ANNALES DE L'I. H. P., SECTION A

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Annales de l'I. H. P., section A, tome 44, nº 2 (1986), p. 173-193
<http://www.numdam.org/item?id=AIHPA_1986__44_2_173_0>

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Ann. Inst. Henri Poincaré,
Vol. 44, n° 2, 1986, p. 173-193. *Physique théorique*

Compact manifold of coherent states invariant by semisimple Lie groups

by

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ABSTRACT. — We characterize the coherent state manifolds invariant by a semisimple Lie group $\mathscr{G}$. Those obtained as orbits of a maximal weight vector of an irreducible representation of $\mathscr{G}$ are Kählerian spaces. We give the list of them for simple compact $\mathscr{G}$. We select those which are symmetric spaces and give two parametrizations of these manifolds.

RÉSUMÉ. — On caractérise les variétés d'états cohérents invariantes pour un groupe de Lie semi simple $\mathscr{G}$. Celles qui sont obtenues comme orbites d'un vecteur de poids maximal d'une représentation irréductible de $\mathscr{G}$ sont des espaces Kähleriens. On en donne la liste pour $\mathscr{G}$ compact simple. On identifie celles qui sont des espaces symétriques et on en donne deux paramétrisations.

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1. INTRODUCTION

Coherent states are the quantum states more closely related to the classical ones: they are one of the best tool to link quantum and classical mechanical descriptions. The usual harmonic oscillator coherent states [*1*] are sharply localized in position and momentum: during their time evolution they preserve their shape and follow the classical path in phase space. For a general dynamical system one can try to satisfy the two conditions: 1) the manifold of coherent states coincides with the classical phase space; 2) the quantum Schrödinger's paths on this manifold coincide with the Euler Lagrange's ones.

Both requirements can be fulfilled by the application of an algebraic method [*2*]: recognize the dynamical Lie algebra $\mathfrak{g}$ of the system (it contains the Hamiltonian) and construct the $\mathscr{G}$-orbit of a chosen cyclic vector of the Hilbert space $\mathscr{E}$ of states, $\mathscr{G}$ being a Lie group corresponding to $\mathfrak{g}$.

More precisely, in quantum mechanics physical states are represented by rays, i. e. the equivalence classes of collinear vectors in $\mathscr{E}$: the manifold of coherent states $\mathscr{M}_v$ is obtained formally as follows:

$$\mathscr{M}_v \doteqdot \pi\mathscr{U}(\mathscr{G})\,|\,v\rangle \subset \mathscr{S} \tag{1.1}$$

In eq. (1.1) $\mathscr{U}$ denotes a unitary irreducible representation of $\mathscr{G}$ on $\mathscr{E}$, $\mathscr{S}$ is the set of rays, $|\,v\rangle \in \mathscr{E}$ the chosen cyclic vector and finally $\pi : \mathscr{E} \mapsto \mathscr{S}$ the projection (which coincides with normalization of the vectors and forgetting the phase). From eq. (1.1) it follows that the manifold $\mathscr{M}_v$ coincides with the homogeneous space:

$$\mathscr{M}_v \simeq \mathscr{G}/\overline{\mathscr{G}}_v \tag{1.2}$$

where $\overline{\mathscr{G}}_v$ denotes the isotropy group of the vector ray represented by $|\,v\rangle$. If a Kählerian structure can be put on $\mathscr{M}_v$ one obtains a manifold which can be interpreted as a phase space, whose corresponding set of states is suitable for constructing path integrals [*3*]. The dynamical group $\mathscr{G}$ guarantees the preservation of coherence during the time evolution, i. e. the Schrödinger paths starting on $\mathscr{M}_v$ will remain on it for all times [*4*]. Moreover a path on the manifold satisfying the Schrödinger equation minimizes the action [*5*].

In quantum physics with an invariance group, the most frequently occurring states are the weight vectors. For instance for a spin one particle ($\mathscr{G} = 0(3)$, $\mathscr{E}$: three dimensional complex space carrying the adjoint (spin 1) representation), the circularly polarized (or helicity) states and the longitudinally polarized states are weight vectors. They form two orbits of 0(3); the one parameter family of other states orbits have no name [*6*].

More generally in particle physics, with internal $\mathscr{G}$-symmetry, only weight vector states occur; of course it is meaningful to construct coherent states only when $\mathscr{G}$ is an exact symmetry e. g. the color SU(3).

For many non linear equations with soliton solutions the manifold of solutions is the orbit of the maximal weight vector under the action of infinite dimensional Lie groups and solitons can be interpreted as coherent states for a dynamical system with an infinite number of degree of freedom [5] [7].

In this work we describe the coherent state manifold for compact semisimple Lie groups constructed from highest weight vector of their (unitary) irreducible representations.

In sect. 2 we give the detailed structure of the Lie algebra of $\overline{\mathscr{G}}_v$. In sect. 3 we show that for maximal weight vector the manifold $\mathscr{M}_v$ is compact and carries Riemaniann, symplectic and also complex structures: so it is Kählerian and can be thought as a classical phase space. Moreover we give the conditions for this manifold to be a symmetric space (with constant Riemaniann curvature) and we list the corresponding representations. For the other weight vectors, when the manifold has not these properties, we do not study them in detail.

In sect. 4 we give two parametrizations of these manifolds and recall the constructions of Bargmann spaces of holomorphic functions on them. Finally, in sect. 5, we give a family of simple examples to illustrate this general work.

2. THE ISOTROPY GROUPS OF WEIGHT VECTOR RAYS

In order to make this paper more self contained, we first recall some basic facts for compact semisimple (real) Lie groups [8]. We denote by $x \wedge y$, $(x, y \in \mathfrak{g})$ its Lie algebra law. On the vector space of $\mathfrak{g}$ one builds the adjont representation $x \mapsto \mathrm{Ad}(x)$ defining the action of the linear operator $\mathrm{Ad}(x)$ on $\mathfrak{g}$ as: $\mathrm{Ad}(x)y = x \wedge y$. Jacobi identity shows that $\mathrm{Ad}(x)$ are antisymmetric real operators with respect to the Cartan-Killing form

$$(x, y) = -\operatorname{tr} \mathrm{Ad}(x)\mathrm{Ad}(y) \tag{2.1}$$

and that they do form a representation of the Lie algebra $\mathfrak{g}$. According to physicists custom we will use in the following (pure imaginary) Hermitean operators

$$\mathrm{J}(x) = i\,\mathrm{Ad}(x) \tag{2.2}$$

denoting the Lie product by means of the commutator:

$$[\mathrm{J}(x), \mathrm{J}(y)] = i\mathrm{J}(x \wedge y) \tag{2.3}$$

A Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is a maximal Abelian subalgebra. One shows that all $\mathfrak{h}$'s are conjugated by the group $\mathscr{G}$: their common dimension l is the rank of $\mathscr{G}$. The set $\{ J(h); h \in \mathfrak{h} \}$ of commuting Hermitean operators has in common an orthonormal basis $| z_\alpha \rangle$ of eigenvectors in the complexified vector space $\mathfrak{g}^{\mathbb{C}}$ equipped with the Hermitean product $\langle z_\alpha | z_\beta \rangle = (\bar{z}_\alpha, z_\beta)$, $z_\alpha, z_\beta \in \mathfrak{g}^{\mathbb{C}}$. The eigenvalues of $J(h)$ on the $| z_\alpha \rangle$ depend linearly on h and hence they can be written as a scalar product with a fixed vector $\mu_\alpha \in \mathfrak{h}$ (as the Cartan-Killing form is non degenerate on $\mathfrak{g}$ and $\mathfrak{h}$ too):

$$J(h) | z_\alpha \rangle = (\mu_\alpha, h) | z_\alpha \rangle \tag{2.4}$$

Eq. (2.4) defines the roots $\mu_\alpha \in \mathfrak{h}$ of $\mathfrak{g}$. Since $J(h) = i \operatorname{Ad}(h)$ and $\operatorname{Ad}(h)$ is a real antisymmetric operator, $| \bar{z}_\alpha \rangle$ is also an eigenvector that we denote also by $| z_{-\alpha} \rangle$ and (2.4) shows that its root is $\mu_{-\alpha} = -\mu_\alpha$.

Extending the Lie algebra law to $\mathfrak{g}^{\mathbb{C}}$ eq. (2.4) can be written either $h \wedge z_\alpha = -i(\mu_\alpha, h) z_\alpha$ or:

$$[J(h), J(z_\alpha)] = (\mu_\alpha, h) J(z_\alpha) \tag{2.5}$$

Jacobi identity shows that if $z_\alpha \wedge z_\beta \neq 0$, the corresponding root is $\mu_{\alpha+\beta} = \mu_\alpha + \mu_\beta$; then, up to a normalizing constant

$$z_\alpha \wedge z_\beta = K_{\alpha\beta} z_{\alpha+\beta} \tag{2.6}$$

The usual normalization is fixed by:

$$z_{-\alpha} \wedge z_\alpha = 2i(z_{-\alpha}, z_\alpha)\mu_\alpha = 2i \langle z_\alpha | z_\alpha \rangle \mu_\alpha \tag{2.7}$$

With the simplified notation:

$$J_3^{(\alpha)} = J(\mu_\alpha/(\mu_\alpha, \mu_\alpha)) \qquad J_\pm^{(\alpha)} = J(z_{\pm\alpha}) \tag{2.8}$$

one obtains the three generators of a $su(2)$ subalgebra in the standard form:

$$[J_3^{(\alpha)}, J_\pm^{(\alpha)}] = \pm J_\pm^{(\alpha)} \qquad [J_+^{(\alpha)}, J_-^{(\alpha)}] = 2J_3^{(\alpha)} \tag{2.9}$$

As a particular case of eq. (2.4) we obtain:

$$J_3^{(\alpha)} | z_\beta \rangle = \frac{(\mu_\alpha, \mu_\beta)}{(\mu_\alpha, \mu_\alpha)} | z_\beta \rangle \tag{2.10}$$

and from (2.9), by an analysis well known to physicists, one deduce that $J_3^{(\alpha)}$ has eigenvalues m, $-j \leqslant m \leqslant j$ with $2j$, $j - m$ positive integers, so the numbers $\mathfrak{n}_{\alpha\beta} = 2(\mu_\alpha, \mu_\beta)/(\mu_\alpha, \mu_\alpha)$ are integers.

Moreover the $J_3^{(\alpha)}$ eigenvectors are $(J_-^{(\alpha)})^k | z_\beta \rangle$, $0 \leqslant k \leqslant q$ and $(J_+^{(\alpha)})^h | z_\beta \rangle$ $0 \leqslant h \leqslant p$ with $p + q = 2j$, $q - p = \mathfrak{n}_{\alpha\beta}$ corresponding to the roots $\mu_\beta + \mathfrak{n}\mu_\alpha$, $-q \leqslant \mathfrak{n} \leqslant p$. Weyl introduced a basis of l roots $\{ \mu_i \}$, $i = 1, \ldots, l$ such that the off diagonal elements of the Cartan matrix:

$$A_{ij} = 2 \frac{(\mu_i, \mu_j)}{(\mu_i, \mu_i)} \tag{2.11}$$

are non positive. In such basis all roots in $\mathfrak{h}$ are linear combination with integer coefficients all positive or all negative: so one can speak about positive or negative roots. The whole structure of $\mathfrak{g}$ is contained in the Cartan matrix which is represented by means of a Dynkin diagram. This is constituted by vertices (o or •) corresponding to (long or short—only two different length are possible) roots: they are joined by $A_{ij}A_{ji}$ bonds. The roots generate an additive group $\mathscr{R}$ which is a $\mathbb{Z}$-lattice (i. e. it is generated via $\mathbb{Z}$-linear combinations of basis vectors). The reflection R_α through the hyperplane orthogonal to the root μ_α transform any root μ_β into:

$$R_\alpha \mu_\beta = \mu_\beta - 2\frac{(\mu_\alpha, \mu_\beta)}{(\mu_\alpha, \mu_\alpha)}\mu_\alpha \equiv \mu_\beta - \mathfrak{n}_{\alpha\beta}\mu_\alpha \tag{2.12}$$

which is the root corresponding to the eigenvalue $-\frac{1}{2}\mathfrak{n}_{\alpha\beta}$ of $J_3^{(\alpha)}$. The reflections R_α generate the Weyl group $\mathscr{W}(\mathscr{G})$ which is the stabilizer in $\mathscr{G}$ of $\mathfrak{h}$. Long and short roots in $\mathfrak{h}$ form two distinct $\mathscr{W}(\mathscr{G})$ orbits. From eq. (2.6) one can see that the z_α's corresponding to positive (resp. negative) roots span the complex subalgebra $\mathfrak{g}_+^{\mathbb{C}}$ (resp. $\mathfrak{g}_-^{\mathbb{C}}$) and $\mathfrak{g}^{\mathbb{C}}$ is decomposed in the direct sum:

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}_-^{\mathbb{C}} \oplus \mathfrak{h} \oplus \mathfrak{g}_+^{\mathbb{C}} \tag{2.13}$$

The Borel algebras $\mathfrak{b}_\pm = \mathfrak{h}^{\mathbb{C}} \oplus \mathfrak{g}_\pm^{\mathbb{C}}$ are maximal solvable subalgebras of $\mathfrak{g}^{\mathbb{C}}$.

The generalization from the adjoint representation to any irreducible representation $x \mapsto F(x)$, with $[F(x), F(y)] = iF(x \wedge y)$ of a simple Lie algebra $\mathfrak{g}$ is straightforward. The roots are replaced by the weights $w_A \in \mathfrak{h}$ and the corresponding weight vectors $|v_A\rangle \in \mathscr{E}$ ($\mathscr{E}$: space of the representation) form an orthonormal basis of $\mathscr{E}$. Similarly to eq. (2.4), the weights w_A are defined by:

$$F(h)\,|v_A\rangle = (h, w_A)\,|v_A\rangle \tag{2.14}$$

With an abbreviated notation corresponding to (2.8), the generators $F(\mu_\alpha/(\mu_\alpha, \mu_\alpha)) = F_3^{(\alpha)}$ and $F(z_{\pm\alpha}) = F_\pm^{(\alpha)}$ satisfy eq. (2.9). So $2F_3^{(\alpha)}$ has integral eigenvalues:

$$\frac{2(\mu_\alpha, w_A)}{(\mu_\alpha, \mu_\alpha)} \in \mathbb{Z} \tag{2.15}$$

Similarly, from eq. (2.9) applied to the $F^{(\beta)}$'s one can build other eigenvectors $(F_\pm^{(\beta)})^n\,|v_A\rangle$ common to all $F(h)$, $h \in \mathfrak{h}$. For instance for $h = \mu_\alpha/(\mu_\alpha, \mu_\alpha)$:

$$F_3^{(\alpha)}(F_\pm^{(\beta)})^n\,|v_A\rangle = \frac{(\mu_\alpha, w_A \pm \mathfrak{n}\mu_\beta)}{(\mu_\alpha, \mu_\alpha)}(F_\pm^{(\beta)})^n\,|v_A\rangle \tag{2.16}$$

so if $(F_{\pm}^{(\beta)})^n \mid v_A \rangle \neq 0$, it corresponds to the weight $w_A \pm n\mu_\beta$. One also verifies that Weyl reflections transform weight into weights:

$$R_\alpha w_A = w_A - 2\frac{(w_A, \mu_\alpha)}{(\mu_\alpha, \mu_\alpha)}\mu_\alpha \tag{2.17}$$

and the weight obtained in eq. (2.17) corresponds to the eigenvalue $-(\mu_\alpha, w_A)/(\mu_\alpha, \mu_\alpha)$ of $F_3^{(\alpha)}$. So the set of weights of a irreducible representation is stable under the Weyl group action and is a union of $\mathscr{W}(\mathscr{G})$ orbits.

Eqs. (2.15) and (2.16) show that the additive group $\mathscr{P}$ generated by the weights is a $\mathbb{Z}$-lattice, invariant by $\mathscr{W}(\mathscr{G})$. As eq. (2.4) is a particular case of eq. (2.14) the adjoint representation is irreducible and its weights are the roots, i. e. $\mathscr{R} \subset \mathscr{P}$. The quotient group of the two $\mathbb{Z}$-lattices:

$$\mathscr{P}/\mathscr{R} = \mathscr{Q} \tag{2.18}$$

is a finite Abelian group given in table 1. Since all weights of an irreducible representation differ by elements of $\mathscr{R}$, they are in a unique coset of $\mathscr{R}$ in $\mathscr{P}$. So the set of irreducible representations of $\mathscr{G}$ decomposes into families, each one corresponding to an element of $\mathscr{Q}$. As the weights of a tensor product $\mathscr{U}_1 \otimes \mathscr{U}_2$ of irreducible representations $\mathscr{U}_1$ and $\mathscr{U}_2$ are of the form $w^{(1)} + w^{(2)}$ ($w^{(i)}$ weight of $\mathscr{U}_i$), the tensor product of representations is compatible with the group law of $\mathscr{Q}$.

In the adjoint representation, since for $h, k \in \mathfrak{h}$, $h \wedge k = 0$ i. e. $J(h) \mid k \rangle = 0$, there is a l-degeneracy in the spectrum of the complete set of commuting operators $J(h)$ at zero (on the contrary all root-spaces are one dimensional). The same degeneracy can appear in an irreducible representation: a $\mathscr{W}(\mathscr{G})$ orbit of weights can be m-times degenerate—Weyl reflections connect weights with the same multiplicity—and there is an arbitrariness in choosing basis for weight spaces or for the Cartan subalgebra $\mathfrak{h}$. On the other hand one proves that $\mathscr{W}(\mathscr{G})$ acts transitively on the set of Weyl basis. As obviously no non trivial element of $\mathscr{W}(\mathscr{G})$ let fixed every vector of the basis, the choice of a Weyl basis chooses C_w, one of the $|\mathscr{W}(\mathscr{G})|$ (= the number of elements of $\mathscr{W}(\mathscr{G})$) Weyl chambers, i. e. a convex connected open cone limited by the reflection hyperplanes (orthogonal to all the roots). $\mathscr{W}(\mathscr{G})$ acts transitively on the set of Weyl chambers and every $\mathscr{W}(\mathscr{G})$-orbit has an unique point in C_w. The dual cone Γ of C_w—i. e. the convex hull of the basic roots—defines a partial order relation in $\mathfrak{h}$: $h \leqslant k \Leftrightarrow k - h \in \Gamma$.

This choice separates the roots into positive $\mu_\alpha \in \Gamma$ and negative $\mu_{-\alpha} = -\mu_\alpha$ as already seen. Raising operators, corresponding to positive roots, elevate the weights. Each irreducible representation has a maximal weight w_M corresponding to a non degenerate weight space $\mathbb{C} \mid v_M \rangle$ which satisfies:

$$\forall \alpha \in \Delta^+, \qquad F_+^{(\alpha)} \mid v_M \rangle = 0 \tag{2.19}$$

where Δ^+ is the set of indices of the positive roots.

Eq. (2.19) characterizes $|v_M\rangle$ completely (up to a multiplicative constant): the representation space $\mathscr{E}$ is built from $|v_M\rangle$ by applying to it words of lowering operators $F_-^{(\alpha)}$ until the minimal weight is reached (annihilated by all $F_-^{(\alpha)}$, $\mu_\alpha > 0$). All the possible maximal weights are contained in the closure of the Weyl chamber $w_M \in \mathscr{P} \cap \overline{C}_w$, i. e. $2(\mu_i, w_M)/(\mu_i, \mu_i) \in \mathbb{Z}^{(+)} \cup \{0\}$, $\{\mu_i\}$ being the Cartan basis. In particular the weights w_j which satisfy:

$$2\frac{(\mu_i, w_j)}{(\mu_i, \mu_i)} = \delta_{ij} \tag{2.20}$$

form the dual root basis (when all roots have the same lenght it is convenient to normalize them by $(\mu_i, \mu_i) = 2$). The basic weights w_j are the maximal weights of the fundamental representations. In this basis the components η_i $\left(w_M = \sum_{i=1}^{l} \eta_i w_i\right)$ of a maximal weight are non negative integers. So every irreducible representation of $\mathfrak{g}$ can be labelled by a set of integers $\eta_i \geqslant 0$ placed on the vertices of the Dynkin diagram of $\mathfrak{g}$ (generally called Dynkin indices).

In particular the irreducible representation with only one Dynkin index, i. e. $w_M = nw_i$ belongs to the completely symmetrized n^{th} tensor power of the fundamental representation of maximal weight w_i.

We are now ready to study the isotropy group of the weight vector rays. Denoting by $\overline{\mathscr{G}^{\mathbb{C}}_{v_A}}$ the isotropy subgroup in $\mathscr{G}^{\mathbb{C}}$ of the ray $\mathbb{C}\,|\,v_A\rangle$, its Lie algebra $\overline{\mathfrak{g}^{\mathbb{C}}_{v_A}}$ is, by definition, given by:

$$\overline{\mathfrak{g}^{\mathbb{C}}_{v_A}} = \{\, x \in \mathfrak{g}^{\mathbb{C}} : F(x)\,|\,v_A\rangle = \lambda\,|\,v_A\rangle,\ \lambda \in \mathbb{C} \,\} \tag{2.21}$$

The annihilator algebra $\mathfrak{g}^{\mathbb{C}}_{v_A}$:

$$\mathfrak{g}^{\mathbb{C}}_{v_A} = \{\, x \in \mathfrak{g}^{\mathbb{C}} : F(x)\,|\,v_A\rangle = 0 \,\} \tag{2.22}$$

is a Lie subalgebra of $\overline{\mathfrak{g}^{\mathbb{C}}_{v_A}}$. From eq. (2.14) one sees that:

$$\mathfrak{h} \subset \overline{\mathfrak{g}^{\mathbb{C}}_{v_A}} \tag{2.23}$$

$$\mathfrak{h} \cap \mathfrak{g}^{\mathbb{C}}_{v_A} \doteqdot \mathfrak{h}_{v_A} \equiv \mathfrak{w}_A^{\perp} = \{\, h \in \mathfrak{h} : (h, w_A) = 0 \,\} \tag{2.24}$$

We will need the two following lemmas.

LEMMA 1. — Let $t = t_- + t_0 + t_+ \in \mathfrak{g}^{\mathbb{C}}$ the decomposition of t according to (2.13), with $t_\pm = \sum_{\alpha \in \Delta^+} \mathscr{S}_{\pm\alpha} z_{\pm\alpha}$, $\mathscr{S}_{\pm\alpha} \in \mathbb{C}$.

Then $F(t)\,|\,v_A\rangle = \lambda\,|\,v_A\rangle \Leftrightarrow (t_0, w_A) = \lambda$ and $\mathscr{S}_{\pm\alpha} F(z_{\pm\alpha})\,|\,v_A\rangle = 0$. The implication $\Leftarrow$ is obvious. We remark that all $F(z_{\pm\alpha})\,|\,v_A\rangle$ either vanish or are orthogonal to $|\,v_A\rangle$ and among each other (they correspond to different points of the spectrum of the set of commuting operators $F(h)$, $h \in \mathfrak{h}$). So

by taking the Hermitean scalar product with $\langle v_A |$ on the left with any of these vectors the lemma is proved. This lemma can also be expressend by the equation:

$$\mathfrak{g}_{v_A}^{\mathbb{C}} = \bigoplus_{\substack{\alpha\in\Delta^+ \\ F(z_{-\alpha})|v_A\rangle=0}} \mathbb{C}z_{-\alpha} \oplus \mathfrak{h}_{v_A} \bigoplus_{\substack{\alpha\in\Delta^+ \\ F(z_\alpha)|v_A\rangle=0}} \mathbb{C}z_\alpha \tag{2.25}$$

LEMMA 2. — If two of the three vectors $F_+^{(\alpha)} | v_A \rangle$, $F_-^{(\alpha)} | v_A \rangle$ $F_3^{(\alpha)} | v_A \rangle$ vanish the third one is also zero.

The $F^{(\alpha)}$'s satisfy the second equation of (2.9) which applied to $| v_A \rangle$ reads

$$F_+^{(\alpha)}F_-^{(\alpha)} | v_A \rangle - F_-^{(\alpha)}F_+^{(\alpha)} | v_A \rangle = 2F_3^{(\alpha)} | v_A \rangle \tag{2.26}$$

So $F_\pm^{(\alpha)} | v_A \rangle = 0$ implies $F_3^{(\alpha)} | v_A \rangle = 0$. Assume now that $F_3^{(\alpha)} | v_A \rangle = 0$ and $F_+^{(\alpha)} | v_A \rangle = 0$, so, by eq. (2.26), $F_+^{(\alpha)}F_-^{(\alpha)} | v_A \rangle = 0$. Since $F_-^{(\alpha)}$ and $F_+^{(\alpha)}$ are Hermitean conjugated each other:

$$0 = \langle v_A | F_+^{(\alpha)}F_-^{(\alpha)} | v_A \rangle = \langle F_-^{(\alpha)}v_A | F_-^{(\alpha)}v_A \rangle,$$

i. e. $F_-^{(\alpha)} | v_A \rangle = 0$, Q. E. D.

Weight vectors of weight on the same $\mathscr{W}(\mathscr{G})$ orbit have conjugated isotropy groups: so, without loss of generality, one can choose them in the Weyl chamber $\overline{C}_w$ i. e.

$$\alpha \in \Delta^+ \qquad (\mu_\alpha, w_A) \geqslant 0 \Leftrightarrow w_A \in \overline{C}_w \tag{2.27}$$

In a way analogous to the derivation of the root ladder one sees that if $(F_-^{(\alpha)})^h | v_A \rangle$, $(F_+^{(\alpha)})^k | v_A \rangle$ are non zero weight vectors with $0 \leqslant h \leqslant q$, $0 \leqslant k \leqslant p$, then $q - p = 2(\mu_\alpha, w_A)/(\mu_\alpha, \mu_\alpha)$: so if $F_+^{(\alpha)} | v_A \rangle \neq 0$ $p \geqslant 1$ and $q \geqslant 1 + 2(\mu_\alpha, w_A)/(\mu_\alpha, \mu_\alpha)$ and also $F_-^{(\alpha)} | v_A \rangle \neq 0$.

Equivalently $F_-^{(\alpha)} | v_A \rangle = 0 \Rightarrow q = 0 \Rightarrow p < 0$ and also $F_+^{(\alpha)} | v_A \rangle = 0$ and, by lemma 2, $F_3^{(\alpha)} | v_A \rangle = 0$ which is equivalent to $(\mu_\alpha, w_A) = 0$.

Summarizing, for $w_A \in \overline{C}_w$ one has:

$$F_+^{(\alpha)} | v_A \rangle \neq 0 \Rightarrow F_-^{(\alpha)} | v_A \rangle \neq 0 \tag{2.28 a}$$

$$F_-^{(\alpha)} | v_A \rangle = 0 \Rightarrow F_+^{(\alpha)} | v_A \rangle = 0 \quad \text{and} \quad (\mu_\alpha, w_A) = 0 \tag{2.28 b}$$

Beware that $(\mu_\alpha, w_A) = 0$ does not imply $F_\pm^{(\alpha)} | v_A \rangle = 0$.

Eq. (2.4) with $h = w_A$ shows that the Lie algebra $\mathfrak{g}_{w_A}$ of the isotropy group $\mathscr{G}_{w_A}$ of the weight w_A in the adjoint representation of $\mathscr{G}$ is given by:

$$\mathfrak{g}_{w_A} = (\mathfrak{g}_{w_A})^{\mathbb{C}} \cap \mathfrak{g}, (\mathfrak{g}_{w_A})^{\mathbb{C}} = \bigoplus_{\substack{\alpha\in\Delta^+ \\ (\mu_\alpha, w_A)=0}} \mathbb{C}z_{-\alpha} \oplus \mathfrak{h}^{\mathbb{C}} \bigoplus_{\substack{\alpha\in\Delta^+ \\ (\mu_\alpha, w_A)=0}} \mathbb{C}z_\alpha \tag{2.29}$$

So for any weight $w_A \in \overline{C}_w$ there is a natural decomposition of $\mathfrak{g}^{\mathbb{C}}$ into the direct sum of three subalgebras:

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{w}_-^A \oplus (\mathfrak{g}_{w_A})^{\mathbb{C}} \oplus \mathfrak{w}_+^A \tag{2.30}$$

where:

$$\mathfrak{w}_\pm^A = \bigoplus_{\substack{\alpha\in\Delta^+ \\ (\mu_\alpha, w_A)>0}} \mathbb{C}z_{\pm\alpha} \tag{2.31}$$

Eqs. (2.25), (2.28 *b*) and (2.30) show that $\mathfrak{g}^{\mathbb{C}}_{v_A}$ is a subalgebra of $(\mathfrak{g}_{w_A})^{\mathbb{C}} \oplus \mathfrak{w}^{A}_{+}$; this latter coincides with the Lie algebra of the isotropy group for the maximal weight vector ray:

$$\overline{\mathfrak{g}^{\mathbb{C}}_{v_M}} = (\mathfrak{g}_{w_M})^{\mathbb{C}} \oplus \mathfrak{w}^{M}_{+} \tag{2.32}$$

Indeed all $F^{(\alpha)}_{+}$ annihilate $| v_M \rangle$ (see eq. (2.19)) i. e. $\mathfrak{w}^{M}_{+} \subset \overline{\mathfrak{g}^{\mathbb{C}}_{v_M}}$ moreover if $F^{(\alpha)}_{3} | v_M \rangle = 0$ i. e. $(\mu_\alpha, w_M) = 0$, since $F^{(\alpha)}_{+} | v_M \rangle = 0$, from lemma 2 also $F^{(\alpha)}_{-} | v_M \rangle = 0$, so $(\mathfrak{g}_{w_M})^{\mathbb{C}} \subset \overline{\mathfrak{g}^{\mathbb{C}}_{v_M}}$.

Finally from eq. (2.32) it is straightforward to verify that $\overline{\mathfrak{g}}_{v_M} = \overline{\mathfrak{g}^{\mathbb{C}}_{v_M}} \cap \mathfrak{g} = \mathfrak{g}_{w_M}$. Given an irreducible representation of $\mathfrak{g}$, we denote by $\mathscr{I}_{w_M}$ the closed convex hull, in the Cartan subalgebra $\mathfrak{h}$, of the orbit of the maximal weight. For a generic degenerate weight w_A the computation of $\overline{\mathscr{G}^{\mathbb{C}}_{v_A}}$ depends on which vector v_A is chosen in the multidimensional eigenspace of the F (h), $h \in \mathfrak{h}$, corresponding to w_A. For a non degenerate weight w_A one has:

$$F(z_\alpha) | v_A \rangle \neq 0 \Leftrightarrow w_A + \mu_\alpha \in \mathscr{I}_{w_M} \tag{2.33}$$

So the smallest possible $\overline{\mathfrak{g}^{\mathbb{C}}_{v_A}}$ is $\mathfrak{h}^{\mathbb{C}}$ and is obtained when w_A is non degenerate and satisfies

$$\overline{\mathfrak{g}^{\mathbb{C}}_{v_A}} = \mathfrak{h}^{\mathbb{C}} \Leftrightarrow \mathscr{R}_{w_A} \subset \mathscr{I}_{w_M} \tag{2.34 a}$$

$$\mathscr{R}_{w_A} = \{ w_A + \mu_\alpha : \mu_\alpha \text{ is a root} \} \tag{2.34 b}$$

$\overline{\mathfrak{g}^{\mathbb{C}}_{v_A}}$ is larger for example when w_A is on the surface of $\mathscr{I}_{w_M}$. In general $\overline{\mathfrak{g}^{\mathbb{C}}_{v_A}}$ would reduce to $\mathfrak{h}^{\mathbb{C}}$ when w_A is in the interior of $\mathscr{I}_{w_M}$ and will grow when w_A is on a k-facet with decreasing dimension k, being maximum when $k = 0$ i. e. it is on a vertex which corresponds to a maximal weight. Note that many fundamental representations have only one weight orbit, so every weight can be maximal for a particular choice of the Weyl basis. These fundamental representations are all those of A_l, the representations with maximal weight w_l for B_l, w_1 for C_l, w_1, w_{l-1}, w_l for D_l, w_1 and w_5 for E_6, w_6 for E_7 (for the labelling of the weights see table 1). We will select again most representations of this list in the next section. Finally we point out an important case: when all components η_i of $w_M = \sum_{i=1}^{l} \eta_i w_i$ are $\neq 0$. Then $\overline{\mathfrak{g}^{\mathbb{C}}_{v_M}} = \mathfrak{b}_+$, the Borel subalgebra already defined.

3. THE MANIFOLD OF COHERENT STATES

Until now we have studied only the Lie algebra $\overline{\mathfrak{g}^{\mathbb{C}}_{v_A}}$ of the isotropy group $\overline{\mathscr{G}^{\mathbb{C}}_{v_A}}$ of a weight vector ray. In general several Lie groups have the same Lie algebra. However for a semisimple Lie algebra $\mathfrak{g}$ (or $\mathfrak{g}^{\mathbb{C}}$) there is a unique simply connected semisimple Lie group $\tilde{\mathscr{G}}$ (or $\tilde{\mathscr{G}}^{\mathbb{C}}$), the universal

covering group, whose Lie algebra is $\mathfrak{g}$ (or $(\mathfrak{g}^{\mathbb{C}})$. Its center is the finite group $\mathcal{Z}$ defined in eq. (2.18) [*9*].

As, by definition, $\mathcal{Z}$ turns to be the kernel of the adjoint representation of $\tilde{\mathcal{G}}$, one has $\tilde{\mathcal{G}}^{\mathbb{C}} \supset \tilde{\mathcal{G}}^{\mathbb{C}}_{v_A} \supset \mathcal{Z}$. Any other group with Lie algebra $\mathfrak{g}$ is the quotient $\tilde{\mathcal{G}}/\mathcal{F}$ where $\mathcal{F} \subset \mathcal{Z}$ and $\mathcal{F}$ is therefore invariant subgroup of $\tilde{\mathcal{G}}^{\mathbb{C}}_{v_A}$, $\tilde{\mathcal{G}}$, $\tilde{\mathcal{G}}^{\mathbb{C}}$. Since

$$\mathcal{G}^{\mathbb{C}}/\mathcal{G}^{\mathbb{C}}_{v_A} = \frac{\tilde{\mathcal{G}}^{\mathbb{C}}}{\mathcal{F}} \Big/ \frac{\tilde{\mathcal{G}}^{\mathbb{C}}_{v_A}}{\mathcal{F}} \tag{3.1}$$

the homogeneous space

$$\mathcal{M}_{v_A} = \mathcal{G}/\overline{\mathcal{G}}_{v_A} \tag{3.2}$$

is independent from the choice among the Lie groups having the same Lie algebra $\mathfrak{g}$.

Let us consider the case of maximal weight vector. We remark that the set of rays $\mathcal{S}$ of a (finite dimensional) Hilbert space is a compact manifold. The isotropy groups of rays corresponding to maximal weight vectors are maximal among the isotropy groups—as shown in sect. 2—so their orbit is closed [*10*] and compact (being contained in $\mathcal{S}$ compact). We can then apply a theorem of Montgomery [*11*] which yields:

$$\tilde{\mathcal{G}}^{\mathbb{C}}/\tilde{\overline{\mathcal{G}}}^{\mathbb{C}}_{v_M} = \tilde{\mathcal{G}}/\tilde{\mathcal{G}}_{w_M} = \mathcal{G}/\mathcal{G}_{w_M} = \mathcal{M}_{v_M} \tag{3.3}$$

since $\tilde{\mathcal{G}}^{\mathbb{C}}$ is simply connected and $\tilde{\mathcal{G}}$ is maximal compact in $\tilde{\mathcal{G}}^{\mathbb{C}}$ and $\overline{\mathcal{G}}^{\mathbb{C}}_{v_A} \cap \tilde{\mathcal{G}} = \tilde{\mathcal{G}}_{w_A}$ (see eqs. (2.29) and (2.30)).

We could also have considered the Iwasawa decomposition [*12*] (in the case of complex semisimple Lie algebra, see theorem 6.3 in ref. 12) $\mathfrak{g}^{\mathbb{C}}$ considered as a real algebra is the sum

$$(\mathfrak{g}^{\mathbb{C}})^{\mathbb{R}} = \mathfrak{g} \oplus \mathfrak{a} \oplus \mathfrak{n} \tag{3.4}$$

where $\mathfrak{a} = \operatorname{Im} \mathfrak{h}^{\mathbb{C}}$ and $\mathfrak{n} = \mathfrak{g}^{\mathbb{C}}_{+}$ has been defined in (2.13). Then for a maximal weight vector, with (2.29) and (2.30) we have

$$(\overline{\mathfrak{g}^{\mathbb{C}}_{v_M}})^{\mathbb{R}} = \mathfrak{g}_{w_M} \oplus \mathfrak{a} \oplus \mathfrak{n} \tag{3.5}$$

The Iwasawa decomposition is valid for any group of Lie algebra $\mathfrak{g}^{\mathbb{C}}$

$$\mathcal{G}^{\mathbb{C}} = \mathcal{G}\mathcal{A}\mathcal{N} \quad \text{with } \mathcal{N} \text{ invariant subgroup of } \mathcal{A}\mathcal{N} \tag{3.6}$$

and every $g \in \mathcal{G}^{\mathbb{C}}$ has a unique decomposition $g = kan$, $k \in \mathcal{G}$, $a \in \mathcal{A}$, $n \in \mathcal{N}$. So

$$\mathcal{M}_{v_M} = \mathcal{G}^{\mathbb{C}}/\overline{\mathcal{G}}^{\mathbb{C}}_{v_M} = \mathcal{G}\mathcal{A}\mathcal{N}/\mathcal{G}_{w_M}\mathcal{A}\mathcal{N} = \mathcal{G}/\mathcal{G}_{w_M} \tag{3.7}$$

Indeed if

$$g = kan$$

from $k \in k\mathcal{G}_{w_M}\mathcal{A}\mathcal{N}$ we obtain

$$g \in k\mathcal{G}_{w_M}\mathcal{A}\mathcal{N}an = k\mathcal{G}_{w_M}\mathcal{A}a\mathcal{N}n = k\mathcal{G}_{w_M}\mathcal{A}\mathcal{N}$$

so

$$g\overline{\mathscr{G}^{\mathbb{C}}_{v_A}} = k\mathscr{G}_{w_M}\mathscr{A}\mathscr{N}$$

i. e. there is a bijective correspondence between the left cosets of $\overline{\mathscr{G}^{\mathbb{C}}_{v_M}}$ in $\mathscr{G}^{\mathbb{C}}$ and those of $\mathscr{G}_{w_M}$ in $\mathscr{G}$.

We can apply these considerations to the case when all Dynkin indices of the representation are non zero. As we have in sect. 2, $\mathscr{M}_{w_M} = \mathscr{G}^{\mathbb{C}}/\mathscr{B} = \mathscr{G}/\mathscr{K}$. It can be shown that this a projective space [9].

Since $\mathscr{M}_{v_M}$ is one of them, we recall the properties of $\mathscr{G}$-orbits in the adjoint representation. By definition of Ad (a), one has

$$\operatorname{Ker} \operatorname{Ad}(a) = \mathfrak{g}_a, \qquad \operatorname{Im} \operatorname{Ad}(a) = \mathrm{T}_a(\mathscr{G}/\mathscr{G}_a) = \mathfrak{g}_a^{\perp} \tag{3.8}$$

where $b + \mathrm{T}_b(\mathscr{G}/\mathscr{G}_a)$ is the tangent space at b of the orbit $\mathscr{G}/\mathscr{G}_a$. The proof of the last equality reads: let $0 \neq a \wedge b \in \operatorname{Im} \operatorname{Ad}(a)$ and let $a \wedge c = 0$ i. e. $c \in \operatorname{Ker} \operatorname{Ad}(a)$; then $(a \wedge b, c) = -(b, a \wedge c) = 0$. The Cartan-Killing metric induces a non degenerate scalar product on each $\mathrm{T}_b(\mathscr{G}/\mathscr{G}_a)$ and therefore a Riemaniann structure on the homogeneous space $\mathscr{G}/\mathscr{G}_a$. We can also introduce a symplectic form on $\mathrm{T}_b(\mathscr{G}/\mathscr{G}_a)$ as follows:

$$\forall x, y \in \mathfrak{g}_b^{\perp} \qquad \sigma_b(x, y) = (b, x \wedge y) = -(x, b \wedge y) \tag{3.9}$$

It is not degenerated i. e. for any $x \in \mathfrak{g}_b^{\perp}$ it cannot vanish for all $y \in \mathfrak{g}_b^{\perp}$. Indeed the restriction of the Cartan-Killing form on $\mathfrak{g}_b^{\perp}$ is non degenerate so there exist $b \wedge y \in \operatorname{Im} \operatorname{Ad}(b) = \mathfrak{g}_b^{\perp}$ such that $0 \neq (x, b \wedge y) = -\sigma_b(x, y)$.

Hence $\sigma_b(x, y)$, $\forall b \in \mathscr{G}/\mathscr{G}_a$ introduces a symplectic structure on the orbit $\mathscr{G}/\mathscr{G}_a$. Incidentally this proves that

$$d_a = \dim \mathscr{G}/\mathscr{G}_a : \text{even} \tag{3.10}$$

Finally we can also establish that $\mathscr{G}/\mathscr{G}_a$ is a complex manifold of dimension $d_a/2$ carrying a $\mathscr{G}$-invariant Hilbert structure:

$$x, y \in \mathrm{T}_b(\mathscr{G}/\mathscr{G}_a) \qquad \langle x, y \rangle = (x, y) + i\sigma_b(x, y) \tag{3.11}$$

Therefore it is a Kählerian manifold. All these properties apply to $\mathscr{M}_{v_M} = \mathscr{G}/\mathscr{G}_{w_M}$, but they cannot be generalized to the orbit of all weight vectors. However this is still true for the weight vectors of non degenerate weight satisfying (2.34). In their case

$$\mathscr{M}_{v_A} = \mathscr{G}^{\mathbb{C}}/\mathscr{K}^{\mathbb{C}} = (\mathscr{G}/\mathscr{K})^{\mathbb{C}} = (\mathscr{G}/\mathscr{G}_x)^{\mathbb{C}} \tag{3.12}$$

where $\mathscr{G}_x$ is the centralizer of a regular group element.

From now on we consider only the case of maximal weight. Dynkin has given a useful rule for obtaining $\mathfrak{g}_{w_M}$ from the Dynkin diagram and the Dynkin indices η_i of the representation. The rule is to remove from the Dynkin diagram all vertices whose $\eta_i > 0$ and replace each of them by a U(1) algebra. The remaining vertices, whose $\eta_i = 0$, form a (in general

non connected) Dynkin diagram which correspond to a semisimple Lie algebra $\mathfrak{k}$ and one has:

$$\mathfrak{g}_{w_M} = \mathfrak{k} \dot{\oplus} (U(1))^k \qquad k = \text{number of } \eta_i > 0 \tag{3.13}$$

One can see that $\mathfrak{g}_{w_M}$ depends only on the $\eta_i = 0$. The $\mathfrak{g}_{w_M}$ of the fundamental representation $w_M = w_i$, or the irreducible component of their symmetrized tensor product of degree m ($w_M = mw_i$) are maximal subalgebras of $\mathfrak{g}$.

We want now select among the Kählerian manifolds $\mathscr{M}_{v_M}$ those which are Cartan symmetric spaces, i. e. they have a constant curvature and in each point have an isometric inversion which is induced by an involutive automorphism α of $\mathscr{G}$ on the orbit $\mathscr{G}/\mathscr{G}^\alpha$ where $\mathscr{G}^\alpha$ is the subgroup of fixed points of α. The corresponding automorphism of the Lie algebra simply multiplies by -1 the vectors of $(\mathfrak{g}^\alpha)^\perp = \mathfrak{m}$. So

$$\mathfrak{g} = \mathfrak{g}^\alpha \oplus \mathfrak{m} \quad a)\ \mathfrak{g}^\alpha \wedge \mathfrak{g}^\alpha \subset \mathfrak{g}^\alpha \quad b)\ \mathfrak{g}^\alpha \wedge \mathfrak{m} \subset \mathfrak{m} \quad c)\ \mathfrak{m} \wedge \mathfrak{m} \subset \mathfrak{g}^\alpha \tag{3.14}$$

In our case $\mathfrak{g}^\alpha = \mathfrak{g}_{w_M}$ and *a*), *b*) are always satisfied because $\mathfrak{g}^\alpha$ is the Lie algebra of an isotropy group. If equations (3.14) hold for the real algebra, they also hold for its complexification. It is easier to study *c*) in the complex version. From eq. (2.30) one sees that:

$$\mathfrak{m}^{\mathbb{C}} = \mathfrak{w}^M_- \oplus \mathfrak{w}^M_+ \tag{3.15}$$

Each summand is a subalgebra: this property is compatible with *c*) only if $\mathfrak{w}^M_\pm \wedge \mathfrak{w}^M_\pm = 0$, i. e. $\mathfrak{w}^M_\pm$ are Abelian subalgebras. For which maximal weights $w_M = \sum_{i=1}^{l} \eta_i w_i$ this necessary condition for *c*) is satisfied? We need first to recall the following lemma. If two basic roots μ_j, μ_k are not orthogonal (i. e. their vertices are connected by a segment in the Dynkin diagram) of the two non diagonal elements A_{jk} or A_{kj} of the Cartan matrix respectively equal to $2(\mu_j, \mu_k)/(\mu_j, \mu_j)$ and $2(\mu_j, \mu_k)/(\mu_k, \mu_k)$, one is equal to -1 (say A_{jk}), so eq. (2.12) for the reflection R_j applied to μ_k shows that $\mu_j + \mu_k$ is a root. Let μ_l be a vertex joined to μ_k (i. e. $(\mu_k, \mu_l) \neq 0$). By the same proof we show that $\mu_j + \mu_k + \mu_l$ is a root and so on... So if w_M has two non vanishing components, say η_j and η_k, denoting by L the line of the Dynkin diagram joining j to h, (these set L may be empty), we have that $\mu_\alpha = \mu_k + \sum_{h \in L} \mu_h$ is a root and $z_j \wedge z_\alpha \neq 0$ since $\mu_j + \mu_\alpha$ is also a root. Since $(\mu_\alpha, w_M) = (\mu_k, w_M) = \eta_k > 0$, z_α and z_k belong to $\mathfrak{w}^M_+$ and it is not Abelian. Hence a necessary condition for $\mathfrak{w}^M_+$ to be Abelian is that w_M has only one non vanishing component i. e. it is a multiple of a fundamental weight $w_M = nw_i$. In that case $\mathfrak{w}^M_+$ contains all z_β corresponding to roots

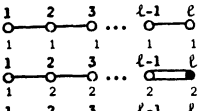
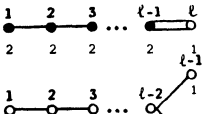
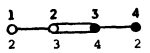
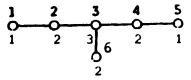
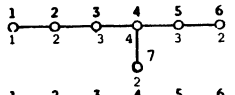
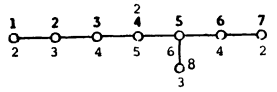
of the form $\mu_\beta = \mu_i + \mu_\alpha$ with $(\mu_\alpha, w_i) = 0$. A further conditions must be satisfied for abelianness

$$\mu_i + \mu_{i+\alpha} = 2\mu_i + \mu_\alpha \qquad \text{is not a root} \tag{3.16}$$

We recall that the positive roots are linear combination $\mu_\alpha = \sum_{i=1}^{l} c_i \mu_i$ of basic roots with non negative integer coefficients which are bounded by those γ_i of the maximal root μ_M (the maximal weight of the adjoint representation) : $c_i \leqslant \gamma_i$. The coefficients γ_i are given in many textbooks and are indicated in table 1. So eq. (3.16) is satisfied for the fundamental

TABLE 1.

Dynkin diagrams of simple Lie algebras with numbering of the vertices used in this paper above each vertex (black dots represent shorter roots). Below each vertex the components of the maximal root μ_M is given [*15*]. On the next column, the finite group $\mathcal{Q}$ (defined in eq. (2.18)) is given. Then the representations giving a symmetric space for $\mathcal{M}_{v_M}$ are listed. The last column gives the nature and the dimension of $\mathcal{M}_{v_M}$ when it is a symmetric space.

ALGEBRA		DIAGRAM	$\mathcal{Q}$	REPRESENTATIONS GIVING $\mathcal{M}_{v_M}$ SYMMETRIC	NATURE OF THE SYMMETRIC SPACE $\mathcal{M}_{v_M}$	COMPLEX DIMENSION OF $\mathcal{M}_{v_M}$
	A_ℓ		$Z_{\ell+1}$	$p = 1$ or $p = \ell$ $1 < p < \ell$	$SU(\ell+1)/U(\ell) = P_\ell \mathbb{C}$ $SU(\ell+1)/S(U(p) \times U(\ell+1-p))$	ℓ $p(\ell+1-p)$
$\ell > 1$)	B_ℓ		Z_2	$p = 1$	$SO(2\ell+1)/SO(2\ell-1) \times SO(2)$	$2\ell-1$
$\ell > 2$)	C_ℓ		Z_2	$p = \ell$	$Sp(2\ell)/U(\ell)$	$\frac{\ell(\ell+1)}{2}$
$\ell > 3$)	D_ℓ		even Z_2^2 odd Z_4	$p = 1$ $p = \ell-1$ or $p = \ell$	$SO(2\ell)/S(O(2\ell-2) \times O(2))$ $SO(2\ell)/U(\ell)$	$2(\ell-1)$ $\frac{\ell(\ell-1)}{2}$
	G2		I		———	——
	F4		I		———	——
	E6		Z_3	$p=1$ or $p=5$	$E_6/S(O(10) \times O(2))$	16
	E7		Z_2	$p = 1$	$E_7/E_6 \times U(1)$	27
	E8		I			

weights w_i corresponding to $\gamma_i = 1$. The list of such representation is given in table 1. We prove now that the condition that $\mathfrak{w}_{\pm}^{\mathrm{M}}$ be Abelian is also sufficient for $\mathscr{M}_{v_{\mathrm{M}}}$ to be symmetric. For a given i such that $\gamma_i = 1$ one has the decomposition:

$$\mathfrak{m}^{\mathbb{C}} = \bigoplus_{\substack{\alpha+i\in\Delta^+ \\ (\gamma, w_i)\ 0}} (\mathbb{C}z_{\alpha+i} \oplus \mathbb{C}z_{-\alpha-i}) \tag{3.17}$$

$\mathfrak{w}_{\pm}^{\mathrm{M}}$ Abelian implies $z_{\pm(\alpha+i)} \wedge z_{\pm(\beta+i)} = 0$. The Lie products such as $z_{i+\alpha} \wedge z_{-(i+\beta)}$, vanish except if $\mu_{i+\alpha} - \mu_{i+\beta} = \mu_{\alpha-\beta}$ is a root. But from eq. (3.17) μ_α and μ_β are orthogonal to w_i, so is their difference and then $z_{\alpha-\beta} \in (\mathfrak{g}_{w_{\mathrm{M}}})^{\mathbb{C}}$. Q. E. D.

In table 1 we list for each simple $\mathscr{G}$ the fundamental weights giving a symmetric space (whose complete dimension is also given).

The generalization from a simple to a semi-simple Lie group is straightforward. The corresponding orbit is the topological product of those for the simple components of the group.

4. PARAMETRIZATION OF THE MANIFOLD AND BARGMANN SPACES

We will give an explicit parametrization of the manifold $\mathscr{M}_{v_{\mathrm{M}}}$ constructing an holomorphic local chart and writing explicitly its metric structure. From eqs. (1.1) and (3.3) one has:

$$\mathscr{M}_{v_{\mathrm{M}}} = \pi\mathscr{T}(\mathscr{G}^{\mathbb{C}}) \mid v_{\mathrm{M}} \rangle = \pi\mathscr{U}(\mathscr{G}) \mid v_{\mathrm{M}} \rangle \tag{4.1}$$

where $\mathscr{T}$ denotes the holomorphic extension of the unitary representation of the group $\mathscr{G}$. The holomorphic $\mathscr{G}^{\mathbb{C}}$ orbit is obtained using the Gauss decomposition [*13*] of the $\mathscr{T}$-regular elements $g \in \mathscr{G}^{\mathbb{C}}$ (we will omit the $\mathscr{T}$ symbol in the following, unless specified):

$$g = z^- k z^+ \qquad \begin{array}{l} z^{\pm} \in \mathscr{G}_{\pm}^{\mathbb{C}} = \exp(\mathfrak{g}_{\pm}^{\mathbb{C}}) \\ k \in \mathscr{K}^{\mathbb{C}} = \exp(\mathfrak{h}^{\mathbb{C}}) \\ g \in \mathscr{G}_{\mathrm{reg}}^{\mathbb{C}} \end{array} \tag{4.2}$$

$\mathscr{G}_{\pm}^{\mathbb{C}}$, $\mathscr{K}^{\mathbb{C}}$ being the simply connected Lie subgroups corresponding to the Cartan decomposition (2.13) and $\mathscr{G}_{\mathrm{reg}}^{\mathbb{C}}$ denoting the subset of $\mathscr{G}^{\mathbb{C}}$ of $\mathscr{T}$-regular elements: a $\mathscr{T}$-regular element is defined by the requirement that all its principal minors in the $\mathscr{T}$ representation are non vanishing. As the set $\mathscr{G}_{\mathrm{reg}}^{\mathbb{C}}$ is open and everywhere dense in $\mathscr{G}^{\mathbb{C}}$, the set $\mathscr{G}^{\mathbb{C}} \backslash \mathscr{G}_{\mathrm{reg}}^{\mathbb{C}}$ of elements non Gauss decomposable has zero measure (the measure is the Haar invariant measure on $\mathscr{G}^{\mathbb{C}}$). By means of the Gauss decomposition (4.2) the Borel subgroup $\mathscr{B} = \exp(\mathfrak{b}_+)$ of the isotropy group $\mathscr{G}_{w_{\mathrm{M}}}^{\mathbb{C}}$ can be factorized out.

The complete factorization of the stability term of g can be obtained using the following further decomposition:

$$z^- = p^- o^- \qquad p^- \in \exp(\mathfrak{w}_-^{\mathrm{M}}),\ o^- \in \exp(\mathfrak{o}_-^{\mathrm{M}}) \tag{4.3 a}$$

$$\mathfrak{g}_-^{\mathbb{C}} = \mathfrak{w}_-^{\mathrm{M}} \oplus \mathfrak{o}_-^{\mathrm{M}} \qquad \mathfrak{o}_-^{\mathrm{M}} = \bigoplus_{\substack{\alpha\in\Delta^+ \\ (\mu_\alpha, w_{\mathrm{M}})=0}} \mathbb{C} z_\alpha \tag{4.3 b}$$

The decomposition (4.3) is possible as $\mathfrak{w}_-^{\mathrm{M}}$ and $\mathfrak{o}_-^{\mathrm{M}}$ are subalgebras decomposing $\mathfrak{g}_-^{\mathbb{C}}$ and $\mathfrak{w}_-^{\mathrm{M}}$ is an ideal [*13*]. So a local chart of $\mathscr{M}_{v_{\mathrm{M}}}$ in $\mathbb{C}^s$ ($s = \dim \mathfrak{w}_-^{\mathrm{M}}$, $\mathfrak{w}_-^{\mathrm{M}} = \mathrm{T}_{w_{\mathrm{M}}}(\mathscr{G}^{\mathbb{C}}/\mathscr{G}^{\mathbb{C}}_{w_{\mathrm{M}}})$) is given by identifying the coset $g\mathscr{G}^{\mathbb{C}}_{w_{\mathrm{M}}}$ ($g \in \mathscr{G}^{\mathbb{C}}_{\mathrm{reg}}$) with the vector ray:

$$|\zeta\rangle = \exp\left\{\sum_{\alpha\in\Delta^+} \zeta^\alpha \mathrm{F}_-^\alpha\right\} |v_{\mathrm{M}}\rangle \in \mathscr{S} \qquad \begin{array}{l} \zeta^\alpha \in \mathbb{C} \\ \zeta = \{\zeta^\alpha\} \end{array} \tag{4.4}$$

The chart defined by eq. (4.4) cannot cover the whole manifold $\mathscr{M}_{v_{\mathrm{M}}}$ as the cosets composed of non $\mathscr{T}$-regular elements $g\mathscr{G}^{\mathbb{C}}_{w_{\mathrm{M}}} \subset \mathscr{G}^{\mathbb{C}} \backslash \mathscr{G}^{\mathbb{C}}_{\mathrm{reg}}$ are not mapped. However these constitute a $\mathscr{M}_{v_{\mathrm{M}}}$-subvariety of lower dimension and the chart can be used as a domain of integration instead of a complete atlas, giving the useful property that the set of states $\{|\zeta\rangle, \zeta \in \mathbb{C}^s\}$ is overcomplete (see also the following). On the other hand an atlas can be obtained using the transitive action of the group on the homogeneous space $\mathscr{M}_{v_{\mathrm{M}}}$.

Explicitly, the Kählerian structure on $\mathscr{M}_{v_{\mathrm{M}}}$, is given in this coordinate system by:

$$ds^2 = 2\sum_{\alpha,\beta} g_{\alpha\bar\beta} d\zeta^\alpha d\bar\zeta^\beta \tag{4.5 a}$$

$$g_{\alpha\beta} = \partial^2 f / \partial\zeta^\alpha \partial\bar\zeta^\beta \tag{4.5 b}$$

and the function f:

$$f = \ln\left\{\frac{\langle\zeta|\zeta\rangle}{\langle v_{\mathrm{M}}|v_{\mathrm{M}}\rangle}\right\} \tag{4.5 c}$$

is positive definite because of the Schwartz inequality

$$\langle\zeta|\zeta\rangle \geqslant \langle\zeta|v_{\mathrm{M}}\rangle|^2/\langle v_{\mathrm{M}}|v_{\mathrm{M}}\rangle \qquad \text{and} \qquad \langle\zeta|v_{\mathrm{M}}\rangle = \langle v_{\mathrm{M}}|v_{\mathrm{M}}\rangle$$

so (4.5 *a*) is a Kählerian metric. The metric (4.5 *a*) provides a measure for $\mathscr{M}_{v_{\mathrm{M}}}$:

$$d\mu(\zeta,\bar\zeta) = \mathrm{N}\det(g_{\alpha\bar\beta}) d\zeta^1 \wedge \ldots \wedge d\zeta^s \wedge d\bar\zeta^1 \wedge \ldots \wedge d\bar\zeta^s \tag{4.6}$$

which can be normalized, by a suitable choice of $\mathrm{N} \in \mathbb{C}$, in such a way to obtain $\int_{\mathbb{C}^s} d\mu(\zeta,\bar\zeta) = 1$, as $\mathscr{M}_{v_{\mathrm{M}}}$ is compact.

One can also give a parametrization of $\mathscr{M}_{v_{\mathrm{M}}}$ by compact coordinates

using the restriction of the holomorphic representation $\mathscr{T}$ of $\mathscr{G}^{\mathbb{C}}$ to the unitary one $\mathscr{U}$ of $\mathscr{G}$ and identifying $\mathscr{M}_{v_M} = \mathscr{G}^{\mathbb{C}}/\mathscr{G}^{\mathbb{C}}_{w_M} = \mathscr{G}/\mathscr{G}_{w_M}$.

The relation between the two coordinate systems (the old ones, which are non compact, and the new which are compact) can be obtained Gauss —decomposing the $\mathscr{U}$-regular representatives of the coset $g\mathscr{G}_{w_M}$:

$\xi^\alpha, \zeta^\alpha, \vartheta^\alpha, \beta^i, \eta^\alpha \in \mathbb{C}$

$$g = \exp\left\{ \sum_{\substack{\alpha\in\Delta^+ \\ (\mu_\alpha, w_M)>0}} \left(\xi^\alpha F_-^{(\alpha)} - \bar{\xi}^\alpha F_+^{(\alpha)}\right) \right\} =$$

$$= \exp\left\{ \sum_{\substack{\alpha\in\Delta^+ \\ (\mu_\alpha, w_M)>0}} \zeta^\alpha F_-^{(\alpha)} \right\} \exp\left\{ \sum_{\substack{\alpha\in\Delta^+ \\ (\mu_\alpha, w_M)=0}} \vartheta^\alpha F_-^{(\alpha)} \right\}$$

$$\exp\left\{ \sum_{i=1}^{l} \beta^i F_3^{(i)} \right\} \exp\left\{ \sum_{\alpha\in\Delta^+} \eta^\alpha F_+^{(\alpha)} \right\} \tag{4.7}$$

Because of the uniqueness of the Gauss decomposition, one constructs a one to one correspondence between representatives of the $\mathscr{U}$-regular cosets in $\mathscr{G}/\mathscr{G}_{w_M}$ and $\mathscr{G}^{\mathbb{C}}/\mathscr{G}^{\mathbb{C}}_{w_M}$

$$\exp\left\{ \sum_{\substack{\alpha\in\Delta^+ \\ (\mu_\alpha, w_M)>0}} \left(\xi^\alpha F_-^{(\alpha)} - \bar{\xi}^\alpha F_+^{(\alpha)}\right) \right\} \leftrightarrow \exp\left\{ \sum_{\substack{\alpha\in\Delta^+ \\ (\mu_\alpha, w_M)>0}} \zeta^\alpha F_-^{(\alpha)} \right\} \tag{4.8}$$

In general the explicit relation $\xi^\alpha = \xi^\alpha(\zeta, \bar{\zeta})$ between the two coordinate systems (and the bounds for the compact coordinates ξ^α) is obtained solving the Baker-Campbell-Hausdorff problem for the group $\mathscr{G}^{\mathbb{C}}$.

Using the compact coordinate system the overcomplete set of coherent states (4.4) is written:

$$|\,\xi\,\rangle = \exp\left\{ \sum_{\substack{\alpha\in\Delta^+ \\ (\mu_\alpha, w_M)>0}} \zeta^\alpha(\xi, \bar{\xi}) F_-^{(\alpha)} \right\} |\,v_M\,\rangle$$

$$= K(\xi, \bar{\xi}) \exp\left\{ \sum_{\substack{\alpha\in\Delta^+ \\ (\mu_\alpha, w_M)>0}} \left(\xi^\alpha F_-^{(\alpha)} - \bar{\xi}^\alpha F_+^{(\alpha)}\right) \right\} |\,v_M\,\rangle \tag{4.9}$$

where the normalization $K(\xi, \bar{\xi})$ is given by:

$$|\,K(\xi, \bar{\xi})\,|^2 = \langle\, \xi \,|\, \xi \,\rangle \tag{4.10 a}$$

$$K(\xi, \bar{\xi}) = \exp\left\{ -\sum_{i=1}^{l} \beta^i(\xi, \bar{\xi})(w_M, \mu_i) \right\} \equiv \langle\, v_M \,|\, \mathscr{U}(g) \,|\, v_M \,\rangle^{-1} \tag{4.10 b}$$

Using the irreducibility and unitarity of the representation one can derive the completeness relation:

$$\mathbb{1} = \int_{\mathcal{M}_{v_M}} d\mu(\xi, \bar{\xi}) \,|\, \xi \times \xi \,| \tag{4.11 a}$$

$$d\mu(\xi, \bar{\xi}) = d_{\mathscr{E}} e^{-f(\xi,\bar{\xi})} d\xi \tag{4.11 b}$$

$d_{\mathscr{E}} = \dim(\mathscr{E})$ being the dimension of the Hilbert space of the representation and f given by (4.5 c).

Using the Kählerian structure of $\mathcal{M}_{v_M}$ and its compact parametrization it is possible to realize the Hilbert space $\mathscr{E}$ of the states as a space of holomorphic functions defined on $\mathcal{M}_{v_M}$ (more precisely on the open subset mapped by the local chart). For every $|\,\psi\,\rangle \in \mathscr{E}$, let us consider the function:

$$\psi(\xi) = \langle\, \xi \,|\, \psi \,\rangle \tag{4.12}$$

The function $\psi(\xi)$ is holomorphic by construction:

$$\frac{\partial}{\partial \bar{\xi}^{\alpha}} \psi(\xi) = \langle\, v_M \,|\, F_+^{(\alpha)} \exp\left\{ \sum_{\substack{\beta \in \Delta^+ \\ (\mu_\beta, w_M) > 0}} \left(\xi^\beta F_-^{(\beta)} - \bar{\xi}^\beta F_+^{(\beta)} \right) \right\} |\, \psi \,\rangle = 0 \tag{4.13}$$

In particular the function $v_M(\xi) = \langle\, \xi \,|\, v_M \,\rangle$ is everywhere constant on $\mathcal{M}_{v_M}$. Using the completeness relation (4.11) one can write the scalar product between two vectors $|\,\psi\,\rangle$, $|\,\varphi\,\rangle$ in the form of an integral over $\mathcal{M}_{v_M}$:

$$\langle\, \psi \,|\, \varphi \,\rangle = \int_{\mathcal{M}_{v_M}} d\mu(\xi, \bar{\xi}) \bar{\psi}(\xi) \varphi(\xi) \tag{4.14}$$

where to the « bra » $\langle\, \psi \,|$ corresponds the antiholomorphic function $\langle\, \psi \,|\, \xi \,\rangle = \bar{\psi}(\xi)$. On this Hilbert space of functions—which is the generalization of the Bargmann-Hilbert [*14*] space of entire functions on $\mathbb{C}$—the group $\mathscr{G}$ acts as follows (see eqs. (4.9) and (4.10 a)):

$$[\mathscr{U}(g')\psi](\xi) = \langle\, \xi \,|\, \mathscr{U}(g') \,|\, \psi \,\rangle = \frac{\langle\, v_M \,|\, \mathscr{U}^+(g) \mathscr{U}(g') \,|\, \psi \,\rangle}{\langle\, v_M \,|\, \mathscr{U}^+(g) \,|\, v_M \,\rangle} =$$

$$= \frac{\langle\, v_M \,|\, \mathscr{U}^+(g'^{-1} g) \,|\, v_M \,\rangle}{\langle\, v_M \,|\, \mathscr{U}^+(g) \,|\, v_M \,\rangle} \psi(g'^{-1}\xi) = \mu(g', \xi) \psi(g'^{-1}\xi) \tag{4.15}$$

where g belongs to the coset labelled by ξ and $g' \in \mathscr{G}$. The function $\mu(g', \xi)$ satisfies the group functional relation:

$$\mu(g''g', \xi) = \mu(g'', g'^{-1}\xi) \mu(g', \xi) \tag{4.16}$$

and on the isotropy group $\mathscr{G}_{w_M}$ one has:

$$\mu(s, \xi) = \langle\, v_M \,|\, \mathscr{U}(s) \,|\, v_M \,\rangle = \kappa(s) \qquad s \in \mathscr{G}_{w_M} \tag{4.17}$$

i. e. it gives a one dimensional representation of $\mathscr{G}_{w_M}$. We have shown in

eq. (3.12) that the group $\mathscr{G}_{w_M}$ is (modulo the division by a finite group in the center) a direct product of a semisimple Lie algebra $\mathfrak{k}$ and $(U(1))^k$ where k is the number of non vanishing w_M components. Since $\mathfrak{k}$ has no non trivial one dimensional representation, $\kappa(s)$ is simply the character of $(U(1))^k$ and then:

$$\kappa(s) = \exp\left\{ i \sum_{i=1}^{l} \alpha_i \eta_i \right\} \qquad \alpha_i \in \mathbb{R} \tag{4.18}$$

and the η_i's label the representation of $(U(1))^k$.

Thinking $\psi(\xi)$ as a function on $\mathscr{G}$ constant on $\mathscr{G}_{w_M}$, one can recognize in eq. (4.15) that the group acts on the Bargmann-Hilbert space as the representation induced by the unitary character κ of $\mathscr{G}_{w_M}$. We emphasizes that the manifold depends only on the zero components η_i; while this induced representation depends on the values of the non vanishing η_i.

5. ONE EXAMPLE

We illustrate the preceeding sections by a simple example. We will consider the compact group $\mathscr{G} = SU(l+1)$ in its natural $(l+1)$ dimensional representation with maximal weight $w_M = w_1$ (or $w_M = w_l$ for the contragradient representation). From table 1 we know that the group orbit of the highest weight vector is the projective space $P_l\mathbb{C}$ which is a Kählerian symmetric space. For $l = 1$ this manifold corresponds to coherent states for spin 1/2, and it is the sphere S^2. For SU(3) in quantum chromodynamics it can describe coherent states of quark colors.

As a complex algebra $su(l+1)$ corresponds to the simple algebra $\mathfrak{g}^{\mathbb{C}} = A_l = sl(l+1, \mathbb{C})$. In the representation that we consider all the weight vectors belong to the same orbit (and can be thought as maximal). All the weights $\neq w_M$ can be put in one-to-one correspondence with the positive roots μ_α not orthogonal to w_M:

$$| v_{A_n} \rangle = \mathrm{col}\,(0, \ldots, 1, \ldots, 0)_{l+1} \tag{5.1}$$

$$| v_M \rangle = \mathrm{col}\,(1, 0, \ldots, 0)_{l+1} \tag{5.2}$$

$$| w_{A_n} \rangle = w_M - \sum_{i=1}^{n} \mu_i \tag{5.3}$$

$$\Delta^{\pm} = \{ \pm (\alpha_i + \alpha_{i+1} + \ldots + \alpha_k); \qquad 1 \leqslant i \leqslant k \leqslant l \} \tag{5.4}$$

(col $(x_1, \ldots, x_k)_k$ denotes the k-dimensional column vector). With the choice of basis (5.1) and (5.2) one has:

$$\begin{aligned} &F_+^{(\alpha)} = E_{i,k+1} \qquad F_-^{(\alpha)} = E_{k+1,i} \qquad \mu_\alpha = \mu_i + \mu_{i+1} + \ldots + \mu_k \\ &F_3^{(\alpha)} = E_{i,i} - E_{k+1,k+1} \end{aligned} \tag{5.6}$$

where E_{ij} denotes the $(l+1) \times (l+1)$ matrix $(E_{ij})_{nm} = \delta_{in}\delta_{jm}$. The Lie algebra of the isotropy group of the highest weight vector ray is simply written:

$$\overline{\mathfrak{g}_{v_M}^{\mathbb{C}}} = \mathfrak{b}_+ \oplus \mathrm{span}_{\mathbb{C}} \left\{ F_-^{(\alpha)} ; \mu_\alpha = \sum_{k=i}^{i} \mu_k ; 2 \leqslant i \leqslant j \leqslant l \right\} \tag{5.7}$$

and the subalgebras:

$$\mathfrak{w}_\pm^M = \mathrm{span}_{\mathbb{C}} \left\{ F_\pm^{(\alpha)} ; \mu_\alpha = \sum_{i=1}^{k} \mu_i ; 1 \leqslant k \leqslant l \right\} \tag{5.8}$$

are Abelian, so giving a symmetric space for $\mathscr{M}_{v_M}$.

The compact and non compact parametrization of the manifold can be obtained using eqs. (4.4) and (4.7):

$$| \xi \rangle = \mathrm{col}\left(1, \frac{\xi}{\| \xi \|} \mathrm{tg}(\| \xi \|)\right)_{l+1} \tag{5.9 a}$$

$$| \zeta \rangle = \mathrm{col}(1, \zeta)_{l+1} \tag{5.9 b}$$

where the compact coordinates $\xi = \mathrm{col}(\xi_1, \ldots, \xi_l)_l$, $\xi_i \in \mathbb{C}$

$$\| \xi \| = \left(\sum_{i=1}^{l} | \xi_i |^2 \right)^{1/2} < \frac{\pi}{2},$$

are linked to the non compact ones $\zeta = \mathrm{col}(\zeta_1, \ldots, \zeta_l)_l$, $\zeta_i \in \mathbb{C}$ by the relation:

$$\zeta_i = \frac{\xi_i}{\| \xi \|} \mathrm{tg}(\| \xi \|) \tag{5.10}$$

which is the generalization to many dimensions of the stereographic projection of the sphere S^2 $\left(\xi = \frac{\vartheta}{2} e^{i\varphi}, \vartheta \text{ and } \varphi \text{ polar angles} \right)$ on the projective complex palne. The boundary of the chart (5.9) $\| \xi \| = \frac{\pi}{2}$ is a $(l-1)$ dimen-

sional manifold which corresponds to the non $\mathscr{U}$-regular representatives of cosets in $\mathscr{G}/\mathscr{G}_{w_M}$ given by:

$$\mathscr{U}(g) = \begin{vmatrix} 0 & -x^+ \\ x & (\mathbb{1}_l - xx^+) \end{vmatrix} \tag{5.11}$$

where $x = \mathrm{col}\,(x_1, \ldots, x_l)_l$ and $\| x \| = 1$.

The Bargmann space of the holomorphic functions on the bounded open domain of $\mathscr{M}_{w_M}$ corresponding to the chart (5.9), for this $l + 1$ dimensional representation of SU $(l + 1)$, is given by the linear polynomials in the compact coordinates or, for the compact ones by:

$$\psi(\xi) = \psi_0 + \sum_{n=1}^{l} \psi_n \xi_n \frac{\mathrm{tg}\,(\| \xi \|)}{\| \xi \|} \tag{5.12}$$

where:

$$\psi(\xi) = \langle \psi \mid \xi \rangle \tag{5.13}$$

$$| \psi \rangle = \sum_{n=0}^{l} \overline{\psi}_n \mid v_{A_n} \rangle \tag{5.14}$$

and we expand any such a function on the basis:

$$v_{A_n}(\xi) = \langle v_{A_n} \mid \xi \rangle = \begin{cases} 1 & \text{if} \quad n = 0 \\ \xi_n \dfrac{\mathrm{tg}\,(\| \xi \|)}{\| \xi \|} & \text{if} \quad 1 \leqslant n \leqslant l \end{cases} \tag{5.15}$$

With the action of the group $\mathscr{G} = \mathrm{SU}(l + 1)$ on $\psi(\xi)$ we have shown in the previous section that we obtain the representation induced by the natural representation of the U(1) in the Cartan subgroup which is generated by the root μ_1. Frobenius reciprocity theorem tells us the contents of the induced representation as the direct sum of irreducible representations: for instance, for SU(2), it is the direct sum with multiplicity one of all representations of spin J (Dynkin index 2J) such that $2\mathrm{J} - \eta$ is a non negative integer (where η is the Dynkin index of w_M).

ACKNOWLEDGMENTS

We whish to thank prof. M. Rasetti for stimulating this work and for useful discussions.

One of us (L. M.) is grateful to the Institute for Scientific Interchange for its hospitality.

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(Manuscrit reçu le 5 août 1985)