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Embedding set configuration spaces into those of Ruelle's point configurations

by

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ABSTRACT. — In this paper we study configuration spaces with finite sets (« molecules ») as elementary ingredients and transform them into those which have one-point sets (« atoms ») as elementary components including certain local restrictions concerning the alphabets. This is carried out for arbitrary countable sets and $\mathbb{Z}^\nu$-lattices. The topological properties of transitivity and mixing of the corresponding translations are considered for such configuration spaces and we show that this notion is useful for questions originally posed in terms of molecules.

RÉSUMÉ. — On étudie des espaces de configurations ayant des ensembles finis (« molécules ») comme composants élémentaires et on les transforme en espaces de configurations ayant des parties réduites à un seul point comme ingrédients élémentaires. Cette construction est faite pour des ensembles dénombrables arbitraires et pour des réseaux $\mathbb{Z}^\nu$. En outre, on considère les propriétés de transitivité et de mélange des translations correspondantes dans de tels espaces et on démontre que cette notion est utile pour des problèmes posés originellement au moyen de molécules.

1. INTRODUCTION

In his book [*9*] D. Ruelle presented lattice thermodynamics on the base of a rather general notion of configuration spaces. His setting works with arbitrary countable sets L as underlying spaces, including, of course, all lattices $\mathbb{Z}^\nu$. For each point $x \in \mathrm{L}$ an arbitrary finite alphabet is allowed and in the large variety of resulting configuration spaces there are found those without any neighbour conditions as well as those with e. g. hard cores. This is possible by means of a locally finite system $\mathscr{F}$ of subsets $\Lambda \subset \mathrm{L}$ and a certain system of restrictions concerning the alphabets on the sets Λ. In the following we denote all configuration spaces of this type as point configuration spaces since the single points of L take the leading part. Many atomic models are well described within this framework. Concerning Ruelle's definition see sect. 2. A closer look shows this approach applicable also to special cases of proper molecular models. The latter are of interest e. g. in the theory of liquid crystals (see e. g. O. J. Heilmann/E. H. Lieb [*5*]), especially in the theory of phase transitions. To include these models into Ruelle's context one attempts to choose $\mathscr{F}$ and the local restrictions Λ in such a way that the arising space describes configurations of molecules. In order to relate *arbitrary* and possibly *complicated* such models with Ruelle's procedures it is desirable to have an independent definition of a set configuration space. It is given in sect. 2 and works again with a countable set L, but now the part of the points $x \in \mathrm{L}$ is played by a fixed set $\mathscr{G}$ of finite subsets G of L which enables partitions of L. We construct for each such set configuration space a certain point configuration space and a bijective mapping between them. This is possible especially cially by the extension of the given alphabet and is known as a usual procedure from other theorems in this branch (see e. g. M. Denker/C. Grillenberger/K. Sigmund [*1*], p. 119). Thus, in a sense, we get the result that up to a homeomorphism set configuration spaces may be considered as point configuration spaces. Further on, if the set configuration space is based on $\mathrm{L} = \mathbb{Z}^\nu$ and is translation invariant, then it is called a $\mathbb{Z}^\nu$-set lattice system and the corresponding point configuration space may be chosen also translation invariant. Thus $\mathbb{Z}^\nu$-set lattice systems prove a part (in fact: a proper part (see remark 3.4)) of the ν-dimensional versions of finite type subshifts (W. Parry [*8*]) which are called $\mathbb{Z}^\nu$-lattice systems in Ruelle's and $\mathbb{Z}^\nu$-*point* lattice systems in our setting.

In the latter case notions like transitivity and mixing make sense if the translation group in $\mathbb{Z}^\nu$ is considered. Although such problems may be handled also after the mentioned mapping into the point configuration space we obtain especially concerning mixing a result (corollary 4.3) being rather illustrative in the initial set configuration space.

While our considerations are topological, it is, of course, possible to define measures and interactions on set configuration spaces and to translate them through the bijective mapping continuously into a certain point configuration space and vice versa. In this way all notions and results of thermodynamic formalism in the latter type of spaces become meaningful also in arbitrary set configuration spaces. For relevant topics the reader is referred to [9] and the original papers [2], [3], [4] of R. L. Dobrushin and [7] of O. E. Lanford/D. Ruelle.

2. GENERAL EMBEDDING CASE

As announced we define a notion of set configuration space suitable for situations where finite sets instead of singletons are the elementary particles. Notation is close to [9]. Let L be a set and $\mathcal{G}$ a system of subsets $G \subset L$. Then each subset $\mathcal{P} \subset \mathcal{G}$, where

$$(G_1, G_2 \in \mathcal{P},\ G_1 \neq G_2) \Rightarrow G_1 \cap G_2 = \emptyset$$
$$\bigcup_{\mathcal{G} \in \mathcal{P}} G = L,$$

is called a partition of L with elements from $\mathcal{G}$. Let $\pi(\mathcal{G})$ denote the set of all such partitions. Of course, $\pi(\mathcal{G}) = \emptyset$ is possible.

$\mathcal{G}$ is called locally finite, if each $x \in L$ is an element of only a finite number of elements G from $\mathcal{G}$.

2.1. Definition. — Let L be an infinite countable set and $\mathcal{G}$ a nonempty locally finite system of finite subsets of L such that

i) $$\pi(\mathcal{G}) \neq \emptyset,$$

ii) $$G \in \mathcal{G} \Rightarrow \exists \mathcal{P} \in \pi(\mathcal{G}) : G \in \mathcal{P}.$$

Further let Ω_G be a nonempty finite set for each $G \in \mathcal{G}$. Then the set

$$\Omega^* = \bigcup_{\mathcal{P} \in \pi(\mathcal{G})} \bigtimes_{G \in \mathcal{P}} \Omega_G \tag{2.1}$$

is called a *set configuration space* (= SCS). Notation: $\Omega^* = (L, (\Omega_G)_{G \in \mathcal{G}})$.

Denoting by ξ^* a single configuration, (2.1) means

$$\xi^* \in \Omega^* \Leftrightarrow (\exists \mathcal{P} \in \pi(\mathcal{G}),\ \xi^*_G \in \Omega_G(G \in \mathcal{P}) : \xi^* = (\xi^*_G)_{G \in \mathcal{P}}). \tag{2.2}$$

Each $\mathcal{P} \in \pi(\mathcal{G})$ consists of an infinite countable number of elements G from $\mathcal{G}$.

2.2. Remark. — Ruelle's notion of configuration spaces, as defined in [9], is based on the following data: an infinite countable set L, finite sets $\Omega_x (x \in L)$, a locally finite set $\mathcal{F}$ of finite subsets $\Lambda \subset L$ and a family $(\overline{\Omega_\Lambda})_{\Lambda \in \mathcal{F}}$ such that $\overline{\Omega_\Lambda} \subset \prod_{x \in \Lambda} \Omega_x \ (\Lambda \subset \mathcal{F})$. Then his configuration space is

$$\Omega = \left\{ \xi \in \prod_{x \in L} \Omega_x \,\middle|\, \forall \Lambda \in \mathcal{F} : \xi|_\Lambda \in \overline{\Omega}_\Lambda \right\} \tag{2.3}$$

and is denoted also by $\Omega = (\mathrm{L}, (\Omega_x)_{x\in \mathrm{L}}, (\overline{\Omega}_\Lambda)_{\Lambda\in\mathscr{F}})$ and in contrast to 2.1 we named it a *point configuration space* (= PCS).

The topology is generated by the system $\mathscr{S}$ of sets

$$_x[\omega] = \{\, \xi \in \Omega \mid \xi_x = \omega \,\} \qquad (x \in \mathrm{L}, \omega \in \Omega_x)$$

as subbasic sets and generation means: each open set is a union of a finite intersection of subbasic sets (see e. g. J. L. Kelley [6]). Sometimes the system $\mathscr{B}$ of basic sets, consisting of all finite intersections of subbasic sets is also useful. Endowing $\bigtimes_{x\in \mathrm{L}} \Omega_x$ with the product topology the above defined topology becomes the trace topology with respect to Ω. Compactness of Ω in this topology follows from the closedness of this set in $\bigtimes_{x\in \mathrm{L}} \Omega_x$.

There is an obvious intersection between the two types (2.1) and (2.3) of configuration spaces, if $\mathscr{G} = \{\{x\} \mid x \in \mathrm{L}\} = \mathscr{F}$, $\Omega_{\{x\}} = \overline{\Omega}_{\{x\}} = \Omega_x$ and $\xi^* \in \Omega^*$ is identified with $\xi \in \Omega$ iff $\xi_x = \xi^*_{\{x\}}$.

If we weaken this connection between SCS and PCS to a homeomorphic map, each SCS may be interpreted as a certain PCS. This is the essential content of proposition 2.4. Therefore a topology in each SCS is needed.

2.3. Definition. — The topology introduced in $\Omega^* = (\mathrm{L}, (\Omega_\mathrm{G})_{\mathrm{G}\in\mathscr{G}})$ is that generated by the system $\mathscr{S}^*$ of subbasic sets

$$_\mathrm{G}[\omega] = \{\, \xi^* = (\xi^*_\mathrm{H})_{\mathrm{H}\in\mathscr{P}} \mid \mathscr{P} \in \pi_\mathrm{G}(\mathscr{G}),\ \xi^*_\mathrm{G} = \omega \,\} \qquad (\mathrm{G} \in \mathscr{G}, \omega \in \Omega_\mathrm{G})$$

where

$$\pi_\mathrm{G}(\mathscr{G}) = \{\, \mathscr{P} \in \pi(\mathscr{G}) \mid \mathrm{G} \in \mathscr{P} \,\}$$

denotes the system of all G-containing partitions (thereby 2.1 (*ii*) provides for $\pi_\mathrm{G}(\mathscr{G}) \neq \emptyset$).

Also in this case it is sometimes useful to deal with the system $\mathscr{B}^*$ of basic sets B, which are the finite intersections of sets in $\mathscr{S}^*$: $\mathrm{B} = \bigcap_{i=1}^{n} \mathrm{S}_i$, $\mathrm{S}_i = {}_{\mathrm{G}_i}[\omega_i]$. Of course, the single $\mathrm{G}_1, \ldots, \mathrm{G}_n$ have to belong to a certain partition $\mathscr{P} \in \pi(\mathscr{G})$, if B is desired nonvoid.

It is convenient to deal with the compactness of this topology later. In the special case mentioned after remark 2.2 it is the product of the discrete topologies of Ω_x $(x \in \mathrm{L})$ and therefore compact.

2.4. Proposition. — *Let $\Omega^* = (\mathrm{L}, (\Omega_\mathrm{G})_{\mathrm{G}\in\mathscr{G}})$ be an SCS. Then there exists a PCS $\Omega = (\mathrm{L}, (\Omega_x)_{x\in\mathrm{L}}, (\overline{\Omega}_\Lambda)_{\Lambda\in\mathscr{F}})$ and a homeomorphism $\mathrm{F} : \Omega^* \to \Omega$* (†).

Proof. — 1. Starting from Ω^* at first a suitable PCS Ω is constructed. We put

$$\Omega_x = \{\, (\mathrm{G}, \omega) \mid x \in \mathrm{G} \in \mathscr{G}, \omega \in \Omega_\mathrm{G} \,\} \qquad (x \in \mathrm{L}).$$

(†) The author's collaborator V. Warstat proved a similar theorem where $\mathscr{G}$ itself receives the part of L in the PCS (to be published elsewhere).

Thus each Ω_x is a finite set since there are only finitely many $G \in \mathcal{G}$ with $x \in G$. The system $\mathcal{F}$ of subsets of L and the restrictions $\Omega_\Lambda(\Lambda \in \mathcal{F})$ are defined by

$$\mathcal{F} = \{ \Lambda_x \mid x \in L \}, \qquad \text{where } \Lambda_x = \bigcup_{x \in G \in \mathcal{G}} G, \tag{2.4}$$

$$\overline{\Omega}_{\Lambda_x} = \left\{ \xi \in \prod_{x' \in \Lambda_x} \Omega_{x'} \mid (\xi_x = (G, \omega)) \wedge (x' \in G) \Rightarrow \xi_{x'} = (G, \omega) \right\}. \tag{2.5}$$

Characterizing the elements of Ω_x as colours in x, (2.5) means that the colour ξ_x determines the colour $\xi_{x'}$ for all $x' \in G$. In this way a PCS Ω is well defined.

2. Now we construct the asserted homeomorphism F. Noting (2.2) for each $\xi^* \in \Omega^*$ there exists a $\mathcal{P} \in \pi(\mathcal{G})$ where $\xi^* = (\xi^*_G)_{G \in \mathcal{P}}$ and $\xi^*_G \in \Omega_G$. Therefore by

$$\xi_x = (G, \xi^*_G) \Leftrightarrow x \in G \in \mathcal{P} \tag{2.6}$$

and the partition property of $\mathcal{P}$, each ξ_x, and with it

$$\xi = (\xi_x)_{x \in L} = F(\xi^*),$$

is well defined. Each ξ fulfils the conditions given by (2.5). Namely, if $\xi_x = (G, \omega)$ for some $x \in L$ then $x \in G \in \mathcal{P}$ holds and each $x' \in G \in \mathcal{P}$ yields by (2.6) the colour $\xi_{x'} = (G, \xi^*_G)$.

3. To show F bijective let $\xi^*_1, \xi^*_2 \in \Omega^*$ and $\xi^*_1 \neq \xi^*_2$. We consider two cases. If ξ^*_1, ξ^*_2 belong to the same partition $\mathcal{P}$, then they must differ in at least one colour: $\xi^*_{1G} \neq \xi^*_{2G}$ for some $G \in \mathcal{P}$. By (2.6)

$$\xi_{1x} = (G, \xi^*_{1G}) \neq (G, \xi^*_{2G}) = \xi_{2x}$$

shows $F(\xi^*_1) \neq F(\xi^*_2)$. If ξ^*_1, ξ^*_2 belong to different partitions $\mathcal{P}_1, \mathcal{P}_2$, an $x \in L$ with $x \in G_i \in \mathcal{P}_i$ $(i = 1, 2)$ and $G_1 \neq G_2$ may be found and

$$\xi_{1x} = (G_1, \xi^*_{1G}) \neq (G_2, \xi^*_{2G}) = \xi_{2x}$$

is valid since ξ_{1x}, ξ_{2x} cannot agree in at least the first coordinate. Thus F is injective.

To show F surjective we take an arbitrary $\xi \in \Omega$, therefore (2.5) holds. If we define a mapping $q: L \to \mathcal{G}$ by

$$q(x) = G \Leftrightarrow \xi_x = (G, \omega),$$

then $\xi_x \in \Omega_x$ and $x \in q(x)$ $(x \in L)$. q generates a partition $\mathcal{P}$, especially we have

$$q(x_1) \cap q(x_2) \neq \varnothing \Rightarrow q(x_1) = q(x_2). \tag{2.7}$$

This is easily shown taking $\Omega_{\Lambda_{x_i}}$ $(i = 1, 2)$ and $x' \in q(x_1) \cap q(x_2)$. Then

$(G', \omega') = (G_i, \omega_i)$ holds, consequently $G_1 = G' = G_2$ and (2.7) follows. To find the corresponding ξ^* choose

$$\mathscr{P} = \{ q(x) \mid x \in L \}$$

and

$$\xi^* = (\xi^*_G)_{G \in \mathscr{P}}$$

such that

$$\xi^*_{q(x)} = \omega \Leftrightarrow \xi_x = (G, \omega).$$

This is only another description of (2.6) and it is obvious that ξ comes out, if F is applied to the just formed ξ^*.

4. To show continuity of F and F^{-1} we point out first, that

$$F({}_G[\omega]) = \bigcap_{x \in G} {}_x[(G, \omega)], \qquad (G \in \mathscr{G}, \omega \in \Omega_G) \tag{2.8}$$

where ${}_G[\omega]$, ${}_x[(G, \omega)]$ are defined in 2.3 and 2.2. If $\xi^* \in {}_G[\omega]$, then the corresponding $\xi = F(\xi^*)$ satisfies

$$\xi_x = (G, \omega) \qquad \text{for all} \quad x \in G$$

and thus we have « $\subset$ » in (2.8). The converse direction follows by the existence of a $\mathscr{P} \in \pi_G(\mathscr{G})$ and $\xi^* = (\xi^*_H)_{H \in \mathscr{P}}$ where $\xi^*_G = \omega$. Considering (2.5) and

$$x_1, x_2 \in G \Rightarrow {}_{x_1}[(G, \omega)] = {}_{x_2}[(G, \omega)],$$

(2.8) may be reduced to

$$F({}_G[\omega]) = {}_x[(G, \omega)], \qquad (x \in G \in \mathscr{G}, \omega \in \Omega_G). \tag{2.9}$$

Using again the notation $\mathscr{S}$, $\mathscr{S}^*$ for the subbasic systems in Ω, Ω^* (see 2.2, 2.3) (2.9) is equivalent to

$$F^{-1}(\mathscr{S}) = \mathscr{S}^*, F(\mathscr{S}^*) = \mathscr{S}.$$

Well known topological techniques yield $F^{-1}(\mathscr{O}) \subset \mathscr{O}^*$, $(F^{-1})^{-1}(\mathscr{O}^*) \subset \mathscr{O}$ and therefore continuity of F and F^{-1}, if $\mathscr{O}$, $\mathscr{O}^*$ denote the corresponding open set systems.

2.5. COROLLARY. — *The topology in $\mathscr{O}^*$ as defined in 2.3 is compact.*

Proof. — $\mathscr{O}^*$ is the continuous map of Ω under F^{-1}, thus compactness of Ω carries that of Ω^*.

3. EMBEDDING OF Z^ν-SET LATTICE SYSTEMS

The main result of sect. 2 can be sharpened if we take $L = \mathbb{Z}^\nu$ and introduce the translations $\tau^a : \mathbb{Z}^\nu \to \mathbb{Z}^\nu$; $\tau^a(x) = a + x$ $(a, x \in \mathbb{Z}^\nu)$. Then the additional assumptions produce additional results. Further it turns out that not every PCS is a homeomorphic map of an SCS.

3.1. DEFINITION. — An SCS Ω^* is called a $\mathbb{Z}^\nu$*-set lattice system* ($= \mathbb{Z}^\nu$-SLS), if the following properties hold:

i) $$L = \mathbb{Z}^\nu,$$

ii) $\mathcal{G}$ is translation invariant, i. e. : $G + a \in \mathcal{G}$ ($G \in \mathcal{G}$, $a \in \mathbb{Z}^\nu$) where $G + a := \{ g + a \mid g \in G \}$,

iii) $(\Omega_G)_{G\in\mathcal{G}}$ is translation invariant, i. e.: $\Omega_{G+a} = \Omega_G$ ($G \in \mathcal{G}$, $a \in \mathbb{Z}^\nu$).

3.2. REMARKS. — 1. Since $\mathcal{G}$ is locally finite each element $G \in \mathcal{G}$ allows a unique representation

$$G = G_0 + l \ (l \in \mathbb{Z}^\nu)$$

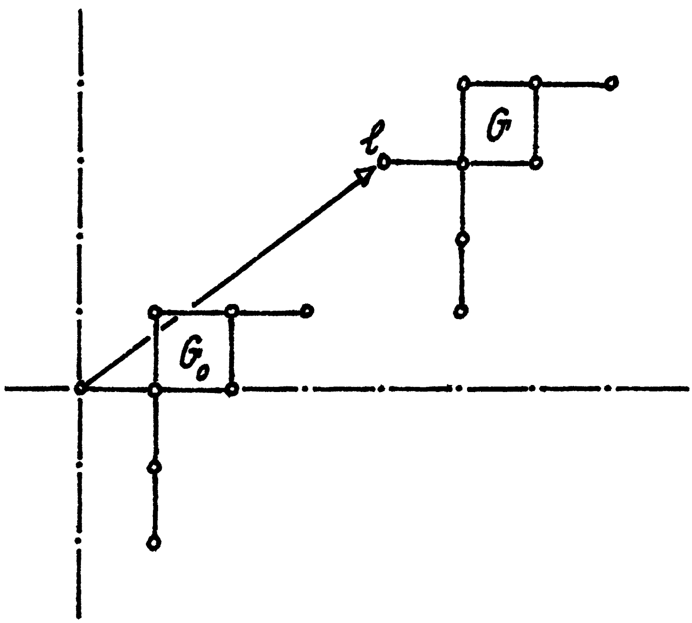

FIG. 1.

such that the first element in G_0 with respect to the lexicographical order is just $0 \in \mathbb{Z}^\nu$. Fig. 1 illustrates this in case of $\mathbb{Z}^2$. If $\mathcal{G}_0$ is the set of all these G_0, we have $\mathcal{G} = \{ G_0 + l \mid G_0 \in \mathcal{G}_0, l \in \mathbb{Z}^\nu \}$ and assuming translation invariance of $\mathcal{G}$ and $(\Omega_G)_{G\in\mathcal{G}}$, $\mathcal{G}_0 \subset \mathcal{G}$ is fulfilled. Thus a $\mathbb{Z}^\nu$-SLS is well defined, if $(\Omega_G)_{G\in\mathcal{G}_0}$ is given and can be written in the form $\Omega^* = (\mathbb{Z}^\nu, (\Omega_G)_{G\in\mathcal{G}_0})$.

2. Our definition 3.1 is similar to Ruelle's notion of a $\mathbb{Z}^\nu$-*(point)-lattice system* ($= \mathbb{Z}^\nu$-PLS) as defined in [*9*], p. 64, which is a specialisation of

his general notion of a PCS, mentioned in 2.2, if there are taken $L := \mathbb{Z}^\nu$, $\Omega_x := \Omega_0\,(x \in L)$ and $\mathcal{F}$, $(\overline{\Omega}_\Lambda)_{\Lambda \in \mathcal{F}}$ are translation invariant. Of course, for these systems $\Omega = (\mathbb{Z}^\nu, \Omega_0, (\overline{\Omega}_\Lambda)_{\Lambda \in \mathcal{F}})$ is a fitting notation.

3. In case of $\nu = 1$ $\mathbb{Z}$-PLS's together with their shift mapping are finite type subshifts (or: topological Markov chains) initiated by W. Parry [*8*] and instructively presented in M. Denker/C. Grillenberger/K. Sigmund [*1*].

3.3. Proposition. — *Let* $\Omega^* = (\mathbb{Z}^\nu, (\Omega_G)_{G \in \mathcal{G}_0})$ *be a* $\mathbb{Z}^\nu$*-SLS.* Then it *exists a* $\mathbb{Z}^\nu$*-PLS* $\Omega = (\mathbb{Z}^\nu, \Omega_0, (\overline{\Omega}_\Lambda)_{\Lambda \in \mathcal{F}})$ *and a homeomorphism* $F : \Omega^* \to \Omega$ *with the property*

$$F \circ \tau^a = \tau^a \circ F \qquad (a \in \mathbb{Z}^\nu) \qquad (\textit{equivariance of } F).$$

Proof. — We confine ourselves to those details being different from the procedure in the proof of 2.4, e. g. to the fact, that it is necessary to introduce new sets $\Omega_x = \Omega_0\,(x \in \mathbb{Z}^\nu)$ to attain a $\mathbb{Z}^\nu$-PLS. Starting from $\Omega^* = (\mathbb{Z}^\nu, (\Omega_G)_{G \in \mathcal{G}_0})$ an appropriate $\mathbb{Z}^\nu$-PLS $\Omega = (\mathbb{Z}^\nu, \Omega_0, (\overline{\Omega}_\Lambda)_{\Lambda \in \mathcal{F}})$ has to be constructed. Now we put

$$\Omega_0 = \{\, (y, G_0, \omega) \mid y \in G_0,\ G_0 \in \mathcal{G}_0,\ \omega \in \Omega_{G_0} \,\} \tag{3.1}$$

and this alphabet is a finite set, since $\mathcal{G}_0$ is finite. The latter follows from $(G \in \mathcal{G}_0 \Rightarrow 0 \in G)$, $\mathcal{G}_0 \subset \mathcal{G}$ and the local finiteness of $\mathcal{G}$. Defining

$$\mathcal{F} = \{\, \Lambda_x \mid x \in \mathbb{Z}^\nu \,\}, \qquad \Lambda_x = \bigcup_{x \in G \in \mathcal{G}} G$$

as in (2.4), the assumed translation invariance of $\mathcal{G}$ implies that of $\mathcal{F}$:

$$\Lambda_x + a = \bigcup_{x \in G \in \mathcal{G}} (G + a) = \bigcup_{x \in H - a \in \mathcal{G}} H = \bigcup_{x + a \in H \in \mathcal{G}} H = \Lambda_{x+a}.$$

According to (2.5) we define

$$\overline{\Omega}_{\Lambda_x} = \left\{ \xi \in \Omega_0^{\Lambda_x} \;\middle|\; \begin{array}{l} \text{from } \xi_x = (y, G_0, \omega) \text{ and } x' \in G_0 + x - y \\ \text{it follows } \xi_{x'} = (y + x' - x, G_0, \omega) \end{array} \right\}. \tag{3.2}$$

By means of $\tau^a \overline{\Omega}_{\Lambda_x} = \overline{\Omega}_{\Lambda_x - a} = \overline{\Omega}_{\Lambda_{x-a}}\,(a \in \mathbb{Z}^\nu)$ translation invariance of $(\overline{\Omega}_{\Lambda_x})_{\Lambda_x \in \mathcal{F}}$ is easily checked and $\Omega = (\mathbb{Z}^\nu, \Omega_0, (\overline{\Omega}_\Lambda)_{\Lambda \in \mathcal{F}})$ is shown to be a $\mathbb{Z}^\nu$-PLS. To define F we consider the two mappings $\sigma : \mathbb{Z}^\nu \times \pi(\mathcal{G}) \to \mathcal{G}_0$, $\rho : \mathbb{Z}^\nu \times \pi(\mathcal{G}) \to \mathbb{Z}^\nu$ defined by

$$\left.\begin{array}{l} \sigma(x, \mathcal{P}) = G_0 \\ \rho(x, \mathcal{P}) = l \end{array}\right\} \Leftrightarrow x \in G \in \mathcal{P} \quad \text{and} \quad G = G_0 + l \tag{3.3}$$

where on the right hand side of this equivalence G_0, l are determined uniquely (see 3.2.1). σ and ρ fulfil the relations

$$\begin{aligned} \sigma(x + a, \mathcal{P}) &= \sigma(x, \mathcal{P} - a), \\ \rho(x + a, \mathcal{P}) &= \rho(x, \mathcal{P} - a) + a, \end{aligned} \tag{3.4}$$

which are easily derived from (3.3). Given $\xi^* = (\xi^*_G)_{G\in\mathscr{P}}$ we define the x-th coordinate of $\xi = F(\xi^*)$ by

$$\xi_x = (x - l, G_0, \xi^*_{G_0+1}) \Leftrightarrow x \in G = G_0 + l \in \mathscr{P}, \qquad (3.5)$$

thus l, G_0 are just the values of ρ, σ depending on x and the partition $\mathscr{P}$ given with ξ^*. The connection of (3.5) with the geometric situation may be enlightened from fig. 2: the letter $(x - l, G_0, \xi^*_{G_0+l})$ from the alpha-

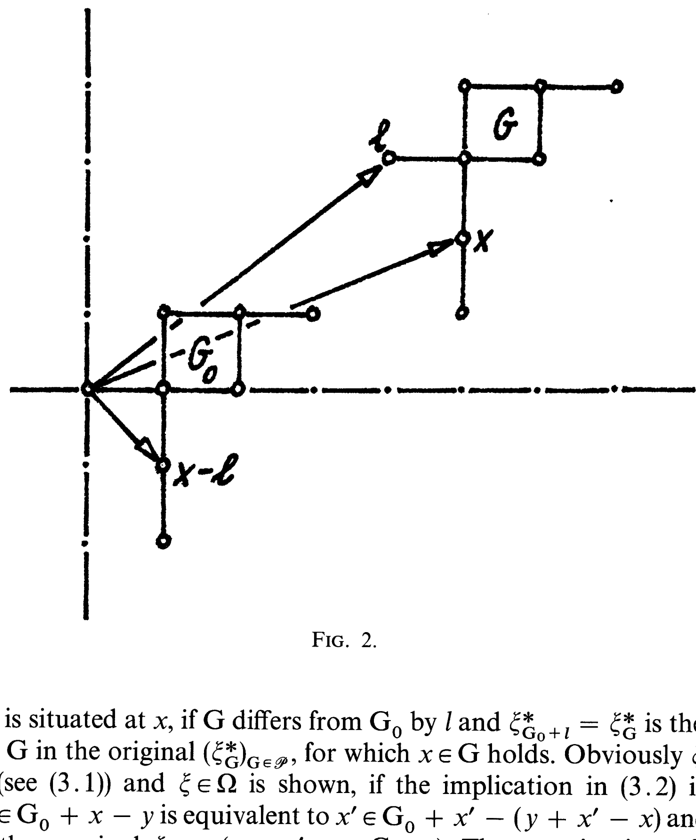

FIG. 2.

bet Ω_0 is situated at x, if G differs from G_0 by l and $\xi^*_{G_0+l} = \xi^*_G$ is the letter of that G in the original $(\xi^*_G)_{G\in\mathscr{P}}$, for which $x \in G$ holds. Obviously $\xi_x \in \Omega_0$ holds (see (3.1)) and $\xi \in \Omega$ is shown, if the implication in (3.2) is true. But $x' \in G_0 + x - y$ is equivalent to $x' \in G_0 + x' - (y + x' - x)$ and (3.5) yields the required $\xi_{x'} = (y + x' - x, G_0, \omega)$. The examination of bijectivity and homeomorphy of F are quite similar to the general case in 2.4. Finally the assumed equivariance of F has to be verified: considering

$$\eta^* = \tau^a\xi^* \quad \Leftrightarrow \quad (\eta^* = (\eta^*_H)_{H\in\mathscr{P}-a} \quad \text{and} \quad \eta^*_H = \xi^*_{H+a}) \qquad (3.6)$$

and translation invariance of $\mathscr{G}$ (thus $\mathscr{P}$-a is a partition of $\pi(\mathscr{G})$ as $\mathscr{P}$) we write using the mappings σ, ρ

$$[F(\tau^a\xi^*)]_x = (x - \rho(x, \mathscr{P}-a), \sigma(x, \mathscr{P}-a), \eta^*_{\sigma(x,\mathscr{P}-a)+\rho(x,\mathscr{P}-a)}),$$
$$[\tau^a F(\xi^*)]_x = [F(\xi^*)]_{x+a} = ((x+a-\rho(x+a, \mathscr{P}), \sigma(x+a, \mathscr{P}), \xi^*_{\sigma(x+a,\mathscr{P})+\rho(x+a,\mathscr{P})}).$$

Equality in the first two coordinates of the right hand side follows by (3.4) and in the third coordinate it follows, if further (3.6) is used. □

3.4. REMARK. — There are $\mathbb{Z}$-PLS's being not homeomorphic and equivariant images of $\mathbb{Z}$-SLS's. To examine this, notice at first from

$$\tau^a\xi^* = \xi^* \Rightarrow \tau^a\xi = F\tau^aF^{-1}\xi = F\tau^a\xi^* = F\xi^* = \xi$$

and the converse implication that periodic configurations of (minimal) period a in Ω and Ω^* are images of each other under the homeomorphic and equivariant maps F resp. F^{-1}. Then forming the special

$$\mathbb{Z}\text{-PLS} \qquad \Omega = (\mathbb{Z}, \Omega_0, (\overline{\Omega}_\Lambda)_{\Lambda\in\mathscr{F}}),$$

where

$$\Omega_0 = \{0, 1\}, \quad \mathscr{F} = \{\{x, x+1\} \mid x \in \mathbb{Z}\}, \quad \overline{\Omega}_\Lambda = \{(0, 0), (0, 1), (1, 1)\} \ (\Lambda \in \mathscr{F}),$$

we get two configurations $\xi = \{\ldots, 0, 0, 0, \ldots\}$, $\eta = \{\ldots, 1, 1, 1, \ldots,\}$ with period $a = 1$. If $\Omega^* = (\mathbb{Z}, (\Omega_G)_{G\in\mathscr{G}_0})$ would be a suitable preimage the implications

$$a = 1 \Rightarrow \{0\} \in \mathscr{G}_0$$

$$\xi \neq \eta, \ \text{homeomorphy} \Rightarrow |\Omega_{\{0\}}| \geqslant 2$$

would yield a configuration ξ^* of minimal period 2, which is no preimage of some $\xi \in \Omega$, since $\overline{\Omega}_\Lambda$ does not allow such ξ's. Another more complicated example of a $\mathbb{Z}$-PLS Ω follows from

$$\Omega_0 = \{0, 1, 2\}, \qquad \mathscr{F} = \{\{x, x+1\} \mid x \in \mathbb{Z}\},$$
$$\overline{\Omega}_\Lambda = \{(0, 1), (0, 2), (1, 0), (1, 2), (2, 0), (2, 1)\} \qquad (\Lambda \in \mathscr{F}),$$

where the different numbers of periodic configurations of minimal period 6 prevent the existence of a corresponding $\mathbb{Z}$-SLS Ω^*.

4. SOME TOPOLOGICAL PROPERTIES IN Z^ν-SET LATTICE SYSTEMS

A $\mathbb{Z}^\nu$-SLS Ω^*, as defined in 3.1, depends highly on the partition $\mathscr{P}$ being possible with elements from $\mathscr{G} = \{G_0 + l \mid G_0 \in \mathscr{G}_0, l \in \mathbb{Z}^\nu\}$ and therefore the representation of topological properties like transitivity and mixing of the group $\{\tau^a \mid a \in \mathbb{Z}^\nu\}$ in terms of $\Omega = (\mathbb{Z}^\nu, (\Omega_G)_{G\in\mathscr{G}_0})$ is expected at least partially. At first we quote the two mentioned notions.

4.1. DEFINITION. — Let $\Omega^* = (\mathbb{Z}^\nu, (\Omega_G)_{G\in\mathscr{G}_0})$ and $T = (\tau^a)_{a\in Z^\nu}$ be the group of translations defined in sect. 3. Then (Ω^*, T) is said to be

i) topologically transitive, if

$$(U, V \text{ open in } \Omega^*) \Rightarrow (\exists a \in \mathbb{Z}^\nu \ \text{ where } \ U \cap \tau^a V \neq \emptyset),$$

ii) topologically mixing, if

$$(\mathrm{U}, \mathrm{V} \text{ open in } \Omega^*) \Rightarrow (\exists \text{ finite } \mathrm{W} \subset \mathbb{Z}^\nu \text{ where } \mathrm{U} \cap \tau^a \mathrm{V} \neq \emptyset \text{ for all } a \in \mathbb{Z}^\nu \backslash \mathrm{W}).$$

In analogy with prop. 7.10 in [*1*] and using the system $\mathscr{B}^*$ of basic sets of the topology in Ω^* as introduced after definition 2.3 we get the easily proved

4.2. PROPOSITION. — (Ω^*, T) *is topologically transitive (resp. topologically mixing) iff for any two basic sets* $\mathrm{B}_1, \mathrm{B}_2 \subset \mathscr{B}^*$

$$\mathrm{B}_1 \cap (\mathrm{B}_2 + a) \neq \emptyset$$

is valid for some $a \in \mathbb{Z}^\nu$ *(resp. for all* $a \in \mathbb{Z}^\nu$ *large enough (e. g. in the euclidean norm)).*

4.3. COROLLARY. — $\{0\} \in \mathscr{G}$ implies (Ω^*, T) to be topologically mixing.

Proof. — Because of the translation invariance $\{t\} \in \mathscr{G}$ $(t \in \mathbb{Z}^\nu)$ holds, i. e. all one-point sets of $\mathbb{Z}^\nu$ belong to $\mathscr{G}$. To check mixing let $\mathrm{B}_1, \mathrm{B}_2 \in \mathscr{B}^*$ have the representations

$$\mathrm{B}_i = \bigcap_{j=1}^{n_i} {}_{\mathrm{G}_{ij}}[\omega_{ij}] \qquad (i = 1, 2). \tag{4.1}$$

Since all G_{ij} are finite sets, we are given two $m_1, m_2 \in \mathrm{N}$ such that

$$m_i \geqslant \sup \left\{ \| x \| \mid x \in \bigcup_{j=1}^{n_i} \mathrm{G}_{ij} \right\}$$

where $\| \cdot \|$ denotes the euclidean norm. Then

$$\left(\bigcup_{j=1}^{n_1} \mathrm{G}_{1j} \right) \cap \left(\bigcup_{k=1}^{n_2} \mathrm{G}_{2k} + a \right) = \emptyset$$

obviously holds for all $a \in \mathbb{Z}^\nu$ fulfilling $\| a \| > m_1 + m_2$. Now we fix such an a. Then the sets

$$\mathrm{G}_{1j} (j = 1, \ldots, n_1); \qquad \mathrm{G}_{2k} + a (k = 1, \ldots, n_2); \qquad \{x\} (x \notin \mathrm{H})$$

where

$$\mathrm{H} = \left(\bigcup_{j=1}^{n_1} \mathrm{G}_{1j} \right) \cup \left(\bigcup_{k=1}^{n_2} \mathrm{G}_{2k} + a \right)$$

form a partition of $\mathbb{Z}^\nu$ and the element $\xi^* \in \Omega^*$ defined by

$$\begin{aligned} \xi^*_{\mathrm{G}} &= \omega_{1j} \qquad (j = 1, \ldots, n_1), \\ \xi^*_{\mathrm{G}_{2k}+a} &= \omega_{2k} \qquad (k = 1, \ldots, n_2), \\ \xi^*_{\{x\}} &= \omega \qquad (x \notin \mathrm{H}, \omega \in \Omega_{\{0\}}) \end{aligned}$$

belongs to $\mathrm{B}_1 \cap (\mathrm{B}_2 + a)$ and therefore 4.2 shows topological mixing. □

4.4. REMARK. — The essential point in the preceding corollary is the possible completion of the finite subpartition

$$\{ G_{11}, \ldots, G_{1n_1}, G_{21} + a, \ldots, G_{2n_2} + a \}$$

by one-points sets, since they all belong to $\mathcal{G}$. Of course, this gives us an idea to generalize this corollary. The following condition on $\mathcal{G}$ is sufficient: there exists an $m \in \mathrm{N}$ such that for each two collections

$$G_{1j}(j = 1, \ldots, n_1), \qquad G_{2k}(k = 1, \ldots, n_2) \tag{4.2}$$

of sets from $\mathcal{G}$, which are subcollections of partitions, and for each $a \in \mathbb{Z}^\nu$ with $\| a \| > m$ there exists a sequence $(H_l)_{l \in \mathrm{N}}$ of sets $H_l \in \mathcal{G}$ forming together with the sets $G_{1j}(j = 1, \ldots, n_1)$, $G_{2k} + a(k = 1, \ldots, n_2)$ a partition of $\mathbb{Z}^\nu$.

But in general it is not easy to decide whether $\mathcal{G}$ has this property or not. A case of intransitivity is given by the following

4.5. COROLLARY. — If $\mathcal{G}$ is a disjoint union of translation invariant nonempty sets $\mathcal{G}_1$ and $\mathcal{G}_2$ and if $\pi(\mathcal{G})$ is a disjoint union of nonempty sets $\pi(\mathcal{G}_1)$ and $\pi(\mathcal{G}_2)$, then (Ω^*, T) is topologically intransitive.

Proof. — Assume topological transitivity of (Ω^*, T) and $G_{i1}, \ldots, G_{in_i} \in \mathcal{G}_i$ chosen in such a way that it is a subpartition of a certain partition from $\pi(\mathcal{G}_i)$ $(i = 1, 2)$. Defining B_i as in (4.1) topological transitivity would imply the existence of $a \in \mathbb{Z}^\nu$ and $\xi^* \in \Omega^*$ with

$$\xi^*_G = \omega_{1j} \ (j = 1, \ldots, n_1), \qquad \xi^*_{G_{2k}+a} = \omega_{2j} \ (k = 1, \ldots, n_2)$$

and further this would imply the existence of a partition $\mathcal{P}$ containing the elements $G_{1j}(j=1, \ldots, n_1)$ from $\mathcal{G}_1$ and the elements $G_{2k}+a(k=1, \ldots, n_2)$ from $\mathcal{G}_2$. This contradicts $\pi(\mathcal{G}_1) \cap \pi(\mathcal{G}_2) = \emptyset$. $\square$

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