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Banach-power-associative algebras and J-B algebras

by

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ABSTRACT. — It is proved that the set of observables of a quantum system, stable under linear combinations and square, and complete with respect to a compatible norm topology, is a J-B algebra. This means that the Jordan identity can be replaced by the power-associativity in the definition of a J-B algebra.

RÉSUMÉ. — On démontre que l'ensemble des observables d'un système quantique, stable par combinaisons linéaires et carré, et complet relativement à une topologie définie par une norme compatible, est une J-B algèbre. Cela signifie que, dans la définition d'une J-B algèbre, l'identité de Jordan peut être remplacée par la propriété de puissance associative.

0. INTRODUCTION

The idea of using algebraic techniques for structuring sets of observables is almost as old as quantum mechanic itself, as can be seen by looking at the early works of Heisenberg, Jordan, Von Neumann, etc.

In spite of some periods of forgetting, this approach made its way up to the present time (see for memory works of Mackey, Segal, Haag, Kastler, Araki, Kadison, etc.) through, essentially, two more or less correlated ways: the first one leading to concrete algebras of operators on a Hilbert space, with emphasis on the weak operator topology and normal states, the second one, basically more abstract and advocating the use of normed algebras, which is sufficient to get a well-behaved functional calculus, and leads to the consideration of non-normal states, which seems to be physically pertinent, for instance in statistical mechanics.

In both cases, it is usually taken as granted that the natural minimal algebraic structure to be put onto a set of observables is a Jordan one.

This seems evident in the concrete approach, as it is well known that the family of bounded self-adjoint operators on a Hilbert space possesses, in a natural way, a special Jordan algebra structure through the composition law $A_0B = (AB + BA)/2$. In the abstract approach, things are less evident. Of course the same can be said for the self-adjoint part of, for instance, an abstract C*-algebra but then remains the question of the pertinence of the *a priori* product in the algebra. So the problem to be solved can be phrased as follows: is it possible, starting from elementary axioms assuming essentially the existence of linear combinations and powers of observables together with an appropriate norm topology, to deduce the existence of a Jordan structure, or is it necessary, as is done for instance in [*7*], to impose it as a supplementary axiom? Of course, the answer depends greatly on the way of choosing the initial axiomatic.

In the Jordan-von Neumann-Wigner axiomatic [*14*], where the finiteness condition has the effect of discarding the role of topology, the Jordan structure is deduced from a reality condition and a power-associativity axiom by using a particularly simple spectral decomposition.

In the infinite dimensional case, von Neumann obtained the same kind of result [*19*] through a rather intricate axiomatic but where the topology is more or less of the weak type, classifying this result in the operator algebra approach rather than in the normed algebra one.

To this last one pertains the Segal axiomatic [*25*] which develops mainly a functional calculus and the notion of compatibility of observables, but does not give an answer to our problem, by lack of distributivity and also of a spectral decomposition. Considering that this axiomatic, by not beeing linked to some representation on a Hilbert space, presents a more intrinsic character, we would like to fill the gap by proving that, even in the presence of a norm topology, elementary axioms imply the Jordan structure.

This is made possible by the present status of the so-called J-B algebras, whose fundamental structure is by now well known [*11*] and which possess, in their W*-type counterpart, a spectral decomposition, insuring a link with [*19*]. For a parallel, though different, approach, see also [*4*] [*5*] [*8*].

I. SETS OF OBSERVABLES AS BANACH-POWER-ASSOCIATIVE SYSTEMS

A pedagogical exposure would necessitate the motivated introduction of a quantum axiomatic involving the concepts of observables and states. As this would go beyond the scope of this work, we will just start from some algebraico-topological definitions containing the minimal hypothesis which are necessary for our purpose, without any conceptual considerations about the difficulty of simultaneously defining the sum and the square of observables.

DEFINITION I.1. — Let $\mathcal{A}$ be a real linear space. A square map is a map from $\mathcal{A}$ into $\mathcal{A}$, denoted

$$A \in \mathcal{A} \rightarrow A^2 \in \mathcal{A}$$

for reasons that will be obvious in the sequel, and such that

$$(-A)^2 = A^2, \qquad A \in \mathcal{A}.$$

Such a map allows to define a « product » in $\mathcal{A}$ according to

$$A.B = [(A + B)^2 - A^2 - B^2]/2, \qquad A, B \in \mathcal{A}$$

and the n^{th}-power recursively by

$$A^n = A^{n-1}.A, \qquad A \in \mathcal{A}$$

where n is any positive integer such that $n > 2$. Of course one can note

$$A^1 = A.$$

DEFINITION I.2. — A system of observables is a real Banach space $\mathcal{A}$ with a square map such that

$H_0)$ $\| A^2 \| = \| A \|^2$, $A \in \mathcal{A}$.

A subsystem of observables is a real closed subspace of $\mathcal{A}$ stable under squaring. The system of observables $\mathcal{A}$ will be said « with unit » if there exists some element in $\mathcal{A}$, denoted $\mathbb{1}$ and called a unit, such that $A.\mathbb{1} = A$ for any $A \in \mathcal{A}$. We can note $A^0 = \mathbb{1}$.

PROPOSITION I.3. — Let $\mathcal{A}$ be a system of observables. Then

$$A.0 = 0, \qquad A \in \mathcal{A}.$$

Proof. — As $\| 0^2 \| = \| 0 \|^2 = 0$, then $A.0 = -0^2/2 = 0$. ■

If a straightforward computation reveals that the product is commutative but non-associative, we do not know if it is bilinear, continuous, if $A.A = A^2$ or, more generally, if $A^m.A^n = A^{m+n}$.

PROPOSITION I.4. — Let $\mathscr{A}$ be a system of observables and $\mathscr{B} \subset \mathscr{A}$ a subsystem such that

$$A.(-B) = -(A.B), \qquad A, B \in \mathscr{B}.$$

Then, in $\mathscr{B}$, the product is bilinear, continuous with

$$\| A.B \| \leq 2 \| A \| \; \| B \|, \qquad A, B \in \mathscr{B}$$

and such that $A.A = A^2$, $A \in \mathfrak{B}$. If $\mathbb{1} \in \mathscr{B}$, then $\| \mathbb{1} \| = 1$.

Proof. — Because

$$[(A+B)^2 - A^2 - B^2]/2 = A.B = -(A.(-B)) = -[(A-B)^2 - A^2 - B^2]/2,$$

we get the parallelogram law $(A + B)^2 + (A - B)^2 = 2A^2 + 2B^2$. By successive applications of it, we get that

$$\begin{aligned} A.B + A.C &= [(A+B)^2 + (A+C)^2 - B^2 - C^2 - 2A^2]/2 \\ &= [(2A+B+C)^2 - (B+C)^2 - (2A)^2]/4 = [(2A).(B+C)]/2. \end{aligned}$$

As, if $C = 0$, $A.B = [(2A).B]/2$, we obtain the distributivity of the product and, by induction, $A.rB = r(A.B)$ for any rational r. The same will be true for any real λ if we prove the continuity of the product. But once again the parallelogram law allows to write $A.B = [(A+B)^2 - (A-B)^2]/4$ so that, if r is rational

$$\begin{aligned} \| A.B \| = | r |^{-1} \| A.rB \| &= (4 | r |)^{-1} \| (A + rB)^2 - (A - rB)^2 \| \\ &\leq (2 | r |)^{-1} (\| A \| + r \| B \|)^2 . \end{aligned}$$

Extending the last inequality to the reals by density, and putting $r = \| A \| / \| B \|$ we get that $\| A.B \| \leq 2 \| A \| \; \| B \|$, which asserts the claim. Finally the property $A.A = A^2$ is also a direct consequence of the parallelogram law, while $\| \mathbb{1} \| = \| \mathbb{1}^2 \| = \| \mathbb{1} \|^2 = 1$. ■

PROPOSITION I.5. — Let $\mathscr{A}$ be a system of observables and $\mathscr{B} \subset \mathscr{A}$ a subsystem such that

$$A.(-B) = -(A.B) \qquad \text{and} \qquad A^2.A^2 = A^4, \qquad A, B \in \mathscr{B}.$$

Then $\mathscr{B}$ is power-associative in the sense that

$$A^m.A^n = A^{m+n}, \qquad A \in \mathscr{B}$$

for any strictly positive integers m and n (and also $m = 0$ or $n = 0$ if $\mathscr{B}$ has a unit).

Proof. — By proposition I.4, $\mathscr{B}$ is a real commutative but non-associative algebra. It is then a standard result in the theory of such algebras that the condition $A^2.A^2 = A^4$ implies power-associativity [*1*], ([*22*], p. 130). ■

COROLLARY I.6. — Let $\mathscr{A}$ be a system of observables, $A \in \mathscr{A}$ and C(A) the subsystem generated by A, and $\mathbb{1}$ if it exists. Assume the square map is continuous in C(A) and C(A) satisfies to the hypothesis of proposition I.5, then C(A) is a commutative, real, (associative) Banach algebra.

Proof. — Proposition I.4 and I.5 and the continuity of the product insure that C(A) is an associative algebra with $\| B.C \| \leq 2 \| B \| \ \| C \|$ for any $B, C \in C(A)$. But then $\|B.C\|^2 = \|(B.C)^2\| = \|B^2.C^2\| \leq 2\|B^2\| \ \|C^2\|$, or $\|B.C\| \leq \sqrt{2}\|B\| \ \|C\|$ and, by induction, $\|B.C\| \leq \|B\| \ \|C\|$. ■

Remark. — The proofs of distributivity and power-associativity do not rely on H_0) if we assume that $O^2 = 0$.

In order to get a well behaved functional calculus in $\mathscr{A}$, it would be interesting to know that C(A) is isometrically isomorphic to an algebra of continuous functions on some compact space. But as the situation is more intricate for real C*-algebras than for complex ones, we need some additional hypothesis, which motivates the following definition.

DEFINITION I.7. — A Banach-power-associative system is a system of observables $\mathscr{A}$ such that, for any $A, B \in \mathscr{A}$,

H_1) $A^2.A^2 = A^4$,

H_2) $A^m.(-A^n) = -(A^m.A^n)$, $m, n \in \mathbb{N}^*$ (or $\mathbb{N}$ if there is a unit),

H_3) $\| A^2 - B^2 \| \leq \max \{ \| A^2 \|, \| B^2 \| \}$,

H_4) the square map is continuous in C(A).

If moreover H_2) is replaced by $A.(-B) = -(A.B)$ then $\mathscr{A}$ is said a Banach-power-associative algebra.

PROPOSITION I.8. — The hypothesis H_3) implies

H_3') $\| A^2 \| \leq \| A^2 + B^2 \|$, $A, B \in \mathscr{A}$. □

Proof. — Let us assume that $\| A^2 \| \leq \| B^2 \|$. Then if H_3) is true, the identity $2B^2 = A^2 + B^2 - (A^2 - B^2)$ gives

$$\|A^2+B^2\| \geq 2\|B^2\| - \|A^2-B^2\| \geq 2\|B^2\| - \|B^2\| = \|B^2\| \geq \|A^2\|. \quad ■$$

PROPOSITION I.9. — If $\mathscr{A}$ is a Banach-power-associative algebra, then H_4) is redundant because $\| A.B \| \leq \| A \| \ \| B \|$ for any $A, B \in \mathscr{A}$.

Proof. — The first assertion comes from proposition I.4 which insures also that the product $A.B = [(A + B)^2 - (A - B)^2]/4$ is bilinear in $\mathscr{A}$. Hence, if $\| A \| = \| B \| = 1$,

$$\begin{aligned} \| A.B \| &= \| (A + B)^2 - (A - B)^2 \|/4 \\ &\leq \max \{ \| A + B \|^2, \| A - B \|^2 \}/4 \leq (\| A \| + \| B \|)^2/4 = 1 . \quad ■ \end{aligned}$$

PROPOSITION I.10. — Let $\mathscr{A}$ be a Banach-power-associative system and $A \in C(A)$. Then C(A) is algebraically and isometrically isomorphic to $C_0(X)$, the algebra of real continuous functions vanishing at infinity on some locally compact Haudsorff space X, this last one being compact if and only if $\mathscr{A}$ has a unit.

Proof. — It is sufficient to construct a complex commutative C*-algebra of which C(A) will be the self-adjoint part. But, thanks to proposition I.8, this is a well known procedure ([*15*], 6.6), ([*11*], 3.2.2). ■

Before investigating more deeply into the structure of Banach-power-associative systems, we would like to mention the existence of some examples and counter-examples.

DEFINITION I.11. — A J-B algebra is a real Jordan algebra complete with respect to a norm obeying to H_0), to the equivalent conditions H_3) or H'_3), and to $\| A.B \| \leq \| A \| \; \| B \|$ [*6*] [*11*].

The main result of this work will consist in proving the converse of the following proposition which, by the way, asserts the non-vacuity of our definitions.

PROPOSITION I.12. — A J-B algebra $\mathscr{A}$ is a Banach-power-associative algebra. □

Proof. — Let $\{ A, B, C \}$ be the associator equal to $(A.B).C - A.(B.C)$. The only point to be proved is H_1), which is equivalent to the nullity of the associateur $\{ A, A, A^2 \}$, and is implied by the Jordan condition $\{ A, B, A^2 \} = 0$, $A, B \in \mathscr{A}$. ■

However there exist examples of Banach-power-associative systems which are not algebras: for instance the class of counter-examples exhibited by Sherman [*23*].

REMARK I.13. — There exist power-associative algebras which are not Jordan algebras (see for instance [*2*], p. 504). But such algebras contains nilpotent elements which are here excluded by the condition $\| A^2 \| = \| A \|^2$. A finite dimensional power-associative algebra is a Jordan algebra if it is formally real: $\sum_{i=1}^{n} A_i^2 = 0$ implies $A_i = 0$ (which is equivalent to H'_3) in that case) (see [*14*]). The conclusion remains valid under the semi-simplicity hypothesis and also in the infinite dimensional case for a simple algebra without any topological considerations ([*1*] [*2*] [*17*]). But as there is no hope, without any additional hypothesis, for a decomposition of a semi-simple infinite dimensional power-associative algebra into a sum of simple ones, the topological structure has to play some role. Moreover there is a semi-simple formally real Jordan norm-complete algebra such that $\| AB \| \leq \| A \| \; \| B \|$, $\| A^2 \| = \| A \|^2$ and which is not a J-B algebra (see [*11*] 3.1.4).

So it remains open the problem of the equivalence of power-associativity and Jordan structure for more general algebras than semi-simple power-associative normed algebras. Let us recall that this has been done in [*19*] for a weak type topology. Here we solve this problem with a Banach-type topology.

II. BANACH-POWER-ASSOCIATIVE SYSTEMS WITH UNIT AS ORDER-UNIT SPACES

Let us introduce an order in $\mathscr{A}$ with the help of the square map.

PROPOSITION II.1. — If $\mathscr{A}$ is a Banach-power-associative system, then $\mathscr{A}^2 = \{ A \in \mathscr{A} ; A = C^2, C \in \mathscr{A} \}$ is a closed proper convex cone inducing a partial order in $\mathscr{A}$, under $H_3')$ only if $\mathscr{A}$ has a unit.

Proof. — Let us first prove that

$$\{ A \in \mathscr{A} ; A = C^2, C \in C(A) \} = \mathscr{A}^2 =$$
$$= \{ A \in \mathscr{A} ; \| \lambda B^2 - A \| \leq \lambda \text{ for some } \lambda \geq \| A \| \text{ and any } B \in \mathscr{A}, \| B \| = 1 \}$$
$$= \{ A \in \mathscr{A} ; \| \lambda \mathbb{1} - A \| \leq \lambda \text{ for some } \lambda \geq \| A \| \} \text{ if } \mathscr{A} \text{ has a unit.}$$

In fact, if $A = C^2$ with $C \in C(A)$, then $A \in \mathscr{A}^2$. If moreover $\lambda \geq \| A \|$ and $\| B \| = 1$, then

$$\| \lambda B^2 - A \| = \| \lambda B^2 - C^2 \| \leq \max \{ \| \lambda B^2 \|, \| C^2 \| \} = \max \{ \lambda, \| A \| \} = \lambda ;$$

if there is a unit, $\| \lambda \mathbb{1} - A \| = \| D^2 \| \leq \| D^2 + C^2 \| = \| D^2 + A \| = \lambda$ with the help of Proposition I.10 and $H_3')$. Finally, if this last property is true, let $D = A/\lambda - B^2$: as $\| D \| \leq 1$, proposition I.10 allows to choose B in C(A) such that $\| B \| = 1$ and $B^2 + D = C^2$ with $C \in C(A)$. Then $A = \lambda(D + B^2) = (\sqrt{\lambda} C)^2$ where $\sqrt{\lambda} C \in C(A)$. The set $\mathscr{A}^2$ is obviously a cone, proper because if $A^2 = - B^2 \in \mathscr{A}^2 \cap \{ - \mathscr{A}^2 \}$, then $\| A^2 \| = \| B^2 \| \leq \| A^2 + B^2 \| = 0$, closed thanks to the preceeding characterization of $\mathscr{A}^2$. Let us end by proving its convexity. If $A, A' \in \mathscr{A}^2$, then $\| \lambda B^2 - A \| \leq \lambda$ and $\| \mu B^2 - A' \| \leq \mu$ for some $\lambda \geq \| A \|$, $\mu \geq \| A' \|$ and any $B \in \mathscr{A}$ with $\| B \| = 1$. Then, if $0 < \tau < 1$,

$$\| (\tau\lambda + (1 - \tau)\mu)B^2 - (\tau A + (1 - \tau)A') \|$$
$$\leq \tau \| \lambda B^2 - A \| + (1 - \tau) \| \mu B^2 - A' \| \leq \tau\lambda + (1 - \tau)\mu,$$

which proves that $\tau A + (1 - \tau)A' \in \mathscr{A}^2$ because $\| \tau A + (1 - \tau)A' \| \leq \tau\lambda + (1 - \tau)\mu$. ■

We are now in position to give an order-unit space characterization of Banach-power-associative systems with unit, paralleling the one for J-B algebras [*6*] [*11*] that will give us the opportunity of proving the equivalence of $H_3)$ and $H_3')$. For definitions, see ([*3*], Chapter II).

PROPOSITION II.2. — If $\mathscr{A}$ is a Banach-power-associative system with unit $\mathbb{1}$, then $\mathscr{A}$ is an order-unit space whose order-norm coincide with the given one and such that

$$(*) \qquad -\mathbb{1} \leq A \leq \mathbb{1} \quad \text{implies} \quad 0 \leq A^2 \leq \mathbb{1}.$$

Conversely, if $\mathscr{A}$ is a real norm-complete order-unit space with order unit $\mathbb{1}$ and a square map satisfying (*) and inducing a product and a power operation such that $O^2 = 0, A.\mathbb{1} = A, A^2.A^2 = A^4, A^m.(-A^n) = -(A^mA^n)$ and the square map is continuous in C(A), then $\mathscr{A}$ is a Banach-power-associative system with unit. Similarly, if the square map is not required to be continuous but $A.(-B) = -(A.B)$ for any A, $B \in \mathscr{A}$, then $\mathscr{A}$ is a Banach-power-associative algebra with unit.

Proof. — If $\mathscr{A}$ is a Banach-power-associative system with unit, we get the result by using the functional representation of C(A). Conversely, if $\|A\| \leq 1$, $\|B\| \leq 1$, then $\|(A \pm B)/2\| \leq 1$, which is equivalent to $-\mathbb{1} \leq (A \pm B)/2 \leq \mathbb{1}$. Thus $0 \leq ((A \pm B)/2) \leq \mathbb{1}$ in such a way that $-\mathbb{1} \leq ((A+B)/2)^2 - ((A-B)/2)^2 \leq \mathbb{1}$, and $\|((A+B)/2)^2 - ((A-B)/2\| \leq 1$. From this we deduce that $\|C^2 - D^2\| \leq \max\{\|C^2\|, \|D^2\|\}$, which proves H_3). Assume next that $\|A^2\| \leq 1$, or else that $0 \leq A^2 \leq \mathbb{1}$. Then thanks to the remark following corollary I.6,

$$A = [A^2 + \mathbb{1} - (A - \mathbb{1})^2]/2 \leq (A^2 + \mathbb{1})/2 \leq \mathbb{1},$$

while

$$A = [(A + \mathbb{1})^2 - A^2 - \mathbb{1}]/2 \geq (-A^2 - \mathbb{1})/2 \geq -\mathbb{1}$$

since, by (*), all squares are positive, which implies that $\|A\|^2 \leq 1$ and, as $\|A^2\| \leq \|A\|^2$, gives H_0). ■

COROLLARY II.3. — Let $\mathscr{A}$ be a Banach-power-associative system with unit. Then H_3) and H'_3) are equivalent on $\mathscr{A}$.

Proof. — Because of Proposition I.8, this will be a direct consequence of Proposition II.2 if one remarks that the first part of its proof, relying on the functional calculus on C(A), is true only under H'_3) (see the proof of proposition I.10).

We can deduce from Proposition II.2 another characterization of Banach-power-associative systems with unit which relies on the fact the dual $\mathscr{A}^*$ of an order-unit space $\mathscr{A}$ is a base-norm space with base the state space $\mathscr{S}(\mathscr{A})$ of (necessarily positive) continuous linear functionals ψ on $\mathscr{A}$ such that $\|\psi\| = \psi(\mathbb{1}) = 1$ ([*3*], Chapter II, paragraphe 1).

COROLLARY II.4. — Let $\mathscr{A}$ be a real Banach space equipped with a square map and a unit for the induced product. Then $\mathscr{A}$ is a Banach-power-associative system with unit if and only if C(A), the smallest closed subspace of $\mathscr{A}$ containing $A \in \mathscr{A}$ and $\mathbb{1}$, and stable under the square map, is, for any

$A \in \mathscr{A}$, a Banach-power-associative system for the induced product. Then C(A) turns out to be a Banach-power-associative algebra.

Proof. — Remark that C(A) is an order-unit space by Proposition II.2. Let $\mathscr{S}(\mathscr{A})$ be the set of continuous linear functionals ψ on $\mathscr{A}$ such that $\|\psi\| = \psi(\mathbb{1}) = 1$, and $\mathscr{A}^+ = \{A \in \mathscr{A},\ \psi(A) \geq 0,\ \psi \in \mathscr{S}(\mathscr{A})\}$. If $A \in \mathscr{A}$, then $A \in \mathscr{A}^+$ if and only if $A \in C(A)^2$ because, by the Hahn-Banach theorem, any state on C(A) is the restriction of elements of $\mathscr{S}(\mathscr{A})$. Using the remark, it turns out that $\mathscr{A}^+$ is a proper convex cone turning $\mathscr{A}$ into an order-unit space, complete for the given norm, and satisfying to all conditions of Proposition II.2. The converse is trivial. ■

The problem of defining a square map on an order-unit space as assumed in Proposition II.2 can be solved positively (see V. 4.1).

III. ADJUNCTION OF A UNIT

If $\mathscr{A}$ is a Banach-power-associative system without unit, we can define a unitization process, inspired from [*26*].

LEMMA III.1. — Let $\mathscr{A}$ be a Banach-power-associative system with unit and $\mathscr{B}$ a Banach-power-associative subsystem with the same unit. Any positive linear form φ on $\mathscr{B}$ is the restriction of a positive linear form on $\mathscr{A}$ with the same norm. If $\mathscr{A}$ has no unit, the same is true for every $\mathscr{B} = C(A)$, $A \in \mathscr{A}$.

Proof. — The first assertion is just an application of the Hahn-Banach theorem on a base-norm space while the proof of the second one is identical to ([*26*], theorem 2.5). ■

LEMMA III.2. — Let $\mathscr{A}$ be a Banach-power-associative system without unit, $\tilde{\mathscr{A}} = \mathscr{A} \times \mathbb{R}$, $A \in \mathscr{A}$ with $C(A) \sim C_0(X)$, $\overline{X} = X \cup \{\omega\}$ the one point compactification of X, and $\mathbb{1}_{\overline{X}}$ the function constant and equal to one on $\overline{X}$. Defining an order on $\tilde{\mathscr{A}}$ by setting $(A, \lambda) \geq 0$ if and only if $A + \lambda \mathbb{1}_{\overline{X}}$ is positive as an element of $C(\overline{X})$, we get a partial order relation on $\tilde{\mathscr{A}}$ extending the one on $\mathscr{A}$ and turning $\tilde{\mathscr{A}}$ into a norm-complete order-unit space with order-unit $\mathbb{1} = (0, 1)$, the order-norm extending the one on $\mathscr{A}$.

Proof. — Let us first show that the positivity condition induces a partial order relation by proving that it is equivalent to the next one: $(A, \lambda) \geq 0$ if and only if $\lambda \geq 0$ and, given $\varepsilon > 0$, there exists some $A_\varepsilon \in \mathscr{A}^2$ such that $\|A_\varepsilon\| < \lambda + \varepsilon$ and $A + A_\varepsilon \geq 0$ (it is easy to check that it is in fact a partial order relation on $\tilde{\mathscr{A}}$). Of course if $A + \lambda \mathbb{1}_X \geq 0$, then $\lambda \geq 0$ because $A(\omega) = 0$. If the Jordan decomposition of A in C(A) is $A = A^+ - A^-$, we choose $A_\varepsilon = A^-$ for any ε and $(A, \lambda) \geq 0$ according to the new definition.

Conversely, let $(A, \lambda) \geq 0$ according to the new definition, φ a state on $C(\overline{X})$, φ_1 its restriction to $C_0(X)$ considered as an ideal of $C(\overline{X})$, ψ_1 its norm preserving extension to $\mathscr{A}$ according to lemma III.1, and ψ the extension of ψ_1 to $\tilde{\mathscr{A}}$ according to $\psi((B, \mu)) = \psi_1(B) + \mu$ where $(B, \mu) \in \tilde{\mathscr{A}}$. Then ψ is positive on $\tilde{\mathscr{A}}$ with respect to the new order because if $B + B_\varepsilon \geq 0$ and $\| B_\varepsilon \| < \mu + \varepsilon$, then $\psi((B, \mu)) = \psi_1(B) + \mu \geq \mu - \psi_1(B_\varepsilon) \geq \mu - \| \psi_1 \| (\mu + \varepsilon) \geq -\varepsilon$ for each ε. Hence $\psi((A, \lambda)) = \varphi(A + \lambda \mathbb{1}_{\overline{X}}) \geq 0$ for any state φ on $C(\overline{X})$, which means that $A + \lambda \mathbb{1}_{\overline{X}} \geq 0$ in $C(\overline{X})$. Using the functional calculus in $C(\overline{X})$, it is straightforward to check that $\mathbb{1} = (0, 1)$ is an order-unit, that the order relation is archimedean and extends the initial one on $\mathscr{A}$, that the order-norm on $\tilde{\mathscr{A}}$ extends the initial one on $\mathscr{A}$. Finally, $\tilde{\mathscr{A}}$ is complete because so is $\mathscr{A}$. ■

PROPOSITION III.3. — Let $\mathscr{A}$ be a Banach-power-associative system without unit. Equipped with the product $(A, \lambda)(B, \mu) = (A . B + \mu A + \lambda B, \lambda \mu)$ and the order-norm, $\tilde{\mathscr{A}}$ is a Banach-power-associative system with unit $\mathbb{1} = (0, 1)$ and positive cone $\tilde{\mathscr{A}}^2$. Any state on $\mathscr{A}$ is the restriction of a state on $\tilde{\mathscr{A}}$.

Proof. — It is clear that all the required algebraic properties in the second part of proposition II.2 are satisfied in $\tilde{\mathscr{A}}$ because the product on $\tilde{\mathscr{A}}$ extends the one on $\mathscr{A}$. Moreover, as $-\mathbb{1} \leq (A, \lambda) \leq \mathbb{1}$ is equivalent to $-\mathbb{1}_{\overline{X}} \leq A + \lambda \mathbb{1}_{\overline{X}} \leq \mathbb{1}_{\overline{X}}$ we get that $0 \leq (A, \lambda)^2 \leq \mathbb{1}$ and $\tilde{\mathscr{A}}_+ = \tilde{\mathscr{A}}^2$. Finally the square map is continuous in $C((A, \lambda))$ because if the sequence $\{ (A_i, \lambda_i) \}_{i \in \mathbb{N}}$ tends to $(0, 0)$ in $C((A, \lambda))$ then $\{ A_i \}_{i \in \mathbb{N}}$ tends to 0 in $C(A)$ and $\{ \lambda_i \}_{i \in \mathbb{N}}$ tends to 0 in $\mathbb{R}$. This is so because, for any ε, there exists i_ε such that $\inf \{ \mu > 0, -\mu \mathbb{1} \leq (A_i, \lambda_i) \leq \mu \mathbb{1} \} < \varepsilon$ for $i > i_\varepsilon$, or else that

$$\sup_{x \in \overline{X}_i} | A_i(x) + \lambda_i \mathbb{1}_{\overline{X}_i} | < \varepsilon$$

where $A_i(x) \in C(\overline{X}_i) = C(X_i \cup \{ \omega_i \})$. In particular, as $A_i(\omega_i) = 0$, $| \lambda_i | < \varepsilon$ which means that $\{ \lambda_i \}_{i \in \mathbb{N}}$ tends to 0, and the same is true for A_i. Any state in $\mathscr{A}$ can be extended to a state on $\tilde{\mathscr{A}}$ by a procedure identical to the extension of ψ_1 to ψ in the proof on lemma III.2. ■

IV. THE BIDUAL OF A BANACH-POWER-ASSOCIATIVE ALGEBRA

Thanks to the Peirce decomposition of a Banach-power-associative algebra with respect to an idempotent, we get (see [*22*] V. 1 and V. 2, Lemma 5.2).

LEMMA IV.1. — Let $\mathscr{A}$ be a Banach-power-associative algebra and P, Q two idempotents such that $P.Q = 0$. Then $\{ P, A, Q \} = 0$ for any A in $\mathscr{A}$.

Thus by a spectral argument it is easy to obtain the Jordan identity $\{ B, A, B^2 \} = 0$ in $\mathscr{A}$ if $\mathscr{A}$ has sufficiently many idempotents. So it is natural to investigate the bidual of $\mathscr{A}$.

Remark first that if $\mathscr{A}$ is a Banach-power-associative algebra with unit, $\mathscr{A}^{**}$ is a complete order-unit space which is order and norm-isomorphic to $A^b(S(\mathscr{A}))$ (Bounded real affine functions on $S(\mathscr{A})$). Moreover

$$(\mathscr{A}^{**})^+ = \{ T \in \mathscr{A}^{**} ; T(\varphi) \geq 0 \qquad \forall \varphi \in S(\mathscr{A}) \}$$

(see [*3*] II.1).

We are in position to prove that the bidual $\mathscr{A}^{**}$ is also a Banach-power-associative algebra using the argument of [*10*].

PROPOSITION IV.2. — Let $\mathscr{A}$ be a Banach-power-associative algebra. Then $\mathscr{A}^{**}$ is a monotone and w^*-complete Banach-power-associative algebra with $S(\mathscr{A})$ as a separating set of normal states (i. e. : $\forall \varphi \in S(\mathscr{A})$ $\lim_\alpha \varphi^{**}(A_\alpha) = 0$ for any increasing net in $\mathscr{A}^{**}$ with 0 as least upper bound where φ^{**} is the w^* extension of φ to $\mathscr{A}^{**}$).

Proof. — Assume first that $\mathscr{A}$ has a unit. For any $\varphi \in S(\mathscr{A})$, let $(A, B)_\varphi$ the bilinear form on $\mathscr{A}$ defined according to $(A, B)_\varphi = \varphi(A.B)$. By a Cauchy-Schwarz-type argument, $(A, B)_\varphi^2 \leq (A, A)_\varphi (B, B)_\varphi$ so that $N = \{ A \in \mathscr{A} ; (A, A)_\varphi = 0 \}$ is a subspace of $\mathscr{A}$ such that $\mathscr{A}/N$ is a real prehilbert space whose completion will be denoted H_φ. If η_φ is the map $\mathscr{A} \to \mathscr{A}/N \subset H_\varphi$, $(\eta_\varphi(A), \eta_\varphi(B)) = (A, B)_\varphi = \varphi(A.B)$ and

$$\| \eta_\varphi(A) \|^2 = \varphi(A^2) \leq \sup_{\varphi \in S(\mathscr{A})} \varphi(A^2) = \| A^2 \| = \| A \|^2 ,$$

so that $\| \eta_\varphi \| \leq 1$. The bitransposed η_φ^{**} is a norm-conserving and w^*-continuous extension of η_φ from $\mathscr{A}^{**}$ to H_φ. For any pair S, $T \in \mathscr{A}^{**}$, let $f_{S,T}(\varphi) = (\eta_\varphi^{**}(S), \eta_\varphi^{**}(T))$: it is a real function on $S(\mathscr{A})$, separately w^*-continuous in S and T. If A, $B \in \mathscr{A} \subset \mathscr{A}^{**}$, $f_{A,B}$ is real, affine on $S(\mathscr{A})$ and, by w^*-density of $\mathscr{A}$ into $\mathscr{A}^{**}$, the same is true for any S, $T \in \mathscr{A}^{**}$. As moreover $f_{S,T}$ is bounded by $\| S \| \, \| T \|$, there exists, thanks to the remark above, a unique element, denoted S.T, in $\mathscr{A}^{**}$ such that $f_{S,T}(\varphi) = \varphi(S.T)$. This « product » in $\mathscr{A}^{**}$ is commutative, separately w^*-continuous and extends the one in $\mathscr{A}$. Hence the identity

$$\begin{aligned} &\{ A, B, C.D \} + \{ C, B, D.A \} + \{ D, B, A.C \} + \{ B, A, C.D \} \\ &\quad + \{ C, A, D.B \} + \{ D, A, B.C \} + \{ C, D, A.B \} + \{ A, D, B.C \} \\ &\quad + \{ B, D, C.A \} + \{ D, C, A.B \} + \{ A, C, B.D \} + \{ B, C, D.A \} = 0, \end{aligned}$$

(which can be shown to be equivalent to $A^2.A^2 = A^4$ through a linearization process ([22], Chapter V, Paragraph 1)) remains valid in $\mathscr{A}^{**}$ and insures the validity of all hypothesis of proposition II.2, but the implication (*). If $-I \leq T \leq I$, $T^2(\varphi) = f_{T,T}(\varphi) = \|\eta_\varphi^{**}(T)\|^2 \leq \|T\|^2 \leq 1 = \varphi(\mathbb{1}) = \mathbb{1}(\varphi)$ and $0 \leq T^2 \leq \mathbb{1}$. Finally, $\mathscr{A}^{**}$ is monotone-complete because so is $A^b(S(\mathscr{A}))$, and w^*-complete by duality. On the other hand, $S(\mathscr{A})$ separates $\mathscr{A}^{**}$ and any $\varphi \in S(\mathscr{A})$ can be extended, through bitransposition, to a w^*-continuous positive linear form φ^{**} on $\mathscr{A}^{**}$. If $\{T_\alpha\}$ is an increasing bounded net in $\mathscr{A}^{**} = A^b(S(\mathscr{A}))$ with least upper bound 0, then $T_\alpha(\varphi)$ tends to 0 and also $\varphi^{**}(T_\alpha)$, which means that φ^{**} is normal. If now $\mathscr{A}$ has no unit, let $\tilde{\mathscr{A}} = \mathscr{A} \times \mathbb{R}$ as in Chapter III: $\mathscr{A}$ is a closed subalgebra of $\tilde{\mathscr{A}}$ and through bitransposition, $\mathscr{A}^{**}$ is a w^*-closed subalgebra of $\tilde{\mathscr{A}}^{**}$, monotone-complete because if $\{T_\alpha\}$ is an increasing bounded net in $\mathscr{A}^{**}$ with least upper bound T in $\tilde{\mathscr{A}}^{**}$, then $\varphi(T_\alpha)$ tends to $\varphi(T)$, which means that $\{T_\alpha\}$ tends w^*-continuously to T, so that $T \in \mathscr{A}^{**}$. Finally, let $\varphi \in S(\mathscr{A})$, $\tilde{\varphi}$ its extension as an element of $S(\tilde{\mathscr{A}})$ according to proposition III.3, $\tilde{\varphi}^{**}$ its extension to $\tilde{\mathscr{A}}^{**}$ and φ^{**} its restriction to $\mathscr{A}^{**}$: φ is normal because, as seen above, an increasing bounded net in $\mathscr{A}^{**} \subset \tilde{\mathscr{A}}^{**}$ has its least upper bound in $\mathscr{A}^{**}$. ∎

Using standard arguments ([11] [20]), we get

COROLLARY IV.3. — Let $\mathscr{A}$ be a Banach-power-associative algebra. For any $A \in \mathscr{A}^{**}$, let W(A) be the w^*-closed subalgebra generated by A (and $\mathbb{1}$ if it exists). Then W(A) is a monotone-complete real Banach-algebra.

V. BANACH-POWER-ASSOCIATIVE ALGEBRAS AS J-B ALGEBRAS. CONSEQUENCES

We can now prove the converse of proposition I.12:

THEOREM V.1. — A Banach-power-associative algebra is a J-B algebra.

Proof. — It is enough to prove the Jordan identity in $\tilde{\mathscr{A}}^{**}$. As W(A) is monotone-complete, $W(A) \approx C_0(X)$ where X is a stonean compact space, which means that $A \in \tilde{\mathscr{A}}^{**}$ can be approximated uniformely by finite linear combinations of orthogonal idempotents in W(A) ([20]). But for such elements and any B in $\mathscr{A}$,

$$\left\{ \sum \lambda_i P_i, B, \left(\sum_j \lambda_j P_j \right)^2 \right\} = \sum_{i,j} \lambda_i \lambda_j^2 \{ P_i, B, P_j \}$$
$$= 0 \qquad \text{by lemma IV.1}.$$

The Jordan identity $\{ A, B, A^2 \} = 0$ follows by continuity. ∎

As a by product we get the following definition of J-B algebra, of course equivalent to the usual one since we have proved that $\| A.B \| \leq \| A \| \; \| B \|$ is redundant (Prop. I.9). This fact was first noticed by Shultz without proof [*24*].

COROLLARY V.2. — A J-B algebra is a real Banach space with a square map inducing a product such that:

$$A.(-B) = -(A.B),\ A^2.A^2 = A^4$$
$$\| A^2 \| = \| A \|^2$$
$$\| A^2 - B^2 \| \leq \max(\| A^2 \|, \| B^2 \|)$$

the last axiom being equivalent to $\| A^2 \| \leq \| A^2 + B^2 \|$ if there exists a unit.

COROLLARY V.3. — Let $\mathscr{A}$ be a real Banach space with a bilinear power-associative product (not necessarily associative) such that

$$\| A^2 \| = \| A \|^2$$
$$\| A^2 - B^2 \| \leq \max \{ \| A^2 \|, \| B^2 \| \}$$

Then $\mathscr{A}$ is a Jordan algebra for the symmetrized product, and then a J-B algebra.

There are many connections between the order structure and the product structure in a Banach-power-associative system. We list in the following some of them, but we introduce first

DEFINITION V.4. — Let $(\mathscr{A}, \mathscr{A}^+, \mathbb{1})$ be a complete order unit space. A square is said to be associated if it satisfies the condition of proposition II.2. In particular $\mathscr{A}^+ = \{ A^2 \mid A \in \mathscr{A} \}$.

Suppose $(\mathscr{A}, \mathscr{A}^+, \mathbb{1})$ is given, then:

CONSEQUENCE V.5:

V.5.1) There always exists on $\mathscr{A}$ an associated square ([*18*]): if $A \in \mathscr{A}$ and $\mathscr{A}_A$ is the vector subspace generated by $\mathbb{1}$ and A, define $\mathscr{A}_A^+ = \mathscr{A}_A \cap \mathscr{A}^+$. Then $(\mathscr{A}_A, \mathscr{A}_A^+, \mathbb{1})$ is a two dimensional Banach lattice, thus isomorphic (for the order and the norm) to $C(\check{S}(\mathscr{A}_A))$ (continuous functions of the Silŏv boundary of $\mathscr{A}_A^+$ ([*21*]), and that space is a Banach-power-associative system for the square of functions.

V.5.2) If the induced product associated to an associated square is bilinear, $\mathscr{A}^{**}$ is a J-B-W-algebra by the previous theorem and [*10*]. Thus the closure of $(\mathscr{A}^{**})^+$ for the deduced norm of a selfpolar form associated to a normal state is a facially homogeneous selfdual cone which is independent of that form ([*12*] VII 1.1).

V.5.3) Note that the bilinearity is not related to the two previous geometrical properties: let X be a set such that card $X = 3$, $\mathscr{A} = C(X)$, $\mathscr{A}^+$ the positive functions and $\mathbb{1} : X \to 1$; then $(\mathscr{A}, \mathscr{A}^+, \mathbb{1})$ has two structures of Banach-power-associative system, the former being bilinear, the latter being not ([*23*]). Thus the bilinearity is not a global notion.

V.5.4) In 1) and in the counterexemple of Sherman, $\mathscr{A}_A$ is a two-dimensional associative J-B-algebra, thus isomorphic to C(X) where card $X = 2$. More generally if an associated square is quadratic (i. e. $\dim(\mathscr{A}_A) = 2$ $\forall A \in \mathscr{A}$) then $\mathscr{A}_A \approx C(X)$ with card $X = 2$ ([*11*] 3.2.2).

V.5.5) If $S(\mathscr{A})$ is strongly spectral ([*3*] [*4*]) (i. e. the associated square is given by a spectral decomposition), then the bilinearity is equivalent to $S(\mathscr{A})$ elliptic or $[P - P', Q - Q']\mathbb{1} = 0$ for all P-projections P and Q ([*4*] [*13*]).

V.5.6) There exists up to an isomorphism at most one bilinear product for an associated square on $(\mathscr{A}, \mathscr{A}^+, \mathbb{1})$: in fact every linear bijection preserving the order and the unity is a J-B-isomorphism ([*16*] with the same proof) and the previous theorem gives the claim.

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