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Non-periodic solutions to relativistic field equations : Hyperbolic case

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ABSTRACT. — Non-periodic solutions to the Einstein field equations are obtained in a spacetime described by a cylindrically symmetric metric with reflection symmetry under the assumption that the source of gravitation is massless attractive scalar fields coupled with electromagnetic fields.

RÉSUMÉ. — On obtient des solutions non périodiques des équations d'Einstein dans un espace-temps décrit par une métrique à symétrie cylindrique avec symétrie de réflexion, sous l'hypothèse que la source de gravitation consiste en champs scalaires attractifs de masse nulle couplés à des champs électromagnétiques.

I. INTRODUCTION

Following Taub and Tabensky [*1*] who have shown that irrotational stiff fluids are equivalent to massless scalar fields through conformal transformations, Tabensky and Letelier [2] solved the relativistic field equations for massless scalar fields when the spacetime is described by the metric

$$ds^2 = e^{A}(dt^2 - dr^2) - e^{B}(r^2 d\theta^2 + dz^2) \tag{1}$$

where A and B are functions of r and t only. Using the same spacetime geometry, we show in Sec. II that when electromagnetic fields coupled with massless scalar fields are considered as source of gravitation, there are two possibilities: either one or four components of the electromagnetic field tensor are non-zero. In the latter case which has been taken up in this paper, the four components are derivable from two non-vanishing components of electromagnetic four-potential that are functionally related by a certain transformation. In Sec. III, a relation is established between the single degree of freedom inherent in the spacetime and the four-potential components. This relation helps us to find a particular solution of the field equations which we show in Sec. IV to be valid outside a hyperbola in the (r, t)-plane. In Sec. V, we obtain a solution to the Klein-Gordon equation of the scalar fields. In Sec. VI and VII, we study the nature of the electromagnetic fields and that of the gravitational and scalar fields respectively. In the concluding section, we mention some general observations regarding the nature of the solutions.

II. FIELD EQUATION

The field equations to be solved are

$$G_{ij} = - T_{ij} - \frac{1}{4\pi} T'_{ij} \tag{2}$$

where

$$T_{ij} = V_{,i}V_{,j} - (1/2)g_{ij}V^{,k}V_{,k} \tag{3 a}$$

and

$$T'_{ij} = - F_{is}F_{j}^{\ s} + (1/4)g_{ij}F^{kl}F_{kl} \tag{3 b}$$

are the scalar and the electromagnetic stress-energy tensor respectively. The sign convention considered is that of Bergmann [3] and the units are so chosen that the velocity of light is unity and the Newtonian constant of gravitation is $1/8\pi$. The electromagnetic field tensor F^{ij} and the scalar fields V satisfy the equations

$$F^{ij}_{\ ;j} = 0 \tag{4 a}$$

and

$$g^{ij}V_{;ij} = 0 \tag{4 b}$$

in regions where charge and current densities are absent. Semicolons represent covariant differentiation whereas commas denote ordinary partial differentiation. We introduce the characteristic coordinates

$$v = t + r \quad \text{and} \quad u = t - r \tag{5}$$

and let (v, θ, z, u) represent the coordinates (x^1, x^2, x^3, x^4). If A_i represent the electromagnetic four-potential, then

$$F_{ij} = A_{i,j} - A_{j,i} \tag{6}$$

and since A_i are independent of θ and z by virtue of cylindrical symmetry

$$F_{23} = 0 . \tag{7}$$

Five of eq. (2) corresponding to vanishing components of G_{ij} can be expressed in the compact forms

$$F_{14}F_{12} = F_{14}F_{13} = F_{14}F_{24} = F_{14}F_{34} = 0 \tag{8 a}$$

$$F_{13}F_{24} = - F_{12}F_{34} . \tag{8 b}$$

As a solution to eq. (8 *a*), we assume $F_{14} = 0$, reserving the other case for a separate paper. This solution and eq. (7) imply that the four-potential can be expressed in the form $A_i \equiv (0, M, N, 0)$. Letting the indices 1 and 4 represent differentiation with respect to v and u respectively in all the equations that follow eq. (4 *a*) become explicitly

$$M_{14} + (1/4r)(M_1 - M_4) = 0 \tag{9 a}$$

and

$$N_{14} + (1/4r)(N_4 - N_1) = 0 . \tag{9 b}$$

In terms of the new potential L defined by the non-singular transformations

$$L_1 = \frac{M_1}{r} \quad \text{and} \quad L_4 = -\frac{M_4}{r} \quad (r \neq 0) \tag{10}$$

eqs. (9 *a*) and (8 *b*) become

$$L_{14} + (1/4r)(L_4 - L_1) = 0 \tag{9 c}$$

and

$$L_1N_4 - L_4N_1 = 0 . \tag{11}$$

Eqs. (9 *b*), (9 *c*) and (11) give the following three cases.

CASE I. — $L_1 = L_4 = 0$; eqs. (9 *c*) and (11) are identities and field equations involve N only.

CASE II. — $N_1 = N_4 = 0$. Eqs. (9 *b*) and (11) are identities.

CASE III. — When L and N are not constants, eq. (11) implies a functional relation between L and N, since the left hand side of this equation is the Jacobian $\partial(L, N)/\partial(u, v)$. Eq. (9 *b*) and (9 *c*) then yield the relation

$$L = aN + b' . \tag{12}$$

In this equation and in what follows, all lower case latin letters except

r, t, u, v, x, y and z represent constants. The remaining field equations (2) can now be put in the forms

$$B_{ii}+\frac{1}{2}B_i^2-A_iB_i+\frac{1}{2r}(-1)^i(A_i-B_i)=-V_i^2-\frac{k^2}{4\pi}N_i^2e^{-B}(i=1,4), \quad (13\,a)$$

$$B_{14}+B_1B_4=\frac{1}{2r}(B_1-B_4)=-\frac{k^2}{2\pi}e^{-B}N_1N_4, \quad (13\,b)$$

and

$$2A_{14}+B_{14}=-2V_1V_4 \quad (13\,c)$$

where $k^2=a^2+1$. If we set in these equations $a=0$, we get the field equations for case I above. On the other hand, to get the field equations for case II, we solve eq. (12) for N, substitute in eqs. (13 *a*) to (13 *c*) and then set $1/a=0$. Therefore it is sufficient to solve the eqs. (13) and (4 *b*), the latter equation now taking the form

$$2V_{14}+\left(\frac{1}{2r}+B_1\right)V_4+\left(-\frac{1}{2r}+B_4\right)V_1=0. \quad (14)$$

III. HYPERBOLIC SOLUTIONS

To solve the non-linear field equations (13), we observe that one way to satisfy eq. (13 *b*) is to assume that e^B is a function of N. Then, on using eqn. (9 *b*), eq. (13 *b*) becomes equivalent to the pair

$$(e^B)_{NN}=-\frac{k^2}{4\pi}, \quad (15\,a)$$

and

$$N_1N_4(e^B)_{NN}=N_{14}(e^B)_N \quad (15\,b)$$

where the subscript N on e^B represents differentiation with respect to N. Integration of eq. (15 *a*) is immediate:

$$e^B=-\frac{k^2}{8\pi}N^2+bN+1. \quad (16)$$

Such types of relations between metric and electromagnetic potentials are found extensively in literature [*4*] [*5*]. Using eq. (16) in eq. (15 *b*), the latter equation can be integrated twice giving a solution of the form

$$-\frac{k^2}{4\pi}N+b=G(u)H(v) \quad (17)$$

where G and H are some functions of their arguments. The forms of these functions can be obtained by substituting eq. (17) in eq. (9 *b*) and integrating the resulting equation. In this way we obtain after readjusting constants,

$$-\frac{k^2}{4\pi}\mathrm{N} + b = \frac{kn}{[2\pi(2u-m)(2v-m)]^{1/2}}, \tag{18 a}$$

and

$$e^{\mathrm{B}} = 1 + \frac{2\pi b^2}{k^2} - \frac{n^2}{(2u-m)(2v-m)}. \tag{18 b}$$

These equations can now be substituted in eq. (13 *a*) and the latter can be integrated giving the result

$$e^{\mathrm{A}} = (\nabla_1 \nabla_4)^{3/4} \exp\left(-\frac{\mathrm{B}}{2} + \mathrm{I}\right) \tag{19 a}$$

where

$$\mathrm{I} = \int \nabla\left[({\mathrm{V}_1}^2/\nabla_1)dv + ({\mathrm{V}_4}^2/\nabla_4)du\right] \tag{19 b}$$

and

$$\nabla \equiv (v-u)e^{\mathrm{B}}. \tag{19 c}$$

The integral (19 *b*) exists because eq. (14), which can be written as

$$2\nabla \mathrm{V}_{14} + \nabla_1 \mathrm{V}_4 + \nabla_4 \mathrm{V}_1 = 0, \tag{20}$$

is the integrating condition for the existence of I. Eqs. (18) and (19) satisfy eq. (13 *c*) identically. Therefore these represent the complete solution of the field equations, once eq. (20) is solved for V.

IV. REGION OF VALIDITY

An inspection of eqs. (18 *a*) and (18 *b*) shows that they are valid if and only if

$$q^2 \equiv l + \frac{2\pi b^2}{k^2} \geqslant 0, \tag{21 a}$$

and

$$(2u-m)(2v-m) \geqslant \frac{n^2}{q^2}. \tag{21 b}$$

The hyperbolic 3-surface with the equation

$$q^2(2u-m)(2v-m) = n^2 \tag{22}$$

divides the spacetime into a region of validity of the solutions, satisfying the inequality (21 *b*) and into another region in which e^{B} would be negative, which is impossible. At the boundary of the region of validity

$$e^{\mathrm{B}} = 0 \quad \text{and when} \quad (u, v) \to (\pm\infty),\ e^{\mathrm{B}} \to q^2.$$

The family of hyperbolae

$$\left(t - \frac{m}{2}\right)^2 - r^2 = \frac{1}{4}p^2 > \frac{1}{4}q^2 \tag{23}$$

on the 2-surface θ = const., z = const. has the peculiarity that no member of the family can be the world line of a real particle, because as can be seen from the parametric representation,

$$t - \frac{m}{2} = \frac{1}{2} p \cosh \delta, \ r = \frac{1}{2} p \sinh \delta$$

of eq. (23), for no real choice of the parameter δ can the magnitude of the four-velocity of the particle be + 1 (Bergmann sign convention [3]).

V. SOLUTION TO THE SCALAR FIELD EQUATION

The scalar field equation (20) can be solved by writing it as a pair of equations given by

$$V_1 \nabla = E(v - u) + I(v), \ V_4 \nabla = E(v - u) + J(u) \tag{24}$$

where E, I and J are arbitrary functions of their arguments. A particular solution to this pair can be obtained by setting

$$I(v) = \nabla_1 \quad \text{and} \quad J(u) = \nabla_4 \tag{25}$$

and using standard methods of integration. The result, within the region of validity, appears as

$$V = \frac{E(2r)}{rq^2} \left(t + \frac{n^2}{2q^2 P} \log \frac{t - \frac{m}{2} - P}{t - \frac{m}{2} + P} \right) + \log \nabla + Q(r) \tag{26}$$

where

$$P^2 = r^2 + \frac{n^2}{4q^2} \tag{26 a}$$

and Q is arbitrary. Using eqs. (24) and (25), eq. (19 a) can be put in the form

$$e^A = \frac{(\nabla_1 \nabla_4)^{3/4}}{(v - u)} \exp\left(- \frac{3B}{2} + 2V + I' \right) \tag{27}$$

where

$$I' = \int \frac{E^2}{\nabla} \left(\frac{dv}{\nabla_1} + \frac{du}{\nabla_4} \right). \tag{27 a}$$

Thus eqs. (12), (18 a), (18 b), (26) and (27) represent a complete set of solutions to the field equations.

VI. THE ELECTROMAGNETIC FIELDS

The discussions in this and the following section are restricted to the region of validity of the solutions. The components of the electromagnetic field tensor, calculated with the aid of eqs. (6), (10), (12) and (18 *a*) are

$$F_{r\theta} = -\left(\frac{2\pi n^2}{k^2}\right)^{1/2} a \frac{r\left(t - \frac{m}{2}\right)}{\left\{\left(t - \frac{m}{2}\right)^2 - r^2\right\}^{3/2}},$$

$$F_{zr} = -\left(\frac{2\pi n^2}{k^2}\right)^{1/2} \frac{r}{\left\{\left(t - \frac{m}{2}\right)^2 - r^2\right\}^{3/2}},$$

$$F_{zt} = \left(\frac{2\pi n^2}{k^2}\right)^{1/2} \frac{t - \frac{m}{2}}{\left\{\left(t - \frac{m}{2}\right)^2 - r^2\right\}^{3/2}},$$

$$F_{\theta t} = -\left(\frac{2\pi n^2}{k^2}\right)^{1/2} a \frac{r^2}{\left\{\left(t - \frac{m}{2}\right)^2 - r^2\right\}^{3/2}}. \tag{28}$$

Correspondingly, the electromagnetic energy density and the Poynting vector components are given by

$$T'_{tt} = \pi n^2 \frac{\left(t - \frac{m}{2}\right)^2 + r^2}{\left\{\left(t - \frac{m}{2}\right)^2 - r^2\right\}^3} e^{-B} \tag{29 a}$$

$$T'_{rt} = -2\pi n^2 \frac{r\left(t - \frac{m}{2}\right)}{\left\{\left(t - \frac{m}{2}\right)^2 - r^2\right\}^3} e^{-B} \tag{29 b}$$

$$T'_{\theta t} = T'_{zt} = 0. \tag{29 c}$$

Thus the energy density is positive and therefore the field is physically viable. There is some flow of energy in the radial direction. The nature of both the energy density and energy flow at any event depends upon the nature of e^{-B} at that event. Within the region of validity, e^{-B} can be expanded binomially into a rapidly converging series. Therefore it follows that the energy density and energy flow get rapidly attenuated with distance and time. Moreover both are singular on the boundary of the region of validity. The nullity function defined [6] by

$$W = [(F_{ij}F^{ij})^2 + (F_{ij}F^{*ij})^2]^{1/2} \tag{30}$$

where

$$F^{*ij} = -(-g)^{1/2}\varepsilon^{ijkl}F_{kl} \tag{30 a}$$

are the dual electromagnetic field components [7], is given by

$$W = \frac{4\pi n^2(v-u)}{(\nabla_1\nabla_4)^{3/4}[(2u-m)(2v-m)]^2}\exp\left(\frac{B}{2} - I' - 2V\right) \tag{31}$$

In eq. (30 *a*) ε^{ijkl} represents the completely antisymmetric symbol having the values + 1, 0 or − 1. The electromagnetic field is thus non-null except on the axis of symmetry ($r = 0$) or at the boundary of the region of validity ($e^B \to 0$). It is also clear from eqs. (28) that it is non-uniform.

VII. THE NATURE OF THE SCALAR AND GRAVITATIONAL FIELD

The solution of the scalar field eq. (20) represented by eq. (26) shows that $V \to \infty$ as $r \to 0$. This is not the only solution of eq. (20) however; indeed, any arbitrary function of time satisfies it. In that case, eq. (27) is modified. The singularity of e^A is however not affected, as can be seen directly from eq. (19 *a*). Thus $e^A \to \infty$ as $r \to 0$ or $e^B \to 0$. The energy density of the massless scalar fields [8] is

$$\sigma = \frac{1}{2}V_t^2 = \left(\frac{E}{r}e^{-B} + B_t\right)^2 \tag{32}$$

which has singularities at $r \to 0$ and $e^B \to 0$.

According to Ref. [2], an irrotational stiff fluid distribution gives the same solutions as massless scalar fields provided the pressure and the density are given by

$$p = w = V_{,s}V^{,s}. \tag{33}$$

By contracting eq. (2), we see that the Ricci scalar $R = -V_{,s}V^{,s}$ is independent of the presence of the electromagnetic fields. From eqs. (24) and (25), we have

$$p = w = \frac{4(E + \nabla_1)(E + \nabla_4)}{(v-u)}e^{-A-2B} \tag{34}$$

which will be physically viable provided the arbitrary function E is so chosen as to make p and w positive. Eq. (34) shows that the quantity $p = w$ is singular as $r \to 0$ and $e^{B} \to 0$. From eq. (18 *b*) and (19), one can see that e^{B} has the form

$$e^{B} = \frac{1}{2r} S(t + r) + \frac{1}{2r} U(t - r) \tag{35}$$

which is equivalent to a superposition of cylindrical wave patterns in the forward and reverse directions, having the same wave shape.

VIII. CONCLUSION

It had been shown by Letelier [*2*] that in the presence of scalar fields alone the metric potential e^{B} is a linear function of time only; but the presence of an electromagnetic field modifies this result. This arises from the fact that e^{B} and the electromagnetic fields are related through eq. (13 *c*). The latter fields, however, get restricted owing to the nature of the spacetime assumed; the linear relation (12) arises due to the same fact. Two basic assumptions have been used; one of them viz., the transformations (10), however, do not unduly restrict the solutions, as long as they are non-singular. The second assumption that e^{B} is a function of N, is a more general way of satisfying eq. (13 *c*) than assuming for example that each side of that equation is equal to a constant. The presence of the electromagnetic fields limits the solution to a restricted region of spacetime; this restriction is independent of the presence of the scalar fields as can be seen by putting $V = 0$ in the field equations. Similarly, the relation (16) between the metric tensor component $g_{33} = - e^{B}$ and the electromagnetic potential component N is independent of the presence of the scalar fields. The solution (19 *a*) is similar to the one obtained by Letelier [*2*].

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