\hat{V}^{(k)}(\mathscr{D}^{(k-1)}, \hat{R}^{(k)})|_{O(\lambda^2)}$ and the second parenthesis of the right hand side reproduce the same global expression (1.*b*) at level $k - 1$; therefore at each step we are allowed to utilize the negative factor $[\mathscr{G}^{(k)}_{1,O(\lambda^2)} + \mathscr{G}^{(k)}_{2,O(\lambda^2)}]$.

ii) The constant A_2 in Theorem 6 is: $c_4\left(\dfrac{\gamma^{2(1-\varepsilon)} - 1}{\gamma^{2(1-\varepsilon)}}\right)$ and $c_1 = \overline{c}$.

APPENDIX C

Looking at the proof of inequalities (8.14) in [2] and in [3] one realizes that in the estimate of the remainder factors like $\delta_2(B; \alpha^2, M, \lambda, h) \mid \tilde{\Lambda} \cap \hat{G} \mid$ appear which, on the scale h, are estimated by

$$(\text{const.})\, \lambda^k \left\{ \begin{matrix} \gamma^{k(R-h)\varepsilon} \\ \gamma^{k(R-h)\left(2-\frac{\alpha^2}{4\pi}\right)} \end{matrix} \right\} B^\rho \gamma^{2h} \mid \Lambda(R) \cap \hat{G}^{(h)} \mid \tag{1.c}$$

where $k = v(\gamma)$.

They originate in various ways, see for instance eq. (6.54), where B^ρ comes from the fact that in the coefficients of the expression (8.1) we used the Hölder continuity of the field $\varphi^{[\leqslant h-1]}$.

Similar terms with $\hat{\mathscr{G}}^{(h)}$ instead of $\hat{G}^{(h)}$ appear as well (see for instance eq. (6.5) of [3] (there called $\hat{R}$)) which are controlled exactly in the same way. The usual way to control these terms is to show that they appear in expressions of the following type (see eq. (4.23) of [3]):

$$\sum_{G_h \subset Q_h} \left(\exp\left[\left((\text{const.})\, \lambda^2 + (\text{const.}) \sum_{4}^{2M} \lambda^k (\gamma^{k(R-h)\omega}) \right) (1 + B^\rho) \right] \gamma^{2h} \mid \Lambda(R) \cap \hat{G}^{(h)} \mid \right) \cdot \left(\int \overset{\circ}{\chi}{}^b_{G_h}(\varphi^{(h)}) P(d\varphi^{(h)}) \right)^{1/2} \tag{2.c}$$

where we use G_h as in this case G_h is not $\varphi^{(h)}$-dependent but is just an arbitrary set of tesserae εQ_h. By ω we denote $\varepsilon > 0$ for $\alpha^2 \geqslant 8\pi$, $(2 - \alpha^2/4\pi)$ for $\alpha^2 < 8\pi$.

Remembering the estimate (4.25) of [3]

$$\left(\int \overset{\circ}{\chi}{}^b_{G_h}(\varphi^{(h)}) P(d\varphi^{(h)}) \right)^{1/2} \leqslant \prod_{\Delta \in G_h} \exp\left[c_1 - c_2 b^2 (1 + d(\Delta, \gamma^h \Lambda(R))) \right] \qquad c_1, c_2 > 0 \tag{3.c}$$

We can estimate (2.c) in the following way

$$\begin{aligned} [(2.c)] &\leqslant \prod_{\Delta \in G_h} (1 + \exp([\quad] + c_1 - c_2 b^2 (1 + d(\Delta)\, \gamma^h \Lambda(R)))), \\ &\leqslant \exp\left[c_3 e^{([\quad] + c_1 - c_2 b^2)} \right] \gamma^{2h} \mid \Lambda(R) \mid \end{aligned} \tag{4.c}$$

where

$$\gamma^{2h} \mid \Lambda(R) \cap \hat{G}^{(h)} \mid = B^{\tilde{\rho}} \gamma^{2h} \mid \Lambda(R) \cap G^{(h)} \mid = B^{\tilde{\rho}} \,\#\, (\Delta \in \Lambda(R) \cap G^{(h)}) \tag{5.c}$$

and we have chosen $\mathscr{B}$ in the last line of (6.5) independent of h and $\sim B^{\tilde{\rho}/2}$ with $\tilde{\rho} > 0$.

If ω were $= 0$ we could write

$$\exp\left[c_3 e^{([\quad] + c_1 - c_2 b^2)} \right] \gamma^{2h} \mid \Lambda(R) \mid \leqslant \exp c_4 e^{-c_5 b^2} \gamma^{2h} \mid \Lambda(R) \mid \tag{6.c}$$

with $c_4, c_5 > 0$, and choosing B large enough and therefore b large enough we would have:

$$[(6.c)] \leqslant O(\lambda^{2(M+1)}) \gamma^{-2(R-h)} \mid I \mid \tag{7.c}$$

which could be safely put into the remainder.

However since $\omega \neq 0$ for $h < h_0$, we cannot bound $([\quad] + c_1 - c_2 b^2)$ with $- c_5 b^2$.

Thus for $h < h_0$, we bound the expression (2.c) in the following way:

$$e^{(\mathrm{const.})\sum_{4}^{2M}{}_k \lambda^k(\gamma^{k(R-h)\omega})(1+B^\rho)\gamma^{-2(R-h)}|I|} \left\{ \sum_{G_h \subset Q_h} \exp\left[(\mathrm{const.})\,\lambda^2 B^{\tilde\rho}\gamma^{2h}\,|\,\Lambda(R)\cap G_h\,|\,\right] \right.$$

$$\cdot \left(\int \overset{\circ}{\chi}{}^b_{G_h} P(d\varphi^{(h)})\right)^{1/2} \leqslant e^{O(\lambda^{2M+1})\gamma^{-2(R-h)}|I|} e^{(\mathrm{const.})\sum_{4}^{2M}{}_k \lambda^k(\gamma^{k(R-h)\omega})(1+B^\rho)\gamma^{-2(R-h)}|I|} \qquad (8.c)$$

and the second factor in the r. h. s. is estimated as $G^{(h)}_{2,O(\lambda>2)}$ in section 6 (see eqs. (6.47),..., (6.51)).

The final result is that all the terms associated to the regions $\hat{G}^{(h)}$ and $\hat{\mathscr{G}}^{(h)}$ give contributions to the remainder which have all the right properties. We are only left with the proof that we can get rid of the factors like $e^{\rho\overline{H}_h B^\rho}$ of eqs. (8.16). The origin of these factors is twofold: referring to [3] one of these factors appears in the estimate in eq. (6.30) needed for the lover bound. This estimate is used in that way only if $h \geqslant h_0$ and it gives (again provided B is large enough) a factor

$$\exp O(\lambda^{2M+1})\gamma^{-2(R-h)}\,|\,I\,| \qquad (9.c)$$

good for our purposes. If $h < h_0$ we use the obvious estimate

$$\int \mu(dx)\chi(x)e^{V\chi} \geqslant e^{-2\|V\chi\|_\infty}\left(\int \mu(dx)\chi(x)\right)\int \mu(dx)e^{V\chi} \qquad (10.c)$$

where in the notations of (3) $e^{-2\|V\chi\|_\infty} = e^{-2\|\Psi_\square \chi^{\square,B}\|_\infty}$ and these factors sum up to

$$e^{-2\sup\|\Psi_\square \chi^{\square,B}\|_\infty \gamma^{2h}|\Lambda(R)|B^\rho} \geq e^{-O(\lambda^2)B^\rho\gamma^{-2(R-h)}|I|} \qquad (11.c)$$

which again can be controlled if $(R - h)$ is sufficiently large and λ sufficiently small.

In fact,

$$e^{c\lambda^2 B^\rho\gamma^{-2(R-h)}|I|} = e^{c\lambda^2 B^\rho\gamma^{-(2-\delta_1)(R-h)}(\gamma^{-\frac{\delta_1}{2}(R-h)}\,|\,I\,|)} \qquad (12.c)$$

and therefore we need

$$\gamma^{-\frac{\delta_1}{2}(R-h)}(c\lambda^2 B^\rho\gamma^{-(2-\delta_1)(R-h)}) \leqslant O(\lambda^{2M+\tau}) \qquad (13.c)$$

which is fulfilled if λ is very small and $R - h \geqslant R - h_0$ is very large. If $h \geqslant h_0$ the only constraint is that

$$(c\lambda^{2k}\gamma^{2k(R-h_0)\omega}B^\rho) \leqslant O(\lambda^{\tilde\varepsilon}), \qquad \forall k \in [1, M] \qquad \text{with} \qquad \tilde\varepsilon > 0\,. \qquad (14.c)$$

Again it is possible to simultaneoulsy satisfy (13.c) and (14.c) when, for instance (6.50) holds and $(2 - \delta_1) < 1$; (6.50) together with this last constraint gives the following condition

$$\left(2(M-1)\left(2 - \frac{\alpha^2}{4\pi}\right) - 1\right) < 0 \qquad (15.c)$$

The other step where a factor $e^{\rho\overline{H}_h B^\rho}$ appears in the proof of inequalities (8.14) given in [3] is in the estimate of the remainder of the following expression

$$\int P(dZ_\square\,|\,\tilde{Z})e^{\Psi_\square \chi^{\square,b}} = e^{\sum_1^{2M}{}_k \frac{1}{k!}\varepsilon_{\tilde{Z}}(\Psi_\square \chi^{\square,b};k) + (\text{Remainder})} \qquad (16.c)$$

In general $\Psi_\square = \sum_{k=1}^{2M} \Psi_k$ where Ψ_k is of order λ^k. One can perform the cumulant expansion in such a way that the remainder is at most of order

$$\lambda^{4M}\gamma^{4M\omega(R-h)}\gamma^{-(2-\delta_1)(R-h)}(\gamma^{-\frac{\delta_1}{2}(R-h)}\,|\,I\,|)\gamma^{-\frac{\delta_1}{2}(R-h)} \tag{17.c}$$

which implies

$$4M\left(2 - \frac{\alpha^2}{4\pi}\right) - 1 < 0 \tag{18.c}$$

and therefore

$$\alpha^2 > 8\pi\left(1 - \frac{1}{8M}\right) \equiv \overline{\alpha}^2_{2M} \tag{19.c}$$

REFERENCES

[1] G. GALLAVOTTI, F. NICOLO, The Screening phase transitions in the two dimensional Coulomb gas, To appear in *J. Stat. Phys.*

[2] G. BENFATTO, G. GALLAVOTTI, F. NICOLO, On the massive Sine-Gordon equation in the first few regions of collapse. *Commun. Math. Phys.*, t. **83**, 1982, p. 387.

[3] G. BENFATTO, M. CASSANDRO, G. GALLAVOTTI, F. NICOLÒ, E. OLIVIERI, E. PRESUTTI, E. SCACCIATELLI, Ultraviolet stability in Euclidean scalar field theories. *Commun. Math. Phys.*, t. **71**, 1980, p. 95.

[4] G. GALLAVOTTI, F. NICOLÒ, Renormalization Theory in four dimensional scalar field theories (I) and (II) to appear in *Commun. Math. Phyis.*

[5] F. NICOLÒ, On the massive Sine-Gordon equation in the higher regions of collapse. *Commun. Math. Phys.*, t. **88**, 1983, p. 681.

[6] F. NICOLÒ, J. RENN, A. STEINMANN, *On the massive Sine-Gordon equation in all regions of collapse (II University of Rome preprint).*

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