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The wave equation in random domains: Localization of the normal modes in the small frequency region

by

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ABSTRACT. — We consider two models of wave propagation in domains of R^d, $d = 2, 3$ with stochastic boundaries. We then show that the normal modes of sufficiently long wave length are exponentially localized and consequently that there is no wave propagation in the same frequency region. The techniques and main ideas needed for the proof are borrowed from the recent analysis of the Anderson localization for the Schroedinger equation made by J. Fröhlich, E. Scoppola, T. Spencer and the author.

RÉSUMÉ. — On considère deux modèles de propagation d'ondes dans des domaines de $\mathbb{R}^d$, $d = 2, 3$, avec des frontières aléatoires. On montre alors que les modes normaux d'assez grande longueur d'onde sont localisés exponentiellement et par conséquent qu'il n'y a pas de propagation d'ondes dans la même région de fréquences. Les techniques et les idées principales nécessaires pour la démonstration sont empruntées à l'analyse récente de la localisation d'Anderson pour l'équation de Schrödinger effectuée par J. Fröhlich, E. Scoppola, T. Spencer et l'auteur.

INTRODUCTION

In this paper we study two models of wave propagation in unbounded domains of R^d, $d = 2, 3$ whose boundaries are stochastic in a sense to be specified later. This problem is relevant in the analysis of waves guides and in hydrodynamics through the shallow water theory [*1*] and it belongs to the more general context of wave propagation in random media which has received in recent years increasing attention both by mathematicians and by physicists.

The most stricking phenomena for this kind of problems is the so called Anderson localization [*2*]. For large disorder of the medium or for some frequency interval:

i) the normal modes are exponentially localized

ii) there is no wave propagation.

Among the various models of wave propagation in random media the interest was concentrated on the Schroedinger equation with a random potential because of its relevance to the theory of disordered crystals expecially in connection with the metal-insulator transition [*3*] [*4*] [*5*].

For such an equation the rigorous discussion of the Anderson localization turned out to be a difficult problem and only recently it has been proved by J. Frohlich, F. Martinelli, E. Scoppola, T. Spencer [*6*] and subsequently by B. Simon and T. Wolff [*7*] and by F. Delyon, B. Souillard and Y. E. Levy [*8*]. Here we will follow the lines of [*6*] where the physical mechanisms leading to localization emerge in a rather transparent way.

The main point of the approach to the Anderson localization developed in [*6*] was the analysis of the quantum tunneling over long distances for typical configurations of the potential. This point of view was already present in the germinal paper [*9*] and it was subsequently considerably developed in [*10*] by G. Jona-Lasinio, F. Martinelli and E. Scoppola in their work on hierarchical potentials. These potentials which allow a rather direct and detailed analysis of the tunneling are the first multidimensional models for which the Anderson localization has been proved, and have represented a key intermediate step in the analysis of more realistic models like the Anderson model.

In this paper we combine the main ideas of [*6*] with some techniques developed in [*11*] and we present a rather general approach to the Anderson localization for the wave equation in random domains in the small frequency region. For similar results in the case of wave guides with boundary modulated by quasi periodic functions see [*12*].

SECTION I

DESCRIPTION OF THE MODELS AND MAIN RESULTS

We now describe precisely the random domains on which we will investigate the wave equation.

MODEL 1. — Let $\{ C_i \}$ $i \in Z^\nu$ $\nu = 1, 2$ be a paving of R^ν with unit squares centered at the sites of the lattice Z^ν with sides parallel to the coordinate axes and let $\{ h_i \}$ be i. i. d. random variables with common distribution:

$$P(dh) = g(h)dh$$

with $\sup g(h) < +\infty$ and $\operatorname{supp} g = [0, 1]$.

A configuration of the random variables $\{ h_i \}$ is an element of the probability space:

$$\Omega = \prod_{i \in \mathbb{Z}^\nu} \{ (0, 1), g(h_i)dh_i \} \tag{1.1}$$

and it will be denoted by ω.

Let now $\bar{h} > 1$ be fixed and define for a given $\omega = \{ h_i \}$ the set $D_\omega \subset R^d$, $d = \nu + 1$ as follows:

$$D_\omega = \{ \underline{x} = (\underline{y}, Z); \underline{y} \in \mathbb{R}^\nu, Z \in \mathbb{R}, h(\underline{y}) \leqslant Z \leqslant \bar{h} \} \tag{1.2}$$

$$\text{where } h(\underline{y}) = h_i \quad \text{if} \quad \underline{y} \in C_i .$$

In order to avoid ambiguities we assume the squares C_i to be half open. The domain D_ω can therefore be thought of as a slab with stochastic bottom.

MODEL 2. — (The stochastic worm).

Let $\{ R_i \}$ $i \in Z$ be i. i. d. random variables with common distribution:

$$dP(R) = \tilde{g}(R)dR$$

with $\operatorname{supp} \tilde{g} = [\bar{R}_1, \bar{R}_2]$, $0 < \bar{R}_1 < \bar{R}_2$ and $\sup \tilde{g}(R) < +\infty$.

As in model 1 we denote by $\omega = \{ R_i \}$ an element of the probability space:

$$\Omega = \prod_{i \in \mathbb{Z}} \{ [\bar{R}_1, \bar{R}_2], \tilde{g}(R_i)dR_i \} \tag{1.3}$$

Let now for a fixed $R > 0$ and $r_0 > 0$ $D^{\pm}_{R,r_0} \subset R^2$ be the half annuli given by:

$$D^{+}_{R,r_0} = \{ (\theta, \rho); 0 \leqslant \theta < \pi; R \leqslant \rho \leqslant R + r_0 \} \tag{1.4}$$

$$D^{-}_{R,r_0} = \{ (\theta, \rho); \pi < \theta \leqslant 2\pi; R \leqslant \rho \leqslant R + r_0 \} \tag{1.5}$$

and let $D^{\pm}_{R,r_0}(x) = D^{\pm}_{R,r_0} + x$, $x \in R$.

For a given configuration $\omega \in \Omega$ we define the random domain $D_\omega \subset \mathbb{R}^2$ as follows:

$$D_\omega = \bigcup_{i \in \mathbb{Z}} D^{(-1)^i}_{R_i,r_0}(x_i) \tag{1.6}$$

with $x_0 = 0$, $x_i = x_{i-1} + R_i + R_{i-1} + r_0$ for $i > 0$ and $x_i = x_{i-1} - R_{i-1} - R_i - r_0$ for $i < 0$ (see (Fig. 1).

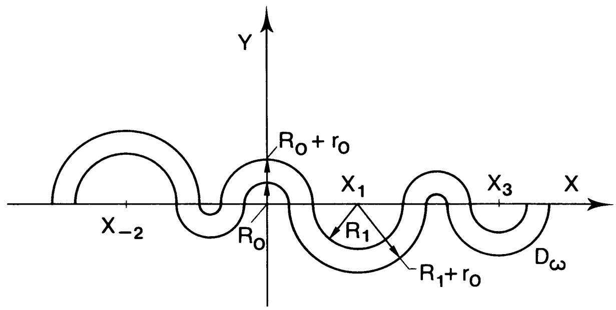

FIG. 1.

Let now D_ω be either defined by (1.2) or by (1.6); then in D_ω we consider the wave equation:

$$\begin{aligned} \partial_{tt}^2 u(x,t) &= \Delta u(x,t) \quad \underline{x} \in D_\omega \\ u(x,t) &= 0 \quad \underline{x} \in \partial D_\omega \\ u(x,0) &= u_0(x) \\ \partial_t u(x,t)\,|_{t=0} &= v_0(x) \end{aligned} \tag{1.7}$$

We assume that the initial data u_0 and v_0 are smooth functions in the interior of D_ω and vanish at ∂D_ω.

In order to analyze the long time behaviour of the solution of (1.7) it is natural to expand the solution $u(x, t)$ in terms of the normal modes of the equation. To do that let us denote by H_ω the operator $-\Delta$ on $L^2(D_\omega)$ with Dirichlet boundary conditions. H_ω is a selfadjoint operator which can be uniquely defined in terms of the corresponding quadratic form on the Sobolev space $H_0^1(D_\omega)$. The normal modes of the equation (1.7) are then the generalized eigenfunctions of H_ω. By this we mean the following:

DEFINITION. — A function φ on D_ω is said to be a generalized eigenfunction of the operator H_ω corresponding to be generalized eigenvalue $E(\omega)$ iff φ is a polynomially bounded solution of the equation:

$$H_\omega \varphi = E(\omega)\varphi$$

vanishing at the boundary ∂D_ω of D_ω.

The closure of the set $\{ E \in R; E \text{ is a generalized eigenvalue of } H_\omega \}$ gives the spectrum $\sigma(H_\omega)$ of H_ω (see [*13*]). It is easy to check using the ergodic theorem [*14*] that $\sigma(H_\omega)$ is almost surely a non random set Σ. Our first result is a characterization of Σ:

PROPOSITION 1.1. — *a*) In model 1 $\Sigma = [\lambda_0(\bar{h}), +\infty)$ where $\lambda_0(\bar{h}) = \pi^2/\bar{h}^2$

is the lowest eigenvalue of $-\dfrac{d^2}{dx^2}$ on $L^2([0, \overline{h}], dx)$ with Dirichlet boundary conditions.

b) In model 2 let $E_0(R)$ be the lowest eigenvalue of $-\dfrac{d^2}{dx^2} - \dfrac{1}{4x^2}$ on $L^2([R, R + r_0], dx)$ with Dirichlet boundary conditions. Then:

i) $$\inf\{\lambda; \lambda \in \Sigma\} = E_0(\overline{R}_1)$$

ii) $$\Sigma \supset [E_0(\overline{R}_1), E_0(\overline{R}_2)].$$

Remark. — *i*) The quantity $E_0(R)$ is easily seen to be a monotone increasing continuous function of R. Therefore $E_0(\overline{R}_2) > E_0(\overline{R}_1)$.

ii) It would be possible to investigate in more detail the structure of the set Σ in model 2 using the methods of [*15*]. However this will not be necessary for our purposes, since we will be interested only in the lowest part of the spectrum of H_ω.

We are now in a position to state our main results on the nature of the spectrum of H_ω near the left edge of Σ and therefore on the long time behaviour of the solution $u(x, t)$ for suitably chosen initial data. Let $E_0 = \inf \Sigma$ and let for $\eta > 0$, $I_\eta = [E_0, E_0 + \eta]$.

Main results

THEOREM 1.1. — Let D_ω be either defined by (1.2) or by (1.6) and let H_ω be the corresponding Dirichlet Laplacian. Then for $\eta > 0$ small enough there exists a set $\Omega_0 \subset \Omega$ with $P(\Omega_0) = 1$ such that if $E(\omega) \in I_\eta$, $\omega \in \Omega_0$, is an arbitrary generalized eigenvalue then the corresponding generalized eigenfunctions decay exponentially fast at infinity.

COROLLARY 1.1. — Let $P_\eta(H_\omega)$ be the spectral projection of H_ω associated to the interval I_η. Then for η small enough there exists a set $\Omega_0 \subset \Omega$ with $P(\Omega_0) = 1$ such that for any $\omega \in \Omega_0$ the following holds:

let $u_0 = P_\eta(H_\omega)\overline{u}_0$, $v_0 = P_\eta(H_\omega)\overline{v}_0$ with $\overline{u}_0$ and $\overline{v}_0$ elements of the space $C_0^\infty(D_\omega)$ and let $u(x, t)$ be the corresponding solution of equation (1.7). Then there exists a constant $C(\omega) > 0$ such that:

$$|u(x, t)| \leqslant C(\omega) \exp(-\sqrt{n}\,|x|^{4/5}) \qquad \forall t > 0, \quad \forall x \in D_\omega.$$

Remark. — The set of generalized eigenvalues in I_η depends on the particular chosen configuration ω although its closure does not as shown in proposition 1.1. In particular following the lines of [*16*] one proves that for η small enough a given energy E is not a generalized eigenvalue with probability one. This in turn implies that the part of the spectrum of H_ω inside I_η has no absolutely continuous component.

The strategy of our proof of theorem (1.1) and of Corollary 1.1 is organized in three steps as follows:

FIRST STEP. — One shows that if η is sufficiently small the random domain D_ω can be decomposed with probability one into two pieces: $D_\omega = D_\omega^0 \cup (D_\omega \backslash D_\omega^0)$ where D_ω^0 is the union of bounded subsets of D_ω very well isolated one from the others and it is such that the restriction of H_ω to $L^2(D_\omega \backslash D_\omega^0)$ with Dirichlet boundary conditions has its spectrum entirely above $E_0 + 2\eta$. Thus $D_\omega \backslash D_\omega^0$ behaves like a barrier for any energy $E \in I_\eta$.

SECOND STEP. — One studies the « tunneling » among the components of D_ω^0 for energies $E \in I_\eta$. Technically this is realized by analyzing the decay properties of the Green's function corresponding to energies in I_η of subsets of D_ω intersecting D_ω^0. This analysis is carried out by means of the inductive perturbation expansion developed in [*9*] and adapted to continuous systems in [*11*]. In particular one finds sufficient conditions on the configuration such that the tunneling over long distances is forbidden for all generalized eigenvalues in the spectral interval I_η.

THIRD STEP. — One shows that for η small enough the conditions found in the second step are satisfied with probability one.

The last two steps of our strategy can be carried out using the general machinery developed in [*6*] and are largely model independent with the exception of a probabilistic estimate, originally due to Wegner [*17*] (see also Lemma 2.4 in [*9*], Lemma 3.2 in [*11*]) which will be proved for the two models in consideration. Therefore our main result on the structure of the typical configurations for η small enough will be stated without proof in the next section. In section 3 we prove proposition 1.1 and Corollary 1.1 while in section 4 we prove the basic probabilistic estimates. Some technical results are collected in Appendices (A), (B) and (C).

SECTION 2

ANALYSIS OF THE TYPICAL CONFIGURATIONS AND PROOF OF THEOREM 1.1

In order to present a unified approach to the proof of our main result in the form of theorem 1.1 we first establish some useful notations.

Let 1 be a non negative integer and let $Z^\nu(1)$ be the lattice $(21+1)Z^\nu$. To each site $j \in Z^\nu(1)$ we associate a half open cube $B(j) \subset R^\nu$ centered at j with sides parallel to the coordinate axes of length $(21 + 1)$ and we will call it the block at site j. The block $B(j)$ is identified with a subset of the domain D_ω which we denote by $B(j, \omega)$ as follows:

a) In Model 1:

$$B_j^\omega = \{ \underline{x} = (\underline{y}, Z) \in \mathbb{R}^d ;\ \underline{y} \in B_j,\ h(\underline{y}) \leqslant Z \leqslant \bar{h} \} \tag{2.1}$$

using the notations of the previous section.

b) In Model 2:

$$B_j^\omega = \bigcup_{i \in \mathbb{Z} \cap B_j} D_{R_i(\omega), r_0}^{(-1)^i}(x_i) \tag{2.2}$$

Given a bounded set $\Lambda \subset \mathbb{Z}^\nu(l)$ we denote by $\Lambda_\omega = \bigcup_{j \in \Lambda} B_j^\omega$ and by $H_\Lambda(\omega)$, $H_\Lambda^N(\omega)$ the restriction of H_ω to $L^2(\Lambda_\omega)$ with Dirichlet and Neumann boundary conditions respectively on $\partial_1 \Lambda_\omega$, where $\partial_1 \Lambda_\omega$ is in Model 1 the part of the boundary of Λ_ω parallel to the z-axis and in Model 2 the part of the boundary of Λ_ω which lies on the x-axis. The selfadjoint operators $H_\Lambda(\omega)$, $H_\Lambda^N(\omega)$ have purely discrete spectrum with eigenvalues $\mu_K(H_\Lambda(\omega))$, $\mu_K(H_\Lambda^N(\omega))$, $k \geqslant 1$, counting multiplicity.

Let now $E_0 = \inf \Sigma$, Σ being the almost surely constant spectrum of H_ω and assume that $\Lambda \subset \mathbb{Z}^\nu(l)$ is such that:

$$\mu_1(H_{B_j}^N(\omega) > E_0 + 2\eta \qquad \forall j \in \Lambda \tag{2.3}$$

Then using the Dirichlet-Neumann bracketing (see Appendix A) we get:

$$\mu_1(H_\Lambda(\omega)) \geqslant \mu_1(H_\Lambda^N(\omega)) \geqslant \mu_1(\bigoplus_{j \in \Lambda} H_{B_j(\omega)}^N(\omega)) \geqslant E_0 + 2\eta \tag{2.4}$$

Next we compute the probability that:

$$\mu_1(H_{B_0}^N(\omega)) \leqslant E_0 + 2\eta \tag{2.5}$$

We have:

THEOREM 2.1. — There exists two positive constants β_1, β_2 such that if $l = [\beta_1/\sqrt{\eta}]$ then for all sufficiently small η:

$$P(\mu_1(H_{B_0}^N(\omega)) < E_0 + 2\eta) \leqslant \exp(-\beta_2 l^\nu)$$

The proof of this theorem will be given in section 4. We observe that a similar result has already been established in [*18*] in the context of random Schroedinger operators using large deviations probabilistic estimates. Our proof will follow closely the ideas of [*18*]. In the sequel the integer l will be kept fixed and equal to $[\beta_1/\sqrt{\eta}]$. Using the above result and (2.4) we obtain that for η small enough a large portion of the set D_ω will substained only energies above $E_0 + 2\eta$ and that the « singular blocks » where (2.3) is violated will from small clusters well isolated one from the other. Furthermore, using the Combes-Thomas argument [*13*] one shows that if $\Lambda \subset \mathbb{Z}^\nu(l)$ satisfies (2.3) and if:

$$G_\Lambda(E, \omega, x, y) = (H_\Lambda(\omega) - E)^{-1}(x, y) \tag{2.6}$$

denotes the Green's function of H_ω then:

$$|G_\Lambda(E, \omega, x, y)| \leqslant \exp(-m|x-y|) \qquad \forall E \in I_\eta \tag{2.7}$$

with $m = \sqrt{\eta}/2$ provided $|x-y| \geqslant 1/\sqrt{\eta}$.

We now turn to the analysis of the tunneling for energies $E \in I_\eta$ among the singular blocks $B_\omega(j)$ where (2.3) is violated. An important observation for this analysis is the following:

let $E(\omega) \in I_\eta$ be a generalized eigenvalue and Ψ be the corresponding generalized eigenfunction. Then for each, $\Lambda \subset \mathbb{Z}^\nu(l)$, Ψ is the unique solution of the Dirichlet problem:

$$\begin{aligned} -\Delta u &= E(\omega)u \qquad \text{in} \qquad \Lambda_\omega \\ u \upharpoonright_{\partial\Lambda_\omega \setminus \partial_1\Lambda_\omega} &= 0, \qquad u \upharpoonright_{\partial_1\Lambda_\omega} = \psi \end{aligned} \tag{2.8}$$

provided $E(\omega) \neq \mu_K(H_\Lambda(\omega))$, $\forall k$.

Therefore by Green's formula:

$$\psi(x) = \int_{\partial_1\Lambda_\omega} d\xi \psi(\xi) \partial_{n_\xi} G_\Lambda(E, \omega, x, \xi) \tag{2.9}$$

where $\partial_{n_\xi} G_\Lambda$ denotes the outward normal derivative of $G_\Lambda(E, \omega, x, \xi)$ at ξ. From (2.9) it is clear that the decay properties of ψ are strongly related to those of $G_\Lambda(E, \omega, x, y)$. To make this idea more precise we introduce the following definition:

DEFINITION. — A set $\Lambda \subset \mathbb{Z}^\nu(l)$ is said to be a k-barrier for the energy $E \in I_\eta$ iff:

i) $\qquad E \neq \mu_n(H\ (\omega\omega) \qquad \forall n$

ii) $\qquad |G_\Lambda(E, \omega, x, y)| \leqslant \exp(-m|x-y|)$

for some constant $m > \dfrac{\sqrt{\eta}}{2}$ and any $x, y \in \Lambda_\omega$ such that $|x-y| \geqslant ld_K/5$ where:

$$d_K = \exp\left(\beta(\eta)\left(\frac{5}{4}\right)^K\right) \tag{2.10}$$

$$\beta(\eta) \sim -\ln \eta$$

Let now Λ_K, $\tilde{\Lambda}_K$ be cubes in $Z^\nu(l)$ centered at the origin with sides of length $[8d_K]$, $[4d_K]$ respectively. Here $[.]$ denotes the integer part and we use the convention that any length in the lattice $Z^\nu(l)$ is measured in the natural length scale l of $Z^\nu(l)$. Then our basic result on the structure of the typical configurations ω which shows the absence of tunneling over long distances for energies in I_η reads as follows:

THEOREM 2.2. — For η small enough there exists a set Ω_0 of full measure such that if $\omega \in \Omega_0$ there exists an integer $k_0(\omega)$ and for any generalized eigenvalue $E(\omega) \in I_\eta$ an integer $K(E, \omega), \omega) \geqslant K_0(\omega)$ such that the following holds:

i) For any $k \geqslant k(E(\omega), \omega)$ the set $A_k = \Lambda_{K+1} \backslash \tilde{\Lambda}_K$ is a $(k\text{-}1)$ – barrier for $E(\omega)$.

ii) Let $\bar{k} \equiv K(E(\omega), \omega) - 1$; then the set $\tilde{\Lambda}_{\bar{K}}$ is a $(\bar{k}-1)$-barrier for $E(\omega)$, provided that $K \geqslant K_0(\omega)$.

Remark. — In the context of random Schroedinger operator this theorem was first proved for the hierarchical models introduced in [*10*] and subsequently for the Anderson model in [*6*]. The proof given in [*10*] was considerably simpler than the one given in [*6*] because of the hierarchical structure of the random potential.

As already anticipated in the introduction we will not give any detail of the proof of theorem 2.2 since it follows step by step the proof of the analogous result for the Anderson model given in [*6*] provided one replaces the inductive analysis of the discretized Green's functions of [*9*] [*6*] with its continuous version discussed in [*11*]. However at the basis of the probabilistic part of the proof there are two estimates which will be proved in section 4. These estimates depend on the specific model in consideration and are necessary in order to apply the general machinery developed in [*6*]. The first of these estimates has been established in theorem 2.1 while the second one allows us to estimate the probability that two separated regions Λ_1, Λ_2 of D_ω are in resonance in the sense that:

$$\text{dist}\,(\sigma(H_{\Lambda_1}(\omega)) \cap I_\eta, \qquad \sigma(H_{\Lambda_2}(\omega)) \cap I_\eta) \leqslant \varepsilon \tag{2.11}$$

We state it in the next lemma:

LEMMA 2.1. — Let $\Lambda \subset \mathbb{Z}^\nu(l)$ be a bounded set. Then for $E \leqslant I_\eta$, and $\varepsilon \leqslant \eta$:

$$P\,(\text{dist}\,(\sigma(H_\Lambda(\omega)), E) \leqslant \varepsilon) \leqslant \text{const.} \sqrt{\varepsilon}\, |\Lambda_\omega|^{\frac{\nu+2}{2}} \exp\left(-\frac{\beta_2}{2} l^\nu\right)$$

where $l = l(\eta)$ is as in theorem 2.1 and $|\Lambda_\omega|$ is the volume of the region Λ_ω.

Using now theorem 2.2 it is easy to complete the proof of theorem 1.1.

Let $\omega \in \Omega_0$ be fixed, Ω_0 as in theorem 2.2, let $E(\omega) \in I_\eta$ be a generalized eigenvalue of H_ω and ψ the corresponding eigenfunction. Let also $x \in D_\omega$ be given, $x \notin \Lambda_{K(E(\omega),\omega)}(\omega)$ where $k(E(\omega), \omega)$ is defined in theorem 2.2. We choose an integer k such that:

$$x \in A_k(\omega) \tag{2.12}$$

and

$$\text{dist}\,(x, \partial_1 A_k(\omega)) \geqslant \frac{1}{3}|x| \geqslant \frac{1}{5} d_{k-1} l \tag{2.13}$$

A simple geometric argument shows that such an integer always exists.

By applying (2.9) to the set A_K we get:

$$\psi(x) = \int_{\partial_1 \Lambda_{K+1}} d\xi \psi(\xi) \partial_{n_\xi} G_{A_K}(E(\omega), \omega, x; \xi)$$
$$+ \int_{\partial_1 \tilde{\Lambda}_K(\omega)} d\xi \psi(\xi) \partial_{n_\xi} G_{A_K}(E(\omega), \omega, x, \xi) \quad (2.14)$$

The normal derivative of the Green's function can be estimated by:

$$| \partial_{n_\xi} G_{A_K(\omega))}(E, \omega, x, \xi) | \leqslant \text{const} \exp\left(- \frac{m}{3} | x | \right) \quad (2.15)$$

for any $\xi \in \partial_1 A_K$ using theorem 2.2 (2.13) and the following technical lemma proved in [*11*]

LEMMA 2.2. — Let $\Lambda \subset \mathbb{Z}^\nu(l)$. Then for any $x \in \Lambda_\omega$ and $\xi \in \partial_1 \Lambda_\omega$ with $| x - \xi | \geqslant 2$ one has:

$$| \partial_{n_\xi} G_\Lambda(E, \omega, x, \xi) | \leqslant \text{const.} \sup_{|y-\xi| \leqslant 1} | G_\Lambda(E, \omega, x, y) |$$

The exponential decay of $\psi(x)$ can now be infered using (2.14), (2.15) and the polynomial boundedness of ψ.

SECTION 3

PROOF OF PROPOSITION 1.1 AND OF COROLLARY 1.1

Proof of Proposition 1.1. — *a*) By the Dirichlet-Neumann bracketing:

$$H_\omega \geqslant H_0 \quad (3.1)$$

in the sense of quadratic forms where H_0 corresponds to the configuration ω_0 in which each random variable h_i is set equal to zero. Therefore:

$$\inf(\lambda; \lambda \in \Sigma) \geqslant \inf(\lambda; \lambda \in \sigma(H_0)) = \lambda_0(\bar{h}) \quad (3.2)$$

It remains to show that any $E \geqslant \lambda_0(\bar{h})$ belongs to Σ. Let us fix $\delta > 0$ and $E \geqslant \lambda_0(\bar{h})$ and choose a cube $\Lambda \subset \mathbb{Z}^\nu$ so large that:

$$\text{dist}(E, \sigma(H_\Lambda(\omega_0))) < \delta \quad (3.3)$$

Since the spectrum of $H_\Lambda(\omega_0)$ can be computed exactly it is easy to see that such a cube always exists. Let us now assume that for all $j \in \Lambda$ one has:

$$0 \leqslant h_j(\omega) \leqslant \varepsilon \quad (3.4)$$

Under this assumption it follows that for all $\lambda > 0$:

$$\text{dist}\,(\sigma(H_\Lambda(\omega)) \cap [0, \lambda], \sigma(H_\Lambda(\omega_0)) \cap [0, \lambda]) \leqslant K(\lambda)\varepsilon \tag{3.5}$$

for a suitable constant $K(\lambda)$. The proof of (3.5) is given in Appendix B. From (3.5) and (3.3) it follows that:

$$\text{dist}\,(\sigma(H_\Lambda(\omega)), E) \leqslant \text{const.}\ \delta \tag{3.6}$$

for ε small enough.

To conclude the proof it is enough to use Weyl's criterium [*19*] (see also [*15*]) and the following standard result based on the ergodic theorem:

$$P\,(\text{there exists a box } \Lambda \subset \mathbb{Z}^\nu \text{ of side L such that } 0 \leqslant h_j \leqslant \varepsilon\, \forall j \in \Lambda) = 1 \tag{3.7}$$

$$\forall L < +\infty, \quad \forall \varepsilon > 0\,.$$

b) We proceed as in *a*). By the ergodic theorem for any $\varepsilon > 0$, any $L < +\infty$ and any $R \in [\overline{R}_1, \overline{R}_2]$:

P (there exists an interval Λ_L of length L in $\mathbb{Z}$ such that

$$| R_i - R | \leqslant \varepsilon \quad \forall i \in \Lambda_L) = 1 \tag{3.8}$$

In Appendix B we will prove in analogy with (3.5) the following result: assume that $| R_i - R | < \varepsilon\, \forall\, | i | \leqslant L$; then for any λ there exists a constant $K(\lambda)$ such that:

$$\text{dist}\,(\sigma(H_{\Lambda_L}(\omega)) \cap [0, \lambda], \sigma(H_{\Lambda_L}(\omega_R)) \cap [0, \lambda]) \leqslant K(\lambda)\varepsilon \tag{3.9}$$

where $\omega_R = \{ R_i = R \}_{i \in \mathbb{Z}}$.

From the Weyl's criterium, (3.8) and (3.9) we obtain (see [*15*] for details):

$$\Sigma \supset \bigcup_{\overline{R}_1 \leqslant R \leqslant \overline{R}_2} \sigma(H_{\omega_R}) \quad . \tag{3.10}$$

We are left with the problem of showing that:

$$\inf \Sigma = E_0(\overline{R}_1) \tag{3.11}$$

and

$$\bigcup_{\overline{R}_1 \leqslant R \leqslant \overline{R}_2} \sigma(H_{\omega_R}) \supset [E_0(\overline{R}_1), E_0(\overline{R}_2)] \tag{3.12}$$

Actually (3.11) is a consequence of (3.12). In fact by the Dirichlet-Neumann bracketing:

$$H_\omega \geqslant \bigoplus_i H_i^N(\omega) \tag{3.13}$$

in the notations of section 2. Therefore:

$$\inf \Sigma \geqslant \inf_i \inf \sigma(H_i^N(\omega)) \tag{3.14}$$

Using the polar coordinates (σ, ρ) centered in $x=x_i$, $H_i^N(\omega)$ is easily seen to be unitarily equivalent to:

$$-\frac{d^2}{d\rho^2} - \frac{1}{4\rho^2} - \frac{1}{\rho_2}\frac{d^2}{d\theta^2} \tag{3.15}$$

on $L^2([R_i, R_i + r_0] \times [0, \pi])$ with Dirichlet boundary conditions at $\rho = R_i$ $\rho = R_i + r_0$ and Neumann boundary conditions at $\theta = 0, \pi$. Thus the lowest eigenvalue of $H_i^N(\omega)$ coincides with $E_0(R_i)$ and thus, using (3.14):

$$\inf \Sigma \geqslant E_0(\overline{R}_1) \tag{3.16}$$

To prove (3.12) we observe that, using the explicit representation (3.15) of the operator H_{ω_R} in the local polar coordinates of each $D_{R,r_0}^{(-1)^i}(x_i)$ it is easy to check that:

$$\inf \sigma(H_{\omega_R}) = E_0(R) \tag{3.17}$$

Thus (3.12) follows from the monotonicity and continuity of $E_0(R)$, $R \in [\overline{R}_1, \overline{R}_2]$.

Proof of Corollary 1.1. — We have to estimate a time evolution and it is natural to expand the solution $u(x, t)$ of the wave equation (1.7) in terms of the generalized eigenfunctions of H_ω. To make this precise we recall for reader's convenience the following general result [*13*]:

THEOREM 3.1. — Let $\alpha > \frac{\nu + 1}{2}$; then there exists a spectral measure $d\rho_\omega^\alpha$ of H_ω such that for any Borel bounded function $g: \mathbb{R} \to \mathbb{R}$

$$g(H_\omega)(x, y) = \int d\rho_\omega^\alpha(E) g(E) F(x, y, \omega, E) \tag{3.18}$$

where $F(x, y, \omega, E)$ is given for almost all E with respect to $d\rho_\omega^\alpha$ by:

$$F(x, y, \omega, E) = (1+|x|^2)^{\alpha/2}(1+|y|^2)^{\alpha/2} \sum_{j=1}^{N(E)} f_j(x) f_j(y) \tag{3.19}$$

Here the functions $\{f_j\}$ are orthogonal functions in $L^2(D_\omega)$ with $\sum_{j=1}^{N(E)} \| f_j \|^2 = 1$ such that the new functions:

$$\varphi_j(x) = (1 + |x|^2)^{\alpha/2} f_j(x) \tag{3.20}$$

are solutions of the equation: $-\Delta\varphi_j = E\varphi_j$.

Using (3.20) and the Harnack inequality (see [*13*]) it is easy to see that the φ_j's are actually polynomially bounded and therefore are generalized eigenfunctions of H_ω. The number N(E) counts the multiplicity.

Let now for $\overline{u}_0, \overline{v}_0 \in C_0^\infty(D_\omega)$, u_0, v_0 given by:

$$u_0(x) = P_\eta(H_\omega)\overline{u}_0(x) = \int_{I_\eta} d\rho_\omega^\alpha(E) \sum_{j=1}^{N(E)} \varphi_j(x) \int dy \overline{u}_0(y)\varphi_j(y) \tag{3.21}$$

$$v_0(x) = P_\eta(H_\omega)\overline{v}_0(x) = \int_{I_\eta} d\rho_\omega^\alpha(E) \sum_{j=1}^{N(E)} \varphi_j(x) \int dy \overline{v}_0(y)\varphi_j(y) \tag{3.22}$$

where $P_\eta(H_\omega)$ is the spectral projection of H_ω associated to I_η. Then we will show that:

$$u(x, t) = \int_{I_\eta} d\rho_\omega^\alpha(E) \cos(t\sqrt{E}) \sum_{j=1}^{N(E)} \varphi_j(x) \int dy \varphi_j(y)\overline{u}_0(y) +$$

$$+ \int_{I_\eta} d\rho_\omega^\alpha(E) \frac{1}{\sqrt{E}} \sin(t\sqrt{E}) \sum_{j=1}^{N(E)} \varphi_j(x) \int dy \varphi_j(y) \overline{v}_0(y) \tag{3.23}$$

is the unique solution of (1.7) and satisfies the stated bound. This follows from theorem 1.1 and from the next estimate on the multiplicity N(E):

LEMMA 3.1. — Let Ω_0 and η be as in theorem 1.1 and let $E = E(\omega) \in I_\eta$ be a generalized eigenvalue of H_ω, $\omega \in \Omega_0$. Then:

$$N(E(\omega), \omega) \leqslant \text{const. } d^\nu_{K(E(\omega),\omega)+1}$$

where $K(E(\omega), \omega)$ is the integer defined in theorem 2.2.

The lemma will be proved in Appendix C. Using the above bound and the results of theorems 1.1, 2.2 we obtain the following basic estimate.

PROPOSITION 3.1. — Let $f: I_\eta \to R$ be bounded and let for $\omega \in \Omega_0$ $g \in C_0^\infty(D_\omega)$. Then there exists an integer $\overline{K}(\omega)$ such that:

$$\int_{I_\eta \cap \{E;\, K(E,\omega) \geqslant K\}} d\rho_\omega^\alpha(E) \mid f(E) \mid \sum_{j=1}^{N(E)} \mid \varphi_j(x) \mid \left| \int dy \varphi_j(y) g(y) \right|$$
$$\leqslant \exp(-m d_K/\sqrt{\eta}) \qquad \forall K \geqslant \overline{K}(\omega) \quad \forall x \in D_\omega$$

and some $m \geqslant \sqrt{\eta}/2$.

The proof of this proposition follows word by word the proof of theorem 5 in [*10*] and it is therefore omitted. However it can be understood in simple terms as follows: if we consider a generalized eigenvalue E such that the corresponding integer $k(E) = k(E(\omega), \omega)$ is very large then, according to theorem 2.2, the box $\tilde{\Lambda}_{K(E)-1}$ will be a $(k(E)-1)$-barrier for E. Therefore, using (2.9) all the eigenstates φ_j's corresponding to E will be exponentially small in the same box which in turn implies that for g of compact support:

$$\left| \int dy g(y) \varphi_j(y) \right| \leqslant \exp\left(- m d_{K(E)-1/\sqrt{\eta}}\right) \tag{3.24}$$

(3.24) together with lemma 3.1 implies the proposition.

We are now in a position to complete the proof of Corollary 1.1. Using the proposition and lemma 3.1 we get that $u(x, t)$ defined by (3.23) satisfies:

i) $\lim_{t \to 0} u(x, t) = u_0(x)$ uniformly in x

ii) $\lim_{t \to 0} \partial_t u(x, t) = v_0(x)$ uniformly in x

Furthermore $u(x, t)$ is easily seen to satisfy the wave equation (1.7). It remains to show the decay in x uniformly in t.

Let us fix x with $|x|$ very large and let $k(x)$ be such that $x \in \Lambda_{K+1}^{(\omega)} \backslash \tilde{\Lambda}_K(\omega)$ Λ_K, $\tilde{\Lambda}_K$ as in section 2. Then we divide the integral in (3.23) into two pieces:

$$\int_{I_\eta \cap \{E;\, K(E,\omega) \leqslant K(x)-1\}} d\rho_\omega^\alpha(\ldots) + \int_{I_\eta \cap \{E;\, K(E,\omega) \geqslant K(x)\}} d\rho_\omega^\alpha(\ldots) \tag{3.25}$$

Using Lemma 3.1 and (2.14) the first term is bounded uniformly in t by:

$$\text{const. } \exp\left(-\frac{m}{2}|x|\right) \tag{3.26}$$

while the second term is estimated using the proposition by:

$$\text{const. } \exp\left(-\frac{m}{2}|x|^{4/5}\right) \tag{3.27}$$

uniformly in t.

In (3.27) we have used the fact that, by definition, $d_{K+1} = d_K^{5/4}$. The proof of the Corollary 1.1 is now complete.

SECTION 4

PROOF OF THE PROBABILISTIC ESTIMATES

In this section we give the proof of theorem 2.1 and of lemma 2.1.

Proof of theorem 2.1. — We first fix a length scale 1 and estimate from below the first eigenvalue $\mu_1(H_{B_0}^N(\omega))$ of $H_{B_0}^N(\omega)$ for a given configuration

$\omega \in \Omega$. Let $\{ C_i \}_{i \in Z^\nu \cap B_0}$ be the unit squares around the sites of $Z^\nu \cap B_0$ and let:

$$\mu_i \equiv \mu_1(H^N_{C_i}(\omega)) \tag{4.1}$$

Then for any $0 < \lambda < 1$ we have:

$$H^N_{B_0}(\omega) \geqslant (1-\lambda)H^N_{B_0}(\omega) + \lambda \sum_{i \in \mathbb{Z}^\nu \cap B_0} \mu_i \chi_{C_i^\omega} \tag{4.2}$$

in the sense of quadratic forms, where $\chi_{C_i^\omega}$ is the characteristic function of the block C_i^ω given by (2.1). In Model 1, using the Dirichlet-Neumann bracketing, we also have:

$$(1-\lambda)H^N_{B_0}(\omega) + \lambda \sum_i \mu_i \chi_{C_i^\omega} \geqslant (1-\lambda)H^N_{B_0}(\omega_0) + \lambda \sum_i \mu_i \chi_{C_i^\omega} \tag{4.3}$$

where $\omega_0 = \{ h_i = 0 \}$.

In Model 2, using the representation of H_ω in polar coordinates (3.15), we have:

$$(1-\lambda)H^N_{B_0}(\omega) + \lambda \sum_i \mu_i \chi_{C_i^\omega} \geqslant$$

$$\geqslant (1-\lambda)\left\{ -\frac{d^2}{d\rho^2} - \frac{1}{4(\rho + \overline{R}_1)^2} - \frac{1}{(\rho + R_2)^2}\frac{d^2}{d\theta^2} \right\} +$$

$$+ \lambda \sum_i \mu_i \chi_{\{0 \leqslant \rho \leqslant r_0 ; (i-1)\pi \leqslant \theta \leqslant i\pi\}} \tag{4.4}$$

where the second operator in the r. h. s. of (4.4) acts on

$$\bigoplus_{i \in Z^\nu \cap B_0} L^2([0, r_0] \times [(i-1)\pi, i\pi])$$

with Dirichlet boundary conditions at $\rho = 0, r_0$ and Neumann boundary conditions at $\theta = l\pi, -l\pi$. In conclusion we have shown that if:

$$\mu_1(H^N_{B_0}(\omega)) < E_0 + 2\eta \tag{4.5}$$

then also the lowest eigenvalue of the operators appearing in the r. h. s. of (4.3), (4.4) are less than $E_0 + 2\eta$.

We now observe that both these operators can be written as:

$$(1-\lambda)H_0 + \lambda \sum_i \mu_i \chi_{C_i} \tag{4.6}$$

with:

i) $\mu_1(H_0) = E_0$ in both models.

ii) $\inf_\omega \mu_i(\omega) = E_0$ in both models.

Thus, by adding and subtracting E_0 in (4.6) we have reduced the proof of the theorem to the estimate of:

$$P\left(\mu_1((1-\lambda)(H_0-E_0)+\lambda\sum_i(\mu_i-E_0)\chi_{C_i})\leqslant 2\eta\right) \tag{4.7}$$

We now take $\lambda = 1/2$ for concreteness and observe that the estimate of (4.7) falls into the cases considered in [18] for the Schrodinger operator:

$$H_\omega = -\Delta + V_\omega \qquad \text{on} \qquad L^2(\mathbb{R}^\nu) \tag{4.8}$$

with $V_\omega(x) = v_i\ \forall x \in C_i,\ i \in \mathbb{Z}^\nu,\ v_i \geqslant 0$ and $\inf_\omega v_i(\omega) = 0$, for which theorem 2.1 was established.

Proof of lemma 2.1. — We proceed as in the original paper by Wegner [17] (see also [6] and [11]). Using theorem 2.1 for $\varepsilon \leqslant \eta$ and $E \in I_\eta$ we have:

$$P(\text{dist}(\sigma(H_\Lambda(\omega)), E) \leqslant \varepsilon) \leqslant P(\mu_1(H_\Lambda(\omega)) \leqslant E_0 + 2\eta)$$
$$\leqslant |\Lambda|\, P(\mu_1(H^N_{B_0}(\omega)) \leqslant E_0 + 2\eta) \leqslant |\Lambda| \exp(-\beta_2 l^{-\nu/2}) \tag{4.9}$$

with $l = l(\eta)$ as in theorem 2.1.

Let now:

$$N_\Lambda(E, \omega) \equiv \#\{K \in \mathbb{N}; \mu_K(H_\Lambda(\omega)) \leqslant E\} \tag{4.10}$$

Then:

$$P(\text{dist}(\sigma(H_\Lambda(\omega)), E) \leqslant \varepsilon) \leqslant \int dP(\omega)\{N_\Lambda(E+\varepsilon, \omega) - N_\Lambda(E-\varepsilon, \omega)\} =$$
$$= \int dP(\omega) \int_{|E'-E| \leqslant} dE' \frac{d}{dE'} N_\Lambda(E', \omega) \tag{4.11}$$

We will estimate the (formal derivative) of $N_\Lambda(E', \omega)$ in terms of the derivative with respect to the random variables $\{h_i\}$ and $\{R_i\}$ respectively in Model 1 and in Model 2. To do that we use the next two lemmas which will be proved in Appendix B.

LEMMA 4.1. — Let $1 = 0$ and let in Model 1 for any configuration $\omega = \{h_i\}$ ω_ε be the configuration corresponding to $\{h_i + \varepsilon\}$, $\varepsilon > 0$. Then if $\Lambda \subset \mathbb{Z}^\nu$ is bounded:

$$(1-\varepsilon)\mu_K(H_\Lambda(\omega_\varepsilon)) \leqslant \mu_K(H_\Lambda(\omega)) \leqslant \mu_K(H_\Lambda(\omega_\varepsilon)) - \frac{2\pi^2}{\bar{h}^3}\varepsilon$$

LEMMA 4.2. — In Model 2 let for $\omega = \{R_i\}$, ω_ε be equal to $\{R_i + \varepsilon\}$. Then for Λ bounded in Z and provided that $\mu_K(H_\Lambda(\omega)) < E_0 + 3\eta$

$$\mu_K(H_\Lambda(\omega_\varepsilon)) - C_1\varepsilon \leqslant \mu_K(H_\Lambda(\omega)) \leqslant \mu_K(H_\Lambda(\omega_\varepsilon)) + C_2(\eta)\varepsilon$$

for suitable constants $C_1(\eta)$, $C_2(\eta)$. Furthermore $C_2(\eta) \leqslant -\dfrac{1}{4R_2^3}$ for η small enough.

Using the above lemmas we get that for η small enough:

$$\frac{d}{dE'} N_\Lambda(E', \omega) \leqslant - \text{const.} \sum_{i \in \Lambda \cap \mathbb{Z}^\nu} \frac{\partial}{\partial h_i} N_\Lambda(E', \omega) \tag{4.12}$$

in Model 1 and:

$$\frac{d}{dE'} N_\Lambda(E', \omega) \leqslant - \text{const.} \sum_{i \in \Lambda \cap \mathbb{Z}} \frac{\partial}{\partial R_i} N_\Lambda(E', \omega) \tag{4.13}$$

in Model 2.

The same proof of [*11*] now gives:

$$P(\text{dist}(\sigma(H_\Lambda(\omega)), E) < \varepsilon) \leqslant \text{const.} \mid \Lambda \mid^{\nu+1} \varepsilon \tag{4.14}$$

The Lemma is proved if we take the geometric mean of (4.9) and (4.14).

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APPENDIX A

THE DIRICHLET-NEUMANN BRACKETING

In this first appendix we review the main results of the so called Dirichlet-Neumann bracketing which have been used in the previous sections.

Let A and B be non negative selfadjoint operators on a Hilbert space $\mathscr{H}$ with form domain Q(A) and Q(B) respectively. We write $0 \leqslant A \leqslant B$ if:

i) $$Q(A) \supset Q(B)$$

ii) For any $\psi \in Q(B)$: $0 \leqslant (\psi, A\psi) \leqslant (\psi, B\psi)$.

Using the min-max principle one proves:

If $0 \leqslant A \leqslant B$ then:

a) $$\dim P_{[0,\lambda]}(A) \leqslant \dim P_{[0,\lambda]}(B) \tag{A.1}$$

b) $$\mu_K(A) \leqslant \mu_K(B).$$

Here $\{ P_{[0,\lambda]}(.) \}$ denotes the family of spectral projections.

Let now $\Lambda \subset \mathbb{R}^d$ be an open set with continuous boundary and let Δ_Λ^D Δ_Λ^N be the Dirichlet and Neumann Laplacians respectively on $L^2(\Lambda)$. Then:

a) If $$\Lambda \subset \Lambda' \qquad 0 \leqslant -\Lambda_{\Lambda'}^D \leqslant -\Delta_\Lambda^D$$

b) $$0 \leqslant -\Delta_\Lambda^N \leqslant -\Delta_\Lambda^D \tag{A.2}$$

c) If Λ_1, Λ_2 are disjoint open subsets of Λ so that $\Lambda = (\overline{\Lambda_1 \cup \Lambda_2})^{\text{int}}$ and $\Lambda \backslash \Lambda_1 \cup \Lambda_2$ has zero Lebesgue measure:

$$0 \leqslant -\Delta_\Lambda^D \leqslant -\Delta_{\Lambda_1 \cup \Lambda_2}^D$$

$$0 \leqslant -\Delta_{\Lambda_1 \cup \Lambda_2}^N \leqslant -\Delta_\Lambda^N$$

APPENDIX B

PROOF OF LEMMA 4.1, 4.2

We give the proof of the two lemmas of section 4. We also observe that they prove estimates (3.5) and (3.9).

Proof of Lemma 4.1. — For notations convenience we denote by $\mu_n(h_i; \bar{h})$ the eigenvalues of $H_\Lambda(\omega)$ with $\omega = \{ h_i \}$. Then the following chains of inequalities follows from the results of Appendix A:

$$\mu_n(h_i; \bar{h}) = \mu_n(h_i + \varepsilon; \bar{h} + \varepsilon) \leqslant \mu_n\left((h_i + \varepsilon)\left(1 + \frac{\varepsilon}{\bar{h}}\right); \bar{h}\left(1 + \frac{\varepsilon}{\bar{h}}\right)\right) \qquad \text{(B.1)}$$

$$\mu_n\left(h_i\left(1 - \frac{\varepsilon}{\bar{h}-1}\right); \bar{h}\left(1 - \frac{\varepsilon}{\bar{h}-1}\right)\right) = \mu_n\left(h_i - \frac{\varepsilon h_i}{\bar{h}-1} + \frac{\bar{h}\varepsilon}{\bar{h}-1}; \bar{h}\right) \geqslant \mu_n(h_i + \varepsilon; \bar{h}) \qquad \text{(B.2)}$$

We are now in a position to estimate $\mu_K(H_\Lambda(\omega))$.

Upper bound: Using (B.1) it is enough to estimate $\mu_n\left((h_i+\varepsilon)\left(1 + \frac{\varepsilon}{\bar{h}}\right); \bar{h}\left(1 + \frac{\varepsilon}{\bar{h}}\right)\right)$ and this can be easily done by scaling the z-coordinate by a factor $(1 + \varepsilon/\bar{h})^{-1}$. This gives that minus the Dirichlet Laplacian on the domain described by $(h_i+\varepsilon)(1+\varepsilon/\bar{h})$ and $\bar{h}(1+\varepsilon/\bar{h})$ is unitarily equivalent to:

$$H_\Lambda(\omega) + \left(1 - \frac{1}{(\varepsilon/\bar{h} + 1)^2}\right) \frac{\partial^2}{\partial Z^2} \qquad \text{(B.3)}$$

on $L^2(\Lambda_{\omega_\varepsilon})$. Again by the results of Appendix A:

$$H_\Lambda(\omega) + \left(1 - \frac{1}{(\varepsilon/\bar{h}+1)^2}\right) \frac{\partial^2}{\partial Z^2} \leqslant H_\Lambda(\omega_\varepsilon) - \left(1 - \frac{1}{(\varepsilon/\bar{h} + 1)^2}\right) \lambda_0(\bar{h}) \qquad \text{(B.4)}$$

which implies:

$$\mu_n\left((h_i + \varepsilon)\left(1 + \frac{\varepsilon}{\bar{h}}\right); \bar{h}\left(1 + \frac{\varepsilon}{\bar{h}}\right)\right) \leqslant \mu_n((h_i + \varepsilon); \bar{h}) - c\varepsilon \qquad \text{(B.5)}$$

with
$$C = \frac{2\lambda_0(\bar{h})}{\bar{h}} = 2\pi^2/\bar{h}^3 .$$

Lower bound: We start from (B.2). By scaling the z-coordinate by $(1 - \varepsilon/\bar{h} - 1)^{-1}$ we get:

$$(1 - \varepsilon/h - 1)^2 \mu_n(H_\Lambda(\omega)) \geqslant \mu_n(H_\Lambda(\omega)) \qquad \text{(B.6)}$$

In (B.6) we have used the obvious fact that:

$$-\frac{\partial^2}{\partial \underline{y}^2} - (1 - \varepsilon/\bar{h} - 1)^{-2} \frac{\partial^2}{\partial Z^2} \leqslant -(1 - \varepsilon/\bar{h} - 1)^2 \Delta \qquad \text{(B.7)}$$

Proof of Lemma 4.2. — As already observed in the proof of theorem (1.1) $H_\Lambda(\omega)$ is unitarily equivalent to:

$$-\frac{d^2}{d\rho_i^2} - \frac{1}{4\rho_i^2} - \frac{1}{\rho_i^2}\frac{d^2}{d\theta_i^2} \qquad \text{(B.8)}$$

on
$$\mathscr{H} \equiv \bigoplus_{i \in \Lambda} L^2([R_i, R_i + r_0], d\rho_i) \otimes L^2([0, \pi], d\theta_i).$$

Let now: $\mu_n(H_\Lambda(\omega)) = \sup_{\psi_1 \dots \psi_{n-1}} \inf_{\varphi \perp (\psi_1 \dots \psi_{n-1}} \langle \varphi, H_\Lambda(\omega)\varphi \rangle$ be less than $E_0 + 3\eta$. Then we infer that the infimum appearing in the definition of μ_n must be taken over functions φ such that:

$$\sum_{i\in\Lambda} \int_0^\pi d\theta_i \int_{R_i}^{R_i+r_0} d\rho_i (\partial_{\theta_i}\varphi(\rho_i, \theta_i))^2/\rho_i^2 \leqslant 3\eta \tag{B.9}$$

since by definition:

$$\sum_{i\in\Lambda} \int_0^\pi d\theta_i \int_{R_i}^{R_i+r_0} d\rho_i \left\{ \left(\frac{\partial\varphi}{\partial\rho_i}\right)^2 - \frac{1}{4\rho_i^2}\varphi^2 \right\} \geqslant E_0 \tag{B.10}$$

Let us make the change of variables: $\rho_i' = \rho_i + \varepsilon$. Then if we set $\varphi(\rho_i', \theta_i) = \varphi(\rho_i - \varepsilon, \theta_i)$ we have:

$$\langle \varphi, H_\Lambda(\omega)\varphi \rangle = \langle \varphi_\varepsilon, H_\Lambda(\omega_\varepsilon)\varphi_\varepsilon \rangle +$$

$$+ \sum_{i\in\Lambda} \int_0^\pi d\theta_i \int_{R_i+\varepsilon}^{R_i+\varepsilon+r_0} d\rho_i' \varphi_\varepsilon^2(\rho_i', \theta_i)(1/4\rho_i'^2 - 1/4(\rho_i' - \varepsilon)^2) \tag{B.11}$$

$$+ \sum_{i\in\Lambda} \int_0^\pi d\theta_i \int_{R_i+\varepsilon}^{R_i+\varepsilon+r_0} d\rho_i'((\rho_i' - \varepsilon)^{-2} - (\rho_i')^{-2})\left(\frac{\partial\varphi_2}{\partial\theta_i}\right)^2 \tag{B.11}$$

Using (B.9) the two sums in (B.11) can be estimated from above by:

$$- (1/2R_2^3)\varepsilon + 6\eta R_2^2\varepsilon/R_1^3 \tag{B.12}$$

and from below by:

$$- \varepsilon/R_1^3 \tag{B.13}$$

The proof is complete if we observe that:

$$\sup_{\psi_1 \dots \psi_{n-1}} \inf_{\substack{\varphi \perp (\psi_1 \dots \psi_{n-1}) \\ \varphi \text{ satisfies (B.9)}}} \varphi_\varepsilon, H_\Lambda(\omega_\varepsilon)\varphi_\varepsilon \rangle =$$

$$= \sup_{\psi_{1,\varepsilon}, \dots, \psi_{n-1,\varepsilon} \in \mathscr{H}_\varepsilon} \inf_{\substack{\varphi_\varepsilon \perp (\psi_{1,\varepsilon} \dots \psi_{n-1,\varepsilon}) \\ \varphi \text{ satisfies (B.9)}}} (\varphi_\varepsilon, H_\Lambda(\omega_\varepsilon)\varphi_\varepsilon) = \mu_n(H_\Lambda(\omega_\varepsilon)) \tag{B.14}$$

Here:

$$\mathscr{H}_\varepsilon = \bigoplus_{i\in\Lambda} L^2([R_i + \varepsilon, R_i + \varepsilon + r_0], d\rho_i) \otimes L^2([0, \pi], d\theta_i).$$

APPENDIX C

PROOF OF LEMMA 3.1

Let η, ω, $E = E(\omega)$ and N(E) be as in the statement of the lemma and let $\{ \varphi_j \}_{j=1}^{N(E)}$ be the generalized eigenfunctions corresponding to E. From our main result it follows that there exists a box $\Lambda \subset \mathbb{Z}^\nu$ of side L centered at the origin such that:

$$\sup_{x \in \Lambda_\omega} | \varphi_j(x) | \leqslant \exp(-mL) \quad \forall j \tag{C.1}$$

and

$$\int_{D_\omega \backslash \Lambda_\omega} dx \, | \varphi_j(x) |^2 \leqslant \exp(-m'L) \quad \forall j \tag{C.2}$$

with $m, m' \geqslant \sqrt{\eta}/2$. Actually using the arguments of section 2 the side L can be taken proportional to:

$$L \sim (1/\sqrt{\eta}) d_{K(E,\omega)} \tag{C.3}$$

where $K(E, \omega)$ is defined in theorem 2.2.

Let now $\tilde{X}_\Lambda$ be a smooth approximation of the characteristic function of the set Λ_ω and let $\tilde{\varphi}_j = \tilde{X}_\Lambda \varphi_j$. Then it is easy to see that:

$$-\Delta \tilde{\varphi}_j = E \tilde{\varphi}_j + g_{\Lambda,j} \tag{C.4}$$

with $\sup_{X \in \Lambda_\omega} | g_{\Lambda,j}(x) | \leqslant \text{const.} \exp(-mL) \quad \forall j$

Using (C.4) it also follows that:

$$\text{dist}(E, \sigma(H_\Lambda(\omega)) \leqslant \text{const.} \exp\left(-\frac{m}{2}L\right) \tag{C.5}$$

Let now $\{ \psi_n \}$ be the eigenfunctions of $H_\Lambda(\omega)$ and let us write:

$$\tilde{\varphi}_j = \sum_n c_n(j) \psi_n \tag{C.6}$$

Using (C.1), (C.2), (C.4) and (C.5) and the orthogonality of the functions φ_j's the Fourier coefficients $c_n(j)$ have the following properties:

i) $$1 - \varepsilon \leqslant \sum_n | c_n(j) |^2 \leqslant 1 \qquad \forall j, \qquad \varepsilon = \exp\left(-\frac{m}{2}L\right)$$

ii) $$\left| \sum_n c_n(j) c_n(K) \right| < \varepsilon \qquad \forall \quad K \neq j$$

iii) There exists an integer $n_0(\Lambda)$ such that:

$$\sum_{n \geqslant n_0} | c_n(j) |^2 \leqslant \text{const.}\ \varepsilon \qquad \forall j$$

furthermore n_0 can be estimated by const. $| \Lambda |$.

Properties *i)* and *ii)* are trivial; *iii)* can be derived as follows: we apply to both sides of (C.6) the operator $H_\Lambda(\omega)$ and take subsequently the scalar product with ψ_n. This gives:

$$| c_n(j) |^2 \leqslant | \langle \psi_n, g_{\Lambda,j} \rangle |^2 / | \mu_n(H_\Lambda(\omega)) - E | \tag{C.7}$$

We now observe that there are at most const. $|\Lambda|$ eigenvalues $\mu_n(H_\Lambda)$ below 2E. Therefore if $n_0(\Lambda) \geqslant$ const. $|\Lambda|$ (C.7) implies:

$$\sum_{n \geqslant n_0} | c_n(j) |^2 \leqslant \frac{1}{E} \| g_{\Lambda,j} \|^2 \leqslant \text{const. } \varepsilon \tag{C.8}$$

Using *i)*, *ii)*, *iii)* and the Grahm-Schmidt procedure we can extract from the sequences $\{ c_n(j) \}$ $1 \leqslant j \leqslant N(E)$ new orthonormal sequences $\{ v_n(j) \}_{n \leqslant n_0}$ $1 \leqslant j \leqslant \min(N(E), \text{const. } \varepsilon)$ (see [*10*] for details). However, since $n_0 \leqslant$ const. $|\Lambda|$ this is possible only if

$$\min (N(E), \text{const. } \varepsilon) \leqslant \text{const. } |\Lambda|$$

that is, using the relation $\varepsilon \simeq \exp\left(-\frac{m}{2} L\right)$, only if $N(E) \leqslant$ const. $|\Lambda|$. The Lemma is proved.

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