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The classical limit of reduced quantum stochastic evolutions

by

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ABSTRACT. — The classical limit of the infinitesimal generators of quantum dynamical semigroups obtained from quantum diffusions are shown to be second order semi-elliptic differential operators and thus (Markov) generators of classical diffusions in phase space. Necessary and sufficient conditions are established, in terms of the quantum generators, under which (*a*) the Markov generator is strictly elliptic, (*b*) the classical diffusion is deterministic, and (*c*) the classical flow is symplectic.

RÉSUMÉ. — On montre que la limite classique du générateur infinitésimal d'un semi-groupe dynamique quantique, obtenu à partir d'une diffusion quantique, est un opérateur différentiel semi-elliptique du second ordre et donc, le générateur (Markov) d'une diffusion classique dans l'espace de phase. Des conditions nécessaires et suffisantes sont établies pour que

a) le générateur soit (positivement) elliptique,
b) la diffusion classique soit déterministe,
c) le flot classique soit symplectique.

§ 0. INTRODUCTION

Quantum Brownian motion [*6*] has recently been used to develop a non-commutative stochastic calculus, generalising the classical Ito calcu-

(*) This work was done while the second named author was supported by the Science and Engineering Research Council.

lus [*10*] [*11*] [*12*]. In this paper we consider the classical limit ($\hbar \to 0$) of quantum diffusions by examining the generators of their dynamical semigroups, and display a qualitative difference between the limiting semigroups according to whether or not the driving Q. B. M. is of minimal variance i. e., the « field » is at zero temperature.

In order to compare the classical with the quantum, we exploit the phase space formulation of quantum mechanics [*20*] [*16*] in which the associative and Lie structures on the set of (smooth) functions on phase space given by pointwise product and Poisson bracket are « deformed » [*3*] to give a new associative product and Lie bracket which, in the case of flat phase space, corresponds to Weyl's quantisation scheme [*19*] These are the Moyal product and sine bracket [*16*]. Uniqueness of the deformed structures has been discussed by Arveson [*2*] who considered the class of polynomials on a symplectic vector space and established uniqueness of the Moyal structures under the assumption of invariance under affine symplectic transformations, and Bayen *et al.* [*3*] who considered formal power series in the Poisson bracket (1).

The paper is arranged as follows: in § 1 we introduce twisted products of measures and functions [cf. *14*] (these encompass the deformations mentioned above) and give some convergence results; in § 2 we describe the appropriate version of the Weyl correspondence; in § 3 we review quantum Brownian motion and describe quantum diffusions and their dynamical semigroups; and in § 4 we show that, under certain technical restrictions, the generators of these semigroups have classical limits which are themselves generators of diffusions in phase space, and we determine the conditions under which these are deterministic (i. e. non-stochastic). All results later referred to have been elevated to the status of theorem.

We shall often assume without mention the identification of Hilbert space operators $A \in B(h_1)$ and their ampliation to operators $A \otimes I \in B(h_1 \otimes h_2)$. The summation convention for repeated indices is adopted throughout.

§ 1. TWISTED PRODUCTS

Let V be a finite dimensional real vector space equipped with its usual topology, then we have the following Banach spaces: C_0—the continuous complex valued functions on V vanishing at infinity; M—the complex Borel measures on V; L^∞ and L^1—the (equivalence classes of) complex valued functions on V that are essentially bounded, respectively absolutely integrable, with respect to Haar (Lebesgue) measure on V. We shall denote

(1) See also R. L. Hudson ; Generalised translation invariant dynamics. D Phil Thesis, Oxford, 1966.

by $\xrightarrow{*}$ weak $*$-convergence with respect to the dualities $C_0^* \cong M$ and $L^{1*} \cong L^\infty$; by M_K the subspace of M consisting of measures with compact support and by T the set of locally bounded Borel measurable functions $V \times V \to \mathbb{C}$.

LEMMA. — For μ, $v \in M_K$, $\omega \in T$ the map $\Phi_{\mu,v,\omega} : C_0 \to \mathbb{C}$ given by

$$f \to \int f(u+v)\omega(u,v)d(\mu \times v)(u,v)$$

is a bounded linear functional vanishing on functions with support outside (supp μ + supp v).

Proof. — The integrand is bounded on the support of $\mu \times v$ by $\| f \| \sup_{\substack{u \in \operatorname{supp} \mu \\ v \in \operatorname{supp} v}} | \omega(u,v) |$ and so the integral is well-defined and determines a bounded linear functional on C_0. The support property is clear. /////

Thus a measure $\mu *_\omega v$ of compact support is determined by the relation

$$\int f d(\mu *_\omega v) = \Phi_{\mu,v,\omega}(f)$$

We call $*_\omega$ the *twisted convolution* with *twist* ω, and note that for $\omega \equiv 1$ it is the usual convolution of measures. In general it is non-associative.

THEOREM 1. — Let $\{ \omega_i, \omega_i^\alpha : i = 1, \ldots, n \ \alpha \in \mathbb{R} \setminus \{ 0 \} \} \subset T$ be such that for each i $\omega_i^\alpha \to \omega_i$ as $\alpha \to 0$ uniformly on compact sets, then for $\mu_i \in M_K$,

$$(\boldsymbol{\mu}, \boldsymbol{\omega}^\alpha) \xrightarrow{*} (\boldsymbol{\mu}, \boldsymbol{\omega})$$

where $(\boldsymbol{\mu}, \boldsymbol{\omega}^\alpha)$ is the m-fold twisted convolution $\mu_1 *_{\omega_1^\alpha} \ldots *_{\omega_1^\alpha} \mu_{n+1}$ the bracketing being arbitrary but fixed throughout.

Proof.

$$\int f d(\boldsymbol{\mu}, \boldsymbol{\omega}) = \int f\Big(\sum_{i=1}^{n+1} u_i\Big) \prod_{j=1}^{n} \omega_j\Big(u_j, \sum_{i>j} u_i\Big) d(\mu_1 \times \ldots \times \mu_{n+1})(u_1, \ldots u_{n+1})$$

when $(\boldsymbol{\mu}, \boldsymbol{\omega}) = \mu_1 *_{\omega_1} (\mu_2 *_{\omega_2} (\ldots *_{\omega_n} \mu_{n+1}) \ldots)$, and may be similarly expressed for each of the alternative bracketings. In any case, if K is a compact set containing $\sum_{i \in I}$ supp μ_i for each subset I of $\{ 1, 2, \ldots, n+1 \}$ then

$$\left| \int f d(\boldsymbol{\mu}, \boldsymbol{\omega}) - \int f d(\boldsymbol{\mu}, \boldsymbol{\omega}^\alpha) \right|$$

$$\leqslant \| f \| \prod_{j=1}^{n+1} \| \mu_j \| \sup_{u_i, v_i \in K} \left| \prod_{i=1}^{n} \omega_i^\alpha(u_i, v_i) - \prod_{i=1}^{n} \omega_i(u_i, v_i) \right| \to 0 \quad \text{as} \quad \alpha \to 0.$$

/////

Now let Λ be a non-degenerate symplectic form on V (i. e. bilinear and skew-symmetric), so that V must be even dimensional. We denote by $\hat{\mu}$ the (symplectic) Fourier transform of μ:

$$\hat{\mu}(u) = \int e^{i\Lambda(u,v)} d\mu(v) \qquad \forall \mu \in \mathrm{M}$$

Since Borel measures on V are uniquely determined by their Fourier transforms, we may define, for each twist ω, a *twisted product* $\circ_\omega$ on $\hat{\mathrm{M}}_{\mathrm{K}}$ by

$$\hat{\mu} \circ_\omega \hat{\nu} = \widehat{\mu *_\omega \nu}$$

THEOREM 2. — For $\mu_\alpha, \mu \in \mathrm{M}$, $\mu_\alpha \xrightarrow{*} \mu \Rightarrow \hat{\mu}_\alpha \xrightarrow{*} \hat{\mu}$ i.e. the Fourier transform is weak*-continuous.

Proof.

$$\begin{aligned} \forall g \in \mathrm{L}^1 \qquad \int g \hat{\mu}_\alpha &= \int g(u) \int e^{i\Lambda(u,v)} d\mu_\alpha(v) du \\ &= \int\int g(u) e^{i\Lambda(u,v)} du d\mu_\alpha(v) \qquad \text{by Fubini} \\ &= \int \hat{g}(-v) d\mu_\alpha(v) \\ &\to \int \hat{g}(-v) d\mu(v) \qquad \text{since } \widehat{\mathrm{L}^1} \subset \mathrm{C}_0 \\ &= \int g \hat{\mu} \qquad \text{reversing the above.} \end{aligned}$$

/////

We now introduce the twisted products of interest to us here. For each $\delta > 0$, let $\omega_\delta = \exp i\Lambda_\delta$, $\rho_\delta = 2\cos\Lambda_\delta$, $\tau_\delta = (2/\delta)\sin\Lambda_\delta$ where $\Lambda_\delta = (\delta/2)\Lambda$ —these all belong to T and converge uniformly on compact sets to 1, 2 and Λ respectively as $\delta \to 0$. $\circ_1$ is just pointwise product and $\circ_\Lambda$ is Poisson bracket, as may be seen for instance by taking symplectic coordinates.

We shall denote $\circ_{\rho_\delta}$, $\circ_{\tau_\delta}$ by $[\ ,\ ]_+^\delta$ and $[\ ,\ ]_-^\delta$ respectively—these are the cosine and sine brackets of Moyal [*16*] and are intimately related to Weyl quantisation [*19*] to which we next turn. By theorems 1 and 2,

$$[f, g]_+^\delta \xrightarrow{*} 2f \cdot g; \quad [f, g]_-^\delta \xrightarrow{*} \{ f, g \}_{\mathrm{P.B.}} \quad \text{as} \quad \delta \to 0 \quad \forall f, g \in \hat{\mathrm{M}}_{\mathrm{K}} \quad (1)$$

and more generally, any combination of sine and cosine brackets weak*-converges to the corresponding combination of Poisson brackets and pointwise products. Incidentally, it is when taking combinations of sine and cosine brackets that convergence *in the point-wise sense* breaks down.

§ 2. WEYL QUANTISATION

Let W be a representation of the canonical commutation relations (C. C. R.) over (V, Λ)—that is a map from V into $\mathfrak{B}(h)$ (the algebra of bounded linear operators on some Hilbert space h) satisfying

i) each $W(v)$ is unitary
ii) $W(u)W(v) = \exp[i\Lambda_\hbar(u, v)]W(u + v) \qquad \forall u, v \in V$
iii) $v \to W(v)$ is strongly continuous.

For each $f \in \hat{M}$ we define the operator $Q_f \in \mathfrak{B}(h)$ by

$$Q_f = \int W(u)d\mu(u) \qquad \text{where} \qquad f = \hat{\mu}$$

—the integral being understood in the weak sense.

LEMMA. — For $f = \hat{\mu} \in \hat{M}$,

$$\| \mu \| \geqslant \| Q_f \| \geqslant \| \hat{\mu}_J \|$$

where $\mu_J = \zeta\mu$, $\zeta(u) = \exp[-\Lambda_\hbar(u, Ju)/2]$ and J is a linear operator giving a Λ-allowed complex Hilbert structure to V (i. e. satisfying $J^2 = -I$, $\Lambda(Ju, Jv) = \Lambda(u, v)$, $\Lambda(u, Ju) \geqslant 0 \ \forall u, v \in V$) [14].

Proof. — The first inequality is clear from the definition of Q_f. Assume that the representation is irreducible and let W_0 denote the Weyl form of the Fock representation corresponding to (V, Λ, J) and Ω the corresponding vacuum state vector. We have, by von Neumann's uniqueness theorem [17]

$$\begin{aligned} \| Q_f \| &= \sup_{\|\phi\| = \|\psi\| = 1} \left| \int \langle \phi, W(u)\psi \rangle \, d\mu(u) \right| \\ &\geqslant \sup_{v \in V} \left| \int \langle W_0(v)\Omega, W_0(u)W_0(v)\Omega \rangle \, d\mu(u) \right| \\ &= \sup_{v \in V} \left| \int \exp[i\hbar\Lambda(u, v)]\zeta(u)d\mu(u) \right| \\ &= \| \hat{\mu}_J \| \end{aligned}$$

The assumption of irreducibility may be removed by taking direct sums.

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THEOREM 3. — The map $f \mapsto Q_f$ is an algebraic $*$-isomorphism of $(\hat{M}_K, \circ_{\omega_\hbar})$ onto a subalgebra $\mathscr{M}$ of $\mathfrak{B}(h)$—the involution on $\hat{M}_K$ being complex conjugation.

Proof. — Linearity is clear and injectivity follows from strict positivity of ζ and uniqueness of the Fourier transform, by the lemma. Abreviating $\circ_{\omega_h}$ to $\circ_\hbar$, we have for $f = \hat{\mu}$, $g = \hat{\nu} \in \mathrm{M_K}$

$$\begin{aligned}\langle \phi, \mathrm{Q}_f\mathrm{Q}_g\psi \rangle &= \int \langle \mathrm{Q}_f^*\phi, \mathrm{W}(v)\psi \rangle \, d\nu(v) \\ &= \int\int \langle \phi, \mathrm{W}(u)\mathrm{W}(v)\psi \rangle \, d\mu(u)d\nu(v) \\ &= \int \langle \phi, \mathrm{W}(u+v)\psi \rangle \exp [i\Lambda_\hbar(u,v)] d(\mu \times \nu)(u,v) \\ &= \langle \phi, \mathrm{Q}_{f \circ_\hbar g}\psi \rangle \qquad \forall \phi, \psi \in h\end{aligned}$$

and since $\hat{\mu}^* = \hat{\tilde{\mu}}$ where $\tilde{\mu}(\mathrm{U}) = \overline{\mu(-\mathrm{U})}$ we have

$$\begin{aligned}\langle \phi, \mathrm{Q}_{f^*}\psi \rangle &= \int \langle \phi, \mathrm{W}(v)\psi \rangle \, d\tilde{\mu}(v) \\ &= \int \overline{\langle \psi, \mathrm{W}(-v)\phi \rangle \, d\mu(-v)} \\ &= \overline{\langle \psi, \mathrm{Q}_f\phi \rangle} \\ &= \langle \phi, \mathrm{Q}_f^*\psi \rangle \qquad \forall \phi, \psi \in h\end{aligned}$$

which completes the proof. /////

COROLLARY.

$$\forall f, g \in \hat{\mathrm{M}}_{\mathrm{K}}, \qquad (i\hbar)^{-1}[\mathrm{Q}_f, \mathrm{Q}_g] = \mathrm{Q}_{[f,g]_-^\hbar}\,; \qquad [\mathrm{Q}_f, \mathrm{Q}_g]_+ = \mathrm{Q}_{[f,g]_+^\hbar}$$

establishing the connection between Moyal's formalism and Weyl quantisation.

§ 3. QUANTUM STOCHASTIC CALCULUS [6] [10] [11] [12]

Fix $n \in \mathbb{N}_+$, $\gamma \geqslant 1$ and let λ, $\mu \geqslant 0$ be such $\lambda^2 + \mu^2 = \gamma^2$, $\lambda^2 - \mu^2 = 1$. Let $\Gamma_\mathrm{N} = \overset{(\mathrm{N})}{\otimes} \mathscr{F}(H)$ where $H = \mathrm{L}^2(0,\infty) \oplus \mathrm{L}^2(0,\infty)$ and $\mathscr{F}(H)$ is the symmetric Fock space over H, let $\Omega_\mathrm{N} = \overset{(\mathrm{N})}{\otimes} \Omega$ where $\Omega = (1, 0, 0, \ldots)$ and let $a(.)$, $a^*(.)$ denote the Fock annihilation and creation operators on $\mathscr{F}(H)$ respectively. Now let $\mathrm{A}_t = a(\lambda\chi_{[0,t]}, 0) + a^*(0, \mu\chi_{[0,t]})$ and

$$\mathrm{A}_t^j = \overbrace{\mathrm{I} \otimes \ldots \otimes \mathrm{I}}^{j-1} \otimes \mathrm{A}_t \otimes \mathrm{I} \otimes \ldots \otimes \mathrm{I} \quad \text{on} \quad \Gamma_\mathrm{N}$$

then $\{ \Gamma_\mathrm{N}, \Omega_\mathrm{N}, \mathrm{A}^j, \mathrm{A}^{j*} : j = 1, \ldots, \mathrm{N}) \}$ is an N-dimensional quantum Brownian motion of variance γ^2—that is, when expectations are determined by Ω_N, and $\mathrm{Q}_t^{j,\theta} := e^{-i\theta}\mathrm{A}_t^j + e^{i\theta}(\mathrm{A}_t^j)^*$, $\quad \theta \in [0, 2\pi)$

i) the process is Gaussian (i. e. the corresponding C. C. R. representation [*6*] is quasi-free) with mean zero
ii) $Q_t^{j,\theta}$ is independent (²) of $Q_s^{k,\phi}$ for $j \neq k$
iii) $Q_t^{j,\theta} - Q_s^{j,\theta}$ is independent of $Q_r^{j,\phi}$ for $r \leqslant s < t$, and

$$\mathbb{E}[(Q_t^{j,\theta} - Q_s^{j,\theta})(Q_t^{j,\phi} - Q_s^{j,\phi})] = \hbar\gamma^2(t - s)e^{i(\theta-\phi)}$$

A stochastic calculus is developped in [*10*, *12*]; in particular a « product formula » for stochastic integrals

$$\int F_j dA^{j*} + G_j dA^j + H dt$$

is established where the operator-valued integrands satisfy certain adaptedness, domain and boundedness conditions. This is summarised by the differential relations

$$\begin{gathered} dA^j dA^{k*} = \delta^{jk}\lambda^2\hbar dt\,; \qquad dA^{j*}dA^k = \delta^{jk}\mu^2\hbar dt \\ dA^j dA^k = dA^{j*}dA^{k*} = dA^j dt = dA^{j*}dt (= (dt)^2) = 0 \end{gathered} \tag{2}$$

generalising the classical Itô differential relations

$$dB^j dB^k = \delta^{jk}\sigma^2 dt\,; \qquad dB^j dt (= (dt)^2) = 0$$

which are satisfied by $B^j := (A^j + A^{j*})$ when $\gamma^2\hbar = \sigma^2$.

By ampliation, we have a quantum Brownian motion on $h_0 \otimes \Gamma_N$ for any « initial Hilbert space » h_0. In particular, consider the stochastic differential equation

$$\begin{cases} dV = V(X_j dA^{j*} + Y_j dA^j + Z dt) \\ V_0 = I \end{cases} \tag{3}$$

where $X_j, Y_j, Z \in \mathcal{N}_0$—a von Neumann sub-algebra of $B(h_0) \otimes I$. It is shown in [*10*, *12*] that there is a unique solution to this and that the solution is *unitary valued* exactly when the coefficients X_j, Y_j, Z are related as follows:

$$\left.\begin{aligned} X_j + Y_j^* &= 0 \\ Z + Z^* + \hbar(\lambda^2 X_j^* X_j + \mu^2 X_j X_j^*) &= 0 \end{aligned}\right\} \tag{4}$$

the necessity of these relations being clear from (2).

We introduce a temperature parameter for the « field » represented by the quantum Brownian motion via the relation $\gamma^2 = \coth(\beta\hbar/2)$—the state determined by the vector Ω_N is then a Gibbs state (for the N-dimen-

(²) A family $(N_\beta : \beta \in B)$ of von Neumann algebras is *independent* if whenever $X_i \in N_{\beta_i}$ with β_i distinct, $\mathbb{E}[X_1 \dots X_k] = \mathbb{E}[X_1] \dots \mathbb{E}[X_k]$ and a family of collections of essentially self-adjoint operators is independent if the family of von Neumann algebras generated by their spectral projectors is independent.

sional harmonic oscillator) at inverse temperature β, for the canonical system $\{ [2(t-s)]^{-1/2}(Q_t^j - Q_s^j),\ [2(t-s)]^{-1/2}(P_t^j - P_s^j) : j = 1, \dots, N \}$ where $P_t^j = Q_t^{j,\pi/2}$ [5].

From (4) we see that the general form of a unitary-valued process satisfying an s. d. e. of the form (3) is

$$\begin{cases} dV = \hbar^{-1} V([L_j dA^{j*} - L_j^* dA^j] + [iH - (2(1-e^{-\beta\hbar}))^{-1}(L_j^* L_j + e^{-\beta\hbar} L_j L_j^*)]dt) \\ V_0 = I \end{cases}$$

where we have put $(2i)^{-1}(Z - Z^*) = \hbar^{-1}H$, $X_j = \hbar^{-1}L_j$.

We call the family of $*$-isomorphisms from $\mathcal{N}_0$ into

$$\mathcal{N} := \mathcal{N}_0 \vee \{ \exp iQ_t^{j,\theta} : t, \theta \geqslant 0, j = 1, \dots, N \}$$

induced by unitary conjugation by such processes *quantum diffusions* and write $\{ v_t : t \geqslant 0 \}$.

The *time zero conditional expectation* $\mathbb{E}_0$ from $\mathcal{N}$ onto $\mathcal{N}_0$ is defined by continuous linear extension of the map [*10*] [*12*]

$$S \otimes T \to \langle \Omega_N, T\Omega_N \rangle S \otimes I$$

This is invariant under any state Φ of the form $\phi \otimes \omega_N$, where ϕ is a normal state on $\mathcal{N}_0 : \Phi \circ \mathbb{E}_0 = \Phi$, and, when Φ is faithful ($\gamma > 1$) this is a conditional expectation in the sense of [*18*].

Now we obtain evolutions in $\mathcal{N}_0$ by considering

$$\begin{aligned} [0, \infty) \times \mathcal{N}_0 &\to \mathcal{N}_0 \\ (t, X) &\mapsto \mathbb{E}_0[V_t X V_t^{-1}] \end{aligned}$$

which is a (uniformly) continuous semigroup of completely positive maps whose infinitesimal generator $\mathcal{L}$ is given by [*10*] [*12*]

$$\mathcal{L}(X) = \hbar^{-1}(i[H, X] + \lambda^2 L_j^* X L_j + \mu^2 L_j X L_j^* - (1/2)[\lambda^2 L_j^* L_j + \mu^2 L_j L_j^*, X]_+)$$

or, in Lindblad form [*15*] $\hbar^{-1}([H, X] + K_k^* X K_k - (1/2)[K_k^* K_k, X]_+)$ where

$$K_k = \begin{cases} \lambda L_k & k = 1, \dots, N \\ \mu L_{k-N}^* & k = N+1, \dots, 2N \end{cases}$$

If we let $L_j = F_{2j-1} + iF_{2j}$ with each F_k self-adjoint, then after some manipulations we obtain

$$\begin{aligned} \mathcal{L}(X) = \hbar^{-1}(i[H, X] &+ (-1)^k (i/2)[F_k, [F_{\tilde{k}}, X]]_+ \\ &- (1/2)\coth(\beta\hbar/2)[F_k, [F_k, X]]) \quad (5) \end{aligned}$$

where

$$\sim : k \mapsto \begin{cases} k+1 & k \text{ odd} \\ k-1 & k \text{ even} \end{cases}$$

H is the Hamiltonian of the evolution in the absence of dissipation terms $\{ F_k \}$.

§ 4. THE CLASSICAL LIMIT

We are now in a position to state the main result.

THEOREM 5. — Let Q be a quantisation of the symplectic space (V, Λ) in $\mathfrak{B}(h_0)$. Given any quantum diffusion $\{ v_t : t \geqslant 0 \}$ determined by the s. d. e.

$$\left\{ \begin{array}{l} dV = \hbar^{-1} V([L_j dA^{j*} - L_j dA^j] + [iH - (2(1-e^{-\beta\hbar}))^{-1}(L_j^* L_j + e^{-\beta\hbar} L_j L_j^*)]dt) \\ V_0 = I \end{array} \right.$$

in which $H, L_j \in \mathscr{M} = Q(\hat{M}_K)$, $H = H^*$, there exists, for each starting point $u \in V$, a (path-wise unique) classical diffusion $\{ Y_t : t \geqslant 0 \}$ on V driven by a 2N-dimensional Brownian motion of variance $2/\beta$ whose Markov generator $\mathscr{C}$ is related to the quantum dynamical generator $\mathscr{L}$ by

$$\mathscr{C}(f) = \text{weak}^* - \lim_{\hbar \to 0} (Q^{-1} \mathscr{L} . Q)(f) \qquad \forall f \in \hat{M}_K$$

Proof. — Writing ℓ for $Q^{-1}\mathscr{L}.Q$, we have from (5) and the corollary to theorem 3

$$\ell(f) = [f, h]_-^\hbar + (\hbar/2) \coth(\beta\hbar/2) [f_k, [f_k, f]_-^\hbar]_-^\hbar$$
$$+ (-1)^k/2 [f_k, [f, f_{\tilde{k}}]_-^\hbar]_+^\hbar \qquad \forall f \in \hat{M}_K$$

whose weak*-limit as $\hbar \to 0$ is, by L'Hopital's rule and (1)

$$\{ f, h \} + (-1)^k f_k \{ f, f_{\tilde{k}} \} + \beta^{-1} \{ f_k \{ f_k, f \} \}$$

and it is easy to see that this is a semi-elliptic operator $\mathscr{C}$ acting on f. Choosing symplecting coordinates $\{ e_i \}$ for V, the coefficient matrix for the second order part of $\mathscr{C}$ is $(2/\beta)\sigma\sigma^t$, where $\sigma_k^j = \Lambda^{ij} \partial_i f_k$ and $\{ \Lambda^{ij} \} = \{ \Lambda \Lambda(e_i, e_j) \}^{-1}$. It follows from the theory of classical diffusion processes [e. g. *13*] that there is a pathwise unique classical diffusion on V, for each $u \in V$, satisfying the Itô equation

$$\left\{ \begin{array}{l} dY^j = \sigma_k^j(Y) dB^k + b^j(Y) dt \\ Y_0^j = (q, p) \end{array} \right.$$

where (q, p) are the coordinates of u, b^j the coefficient vector of the first order part of $\mathscr{C}$ and $\{ B_t : t \geqslant 0 \}$ is a 2N-dimensional Brownian motion of variance $(2/\beta)$ and whose Markov generator is $\mathscr{C}$. The uniform boundedness of the coefficients σ, b precludes the possibility of « explosion » [*13*]. /////

COROLLARY. — The Markov generator obtained above is (strictly) elliptic if and only if the rank of the derivative of the map $F : V \to \mathbb{R}^{2N}$ is maximal, that is 2N.

Next we consider the conditions under which the classical diffusion obtained in the previous theorem is in fact deterministic.

THEOREM 6. — Let Y^u be the classical diffusion (starting at u) corresponding to the quantum diffusion $\{ v_t : t \geqslant 0 \}$ as in theorem 5. Y^u is deterministic if and only if *either* the « field » is at zero temperature ($\beta^{-1} = 0$) *or* the quantum evolution is governed only by a Hamiltonian (i. e. $v_t(X) = e^{it\hbar^{-1}H} X e^{-it\hbar^{-1}H}$).

Proof. — Y^u is deterministic if and only if $(2/\beta)\sigma = 0$. $\sigma^j_k = \Lambda^{ij}\partial_i f_k$, so $\sigma = 0$ if and only if each f_k is constant, in other words, if and only if each L_k is a multiple of the identity. The s. d. e. governing $\{ v_t : t \geqslant 0 \}$ is, in this case,

$$\begin{cases} dV = \hbar^{-1} V\left(z_j dA^{j*} - \bar{z}_j dA^j + \left[iH - (\gamma^2/2) \sum_{j=1}^{N} | z_j |^2 \right] dt \right) \\ V_0 = I \end{cases}$$

so that, by the quantum Itô product formula

$$V_t = e^{i\hbar^{-1}(tH + r.Q_t^\theta)}$$

where $r.Q_t^\theta = r_1 Q_t^{1,\theta} + \ldots + r_N Q_t^{N,\theta_N}$ and $z_j = i r_j e^{i\theta_j}$. The result now follows. /////

Rather than looking at the collection of diffusions $\{ Y^u : u \in V \}$ we can consider the flow $\{ \phi_t := Y_t^{\cdot} \mid t \geqslant 0 \}$, a version of which consists of diffeomorphisms of V, we then have the following result.

THEOREM 7. — The flow of the stochastic dynamical system determined by $\mathscr{C}$ (of theorem 5) is symplectic if and only if

$$\sum_{j=1}^{N} df_{2j} \wedge df_{2j-1} = 0 \tag{6}$$

Proof. — First notice that in Hörmander form,

$$\mathscr{C} = X_0 + \beta^{-1} \sum_{k=1}^{2N} X_k^2$$

where $X_k = X_{f_k}$ and $X_0 = \sum_{j=1}^{N} (f_{2j} X_{f_{2j-1}} - f_{2j-1} X_{f_{2j}}) + X_h$.

Necessary and sufficient conditions for the flow to be symplectic are [*4*] that each of the vector fields X_i ($i = 0, 1, \ldots, 2N$) are locally Hamiltonian. This is clearly true for $i = 1, 2, \ldots, 2N$ and since

$$gX_f - fX_g = I(gdf - fdg)$$

where I is the canonical isomorphism between T^*V and TV determined by Λ, and

$$d(gdf - fdg) = dg \wedge df - df \wedge dg = 2dg \wedge df,$$

X_0 is locally Hamiltonian if and only if (6) is satisfied. /////

Notice in particular that the classical diffusions corresponding to quantum diffusions whose generators are self-adjoint are symplectic.

SUMMARY AND CONCLUSION

Given a quantisation Q of (equivalently, a CCR representation over) a symplectic space (V, Λ), any quantum diffusion $\{ v_t : t \geqslant 0 \}$ determined by an s. d. e.

$$\begin{cases} dV = \hbar^{-1}V(L_j dA^{j*} - L_j dA^j + [iH - (2(1-e^{-\beta\hbar}))^{-1}(L_j^* L_j + e^{-\beta\hbar}L_j L_j^*)]dt) \\ V_0 = I \end{cases} \qquad (1)_{qu.}$$

where $\{ A_t, A_t^* : t \geqslant 0 \}$ is an N-dimensional Q. B. M. of variance $\hbar \coth(\beta\hbar/2)$ and H, $L_j \in \mathscr{M}_Q$, has dynamical semigroup

$$T_t = \mathbb{E}_0 \circ v_t \qquad (2)_{qu.}$$

with generator

$$\mathscr{L} = \hbar^{-1}(i[H, \ ] + (-1)^k(i/2)[F_k, [F_{\tilde{k}}, \ ]]_+ - \coth(\beta\hbar/2)[F_k, [F_k, \ ]] \qquad (3)_{qu.}$$

where $L_j = F_{2j-1} + iF_{2j}$, which has classical limit

$$\mathscr{C} = \beta^{-1}(\Lambda^{ij}\Lambda^{lm}\partial_i f_k \partial_l f_k)\partial_j\partial_m + b^j\partial_j \qquad (3)_{cl.}$$

which is the generator of a classical diffusion Y on V, for each $u \in V$, satisfying

$$\begin{cases} dY^j = (\Lambda^{ij}\partial_i f_k)(Y)dB^k + b^j(Y)dt \\ Y_0 = u \end{cases} \qquad (11)_{cl.}$$

where B is a 2N-dimensional Brownian motion of variance

$$2/\beta = \lim_{\hbar \to 0} \hbar \coth(\beta\hbar/2)$$

and a Markov semigroup on C_0

$$(P_t f)(u) = \mathbb{E}[f(Y_t^u)] \qquad (2)_{cl.}$$

Deterministic classical evolutions arising from quantum diffusions « without heat » and Heisenberg evolutions.

Several questions immediately arise from this work. First of all, is it possible to obtain the classical diffusions directly as a limit ($h \to 0$) of the quantum diffusion? The classical processes would have to be regarded from the point of view of [*1*], that is one composes with a class of functions (e. g. $\hat{M}_K$): $f \circ Y_t^u$. Then one could consider the random functions $(f \circ Y_t)(\omega)$ and quantise these. One would then seek to compare the resulting family of operators on h_0 with the operator $v_t(Q_f)$. A method of quantising classical diffusions is clearly required. Secondly, the deformation theory of Bayen *et al.* [*3*] should permit extension to deal with non-flat phase space.

Davies [*7*] has considered the classical limit of a class of quantum dynamical semigroups arising from weak and singular coupling limits of quantum particles interacting with an infinite free reservoir, and obtained Markov semigroups on phase space. His is a different limiting process, using a scaling similar to one used by Hepp [*9*] and working in state space (Schrödinger picture).

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