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On the completeness in three body scattering

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ABSTRACT. — A time dependent proof of asymptotic completeness for the three body scattering in Quantum Mechanics is given when the pair potentials behave like $(1+|x|)^{-2-\varepsilon}$ at ∞. Also some local singularity is allowed for the potentials.

RÉSUMÉ. — On donne une démonstration « dépendant du temps » de la complétude asymptotique pour la diffusion à trois corps en Mécanique Quantique lorsque les potentiels à 2 corps se comportent en $(1+|x|)^{-2-\varepsilon}$ à l'infini. On permet des singularités locales de ces potentiels.

§ 1. INTRODUCTION

The completeness of the three particle scattering has been studied by Fadeev [*1*], Ginibre and Moulin [*2*], Thomas [*3*], Mourre [*4*] and more recently by Hagedorn and Perry [*5*] using time independent methods. A simplified account of some of these methods can be found in [*6*]. On the other hand Enss [*7*] has given some ideas about the time dependent method of obtaining completeness in three body scattering. Here we use some of Enss's ideas and give complete proof for the case when the pair potentials behave like $(1+|x|)^{-2-\varepsilon}$, $\varepsilon > 0$ as $|x| \to \infty$ and admit some local singularities.

In Section 2 we give some two body results which are needed in later sections. Theorem 2.6 is essentially new. Sections 3 and 5 contain two preliminary results—the local decay and low energy results respectively.

Asymptotic evolution of observables for the three body total group is developed in Section 4 using the results of Section 3. Finally in Section 6 all these together are used to prove the main theorem on completeness viz. Theorem 6.1. The appendix contains a generalisation of Theorems 2.5 and 2.6 when the potential behaves like $(1 + |x|)^{-1-\varepsilon}$ at ∞. However, our proof of Lemma 3.6 restricts the final result (*Theorem* 6.1) to the $2 + \varepsilon$-case only.

We use the notations of Chapter 16 of [*6*] and work in the relative Hilbert space $\mathscr{H}_{\text{rel}} \approx L^2(R^{2\nu})$, $\nu \geq 3$, after removal of the centre of mass.

If x_d and y_d denote respectively the pair distance and the distance of the centre of mass of the pair d from the third particle then we set p_d and k_d to be the momenta canonically conjugate to x_d and y_d respectively. Thus we have in $L^2(R^{2\nu}) \approx L^2(R^\nu_{x_d}) \otimes L^2(R^\nu_{y_d})$, $(d = 1, 2, 3)$ the free Hamiltonian $H_0 = \left(\frac{1}{2}\gamma_d p_d^2 + \frac{1}{2}\eta_d k_d^2\right)$ for $\gamma_d, \eta_d > 0$, the cluster Hamiltonian $H^d = H_0 + V_d$ and the total Hamiltonian $H = H_0 + \sum_{d=1}^{3} V_d$ where V_d is the pair potential for the pair d. We also write $h_0^d = \frac{1}{2}\gamma_d p_d^2$, $h^d = h_0^d + V_d$ and E^d = Orthogonal projection onto the point spectral subspace of h^d. Also $h^d \otimes 1$, $h_0^d \otimes 1$, $E^d \otimes 1$ on $L^2(R^{2\nu})$ will be denoted by h^d, h_0^d, E^d respectively. Now we define the unitary groups $V_t = \exp[-itH]$, $U_t = \exp[-itH_0]$, and $U_t^d = \exp[-itH^d]$. Then the wave operators are defined as

$$\Omega_d^\pm = \underset{t\to\pm\infty}{s\text{-lim}}\, V_t^* U_t^d E^d, \qquad \Omega_0^\pm = \underset{t\to\pm\infty}{s\text{-lim}}\, V_t^* U_t \qquad \text{and} \qquad \tilde{\Omega}_d^\pm = \underset{t\to\pm\infty}{s\text{-lim}}\, V_t^* U_t^d.$$

It has been shown in Chapter 15 of [*6*] that these strong limits exist under a very general class of assumptions on V_d. It is well known (see [*6*]) that $R(\Omega_d^\pm)$ ($\equiv$ Range $\Omega_d^\pm$) and $R(\Omega_0^\pm)$ are pairwise orthogonal and that

$$\sum_{d=1}^{3} \oplus R(\Omega_d^\pm) \oplus R(\Omega_0^\pm) \subseteq \mathscr{H}_{ac}(H), \tag{1.1}$$

the absolutely continuous spectral subspace of the total Hamiltonian H. The problem of completeness is that of proving the equality in (1.1). This is achieved in Theorem 6.1.

We also denote the couple (x_d, y_d) by Q and write $Q^2 = \gamma_d^{-1} x_d^2 + \eta_d^{-1} y_d^2$. Note that $\gamma_d^{-1} x_d^2 + \eta_d^{-1} y_d^2$ is actually independent of d. Then a simple calculation shows that for the dilation operator $A = \frac{1}{2} i [H_0, Q^2]$ on $L^2(R^{2\nu})$ we get $A = A_d + A_d^\perp$ where $A_d = \frac{1}{4}(x_d \cdot p_d + p_d \cdot x_d)$ and

$A_d^{\perp} = \frac{1}{4}(y_d \cdot k_d + k_d \cdot y_d)$. Also $[A, H_0] = iH_0$. By $F(M)$ we mean the characteristic function of the set M as well as the associated multiplication operator in the appropriate Hilbert space.

In order to facilitate the application of the results of Section 2 to Sections 3-6, we make the following canonical transformation, without explicitly mentioning it, viz. $x'_d = \gamma_d^{-\frac{1}{2}} x_d$, $y'_d = \eta_d^{-\frac{1}{2}} y_d$, $p'_d = \gamma_d^{\frac{1}{2}} p_d$, $k'_d = \eta_d^{\frac{1}{2}} k_d$. In such a case we get $H_0 = \frac{1}{2}(p_d'^2 + k_d'^2)$, $H^d = H_0 + V_d(\gamma_d^{\frac{1}{2}} x'_d)$, $Q^2 = x_d'^2 + y_d'^2$, etc. Note that under such transformation the linear relation between x_c, y_c, x_d, y_d $(c \neq d)$ remain invariant.

Throughout this article K will denote a generic constant and $\bigcirc$ for the « big $\bigcirc$ » notation.

§ 2. SOME TWO BODY RESULTS

In this section we shall collect some known results and prove some new ones for two body systems as a preliminary to three body calculations later. Throughout this section we shall denote by $h(h_0)$ a typical two body total (free) Hamiltonian respectively. The first theorem is about the completeness and some spectral properties in a class of two body problems.

THEOREM 2.1. — Let $h_0 = \frac{1}{2}p^2$, $h = h_0 + V$ act on $L^2(R^\nu)$ $(\nu \geq 3)$ and suppose that

i) $(1 + x^2)^\delta V$ is h_0 compact for some δ in $\left(\frac{1}{2}, 2\right]$ and

ii) $(h_0 + 1)^{-1}[A, V](h_0 + 1)^{-1}$ is compact where $A = \frac{1}{4}(p \cdot x + x \cdot p)$.

Then

a) the continuous spectrum $\sigma_c(h) = [0, \infty)$. The wave operators $\omega_\pm = \underset{t \to \pm\infty}{s\text{-lim}} \exp[ith] \exp[-ith_0]$ exist and are complete i. e.

$$\mathcal{H}_{ac}(h) = R(\omega_+) = R(\omega_-).$$

b) the singular continuous subspace $\mathcal{H}_{sc}(h) = \{0\}$,

c) the eigenvalues of h can accumulate only at 0 and any nonzero eigenvalue is of finite multiplicity,

d) h does not have any eigenvalue in $(0, \infty)$,

e) if $\lambda < 0$ is an eigenvalue for h with eigenvector e then

$$\exp(\sqrt{\lambda_1}\,|x|)e(x) \in L^2(R^\nu) \qquad \text{for each } \lambda_1 \text{ in } [0, -\lambda).$$

Proof. — See either of [8] [9] [10], for (*a*) and (*b*); [11] or [12] for (*a*), (*b*) and (*c*); [13] for (*d*) and (*e*). Q. E. D.

REMARK 2.2. — By commutation rules between p and x one can easily verify that $V(h_0 + z)^{-1}(1 + x^2)^\delta$ is compact (respectively bounded) whenever $V(1 + x^2)^\delta(h_0 + z)^{-1}$ is compact (respectively bounded) for any $\delta > 0$.

LEMMA 2.3. — Let h_0, h, V be as in Theorem 2.1 and $\psi \in C_0^\infty(R)$. Then

i) $$\| (1+x^2)^\delta [\psi(h_0) - \psi(h)] \| < \infty ,$$

ii) $\| F(|x| > \alpha r)[\psi(h_0) - \psi(h)] \| \leq K(1+\alpha r)^{-2\delta}$ for any $\alpha > 0$.

Proof. i). — Let $\gamma < \inf \sigma(h)$. Without loss of generality we can assume that $\psi \in C_0^\infty(\gamma, \infty)$. Let $\chi(\lambda) = (\lambda - c)^{-1}$ for $\lambda > \gamma$ with $c < \gamma$. Then $\psi(h) = (\varphi \circ \chi)(h)$ for $\psi \circ \chi^{-1} = \varphi$ in $C_0^\infty(0, (\gamma - c)^{-1})$ where $\circ$ denotes the composition of functions. Clearly $(h-c)^{-1}$ and $(h_0 - c)^{-1}$ are bounded self adjoint operators with their spectra in $[0, (\gamma - c)^{-1}]$. Note that

$$\psi(h) - \psi(h_0) = \varphi((h - c)^{-1}) - \varphi((h_0 - c)^{-1}).$$

Let $\tilde{\varphi}$ be the Fourier transform of φ. Then by the functional calculus and fundamental theorem of calculus we have

$$\begin{aligned} b &= \| (1+x^2)^\delta [\varphi((h_0 - c)^{-1}) - \varphi((h-c)^{-1})] \| \\ &= \left\| \int d\lambda \tilde{\varphi}(\lambda) \int_0^\lambda d\mu (1+x^2)^\delta \exp[-i(\lambda-\mu)(h_0 - i)^{-1}](1+x^2)^{-\delta} \right. \\ &\qquad \left. \cdot (1+x^2)^\delta (h_0 - c)^{-1} V (h-c)^{-1} \exp[-i\mu(h-c)^{-1}] \right\| \\ &\leq K \int d\lambda \, |\tilde{\varphi}(\lambda)| \int_0^\lambda d\mu \, \| (1+x^2)^\delta \exp[-i(\lambda-\mu)(h_0 - c)^{-1}](1+x^2)^{-\delta} \| . \end{aligned}$$

In the last step we have used Remark 2.2. By commutation rules and interpolation theorem we see that

$$\| (1+|x|)^\theta \exp[-ir(h_0 - c)^{-1}](1+|x|)^{-\theta} \| = O((1+|r|)^\theta) \text{ for } \theta \text{ in } [0,2]$$

and so $b < \infty$ proving (*i*),

ii) is an easy consequence of (*i*). Q. E. D.

LEMMA 2.4. — Let $\varphi \in C_0^\infty(R)$

i) If $\frac{1}{2} b^2 = \sup \operatorname{supp} \varphi$ with $b > 0$ and c, a are positive constants with $b + c < a$ then

$$\| F(|x| \geq at) \exp(-ish_0) \varphi(h_0) F(|x| \leq ct) \| \leq K_N (1+t)^{-N}$$

for $0 \leq s \leq t$ and $N = 1, 2, \ldots$.

ii) If $\frac{1}{2}b^2 = \inf \operatorname{supp} \varphi$ with $b > 0$ and c, a are positive constants with $c + a < b$ then

$$\| F(|x| \leq at) \exp(-ith_0)\varphi(h_0)F(|x| \leq ct) \| \leq K_N(1 + t)^{-N}$$

for $0 \leq t$ and $N = 1, 2, \ldots$.

Proof. i). — Heuristically

$$\exp(-ish_0)F(|x| \leq ct) \exp(ish_0) = F(|x-ps| < ct).$$

We compute the lower bound to the gap between the regions $|x'| \geq$ at and $|x| \leq ct$ after evolution as follows:

$$|x'-(x-ps)| \geq |x'|-(|x|+s|p|) \geq [a-(b+c)]t \quad \text{since} \quad s \leq t.$$

This only motivates the result. A rigorous proof of this uses the method of stationary phase and the positive operator valued map developed in [*10*] or [*12*].

ii) The proof is similar to that of (*i*). Q. E. D.

THEOREM 2.5. — Let h_0, h, V be as in Theorem 2.1 with δ in $(1, 2]$. Then for a, b, c as in Lemma 2.4 (*i*)

i) $$\|F(|x| > a(t+s)) \exp(-ish)\varphi(h) \exp(-ith_0)F(|x| < c(t+s)) \| = O\,[(1+t+s)^{-2\delta+1}] \quad \text{for} \quad s, t \geq 0,$$

ii) $$\lim_{t\to\infty} \int_0^\infty ds \| F(|x| > a(t+s)) \exp(-ish)\varphi(h)\exp(-ith_0)F(|x| < c(t+s)) \| = 0.$$

Remark. — A proof of related result even when $V(x) \sim (1+|x|)^{-\delta}, \delta > 1$ has been given by Enss in [*14*]. See the appendix for an alternative proof.

Proof. i). — By triangle inequality we have

$$\begin{aligned} M(s, t) \equiv \| F(|x| > a(t+s)) \exp(-ish)\varphi(h) \exp(-ith_0)F(|x| < c(t+s)) \| \\ \leq \| F(|x| > a(t+s)) [\varphi(h) - \varphi(h_0)] \| \\ + \| F(|x| > a(t+s))\varphi(h_0) \exp(-ish) \exp(-ith_0)F(|x| < c(t+s)) \|. \end{aligned}$$

Using Lemma 2.3 (*ii*) and the formula

$$\exp(-ish) = e^{-ish_0} - i \int_0^s d\tau e^{-i\tau h_0} V e^{-i(s-\tau)h},$$

we have that

$$\begin{aligned}
M(s,t) &\leq O[(1+t+s)^{-2\delta}] \\
&+ \| F(|x| > a(t+s))\varphi(h_0)\exp[-i(t+s)h_0]F(|x| < c(t+s)) \| \\
&+ \int_0^s d\tau \| F(|x| > a(t+s)) \\
&\qquad \varphi(h_0)\exp(-i\tau h_0)V\exp[-i(s-\tau)h]\exp(-ith_0)F(|x|<c(t+s)) \| \\
&\leq O[(1+t+s)^{-2\delta}] + O[(1+t+s)^{-N}] \\
&+ \int_0^s d\tau \| F(|x| > a(t+s))\varphi_1(h_0)\exp(-i\tau h_0)F(|x|<c(t+s)) \|. \\
&\qquad \| (h_0+1)^{-1}V \| \; \| \exp[-i(s-\tau)h]\exp(-ith_0)F(|x|<c(t+s)) \| \\
&+ \int_0^s d\tau \| F(|x|>a(t+s))\varphi_1(h_0)\exp(-i\tau h_0) \| \; \| F(|x|>c(t+s))(1+x^2)^{-\delta} \|. \\
&\| (1+x^2)^{\delta}(h_0+1)^{-1}V \| \; \| \exp[-i(s-\tau)h]\exp(-ith_0)F(|x|<c(t+s)) \|
\end{aligned}$$

where we have used Lemma 2.4 (*i*) and taken $\varphi_1(\lambda)=(1+\lambda)\varphi(\lambda)$. Observe that $\varphi_1 \in C_0^\infty(R)$ with supp $\varphi_1 =$ supp φ, use Lemma 2.4 (*i*) and the conditions on the potential to get the result

$$M(s,t) \leq O[(1+t+s)^{-2\delta}] + K\int_0^s d\tau[(1+t+s)^{-2\delta}] \leq O[(1+t+s)^{-2\delta+1}].$$

ii) Follows from (*i*) since $\delta > 1$. Q. E. D.

The next theorem obtains an estimate similar to that in Lemma 13.3 of [*6*] with total evolution group $\exp(-ith)$ replacing the free group. For this we extend the ideas of [*15*]. The reader is also referred to [*16*] and [*5*] where similar estimates are obtained. In [*5*] time-independent analytic methods are used for a similar class of short range potentials. On the other hand in [*16*] only smooth long range potentials were considered for which the present method is inadequate.

THEOREM 2.6. — Let h be as in Theorem 2.5 with the potential satisfying further more:

$$(1+x^2)^{\frac{1}{2}\delta}(h_0+1)^{-1}(V+i[A,V])(h_0+1)^{-1}(1+x^2)^{\frac{1}{2}\delta} \text{ is bounded.}$$

Then for $\varphi \in C_0^\infty(0,\infty)$ and $t \in R$,

i) $\| (A+i)^{-1}\exp(-ith)\varphi(h)(A+i)^{-1} \| = O[(1+|t|)^{-1}]$,

ii) $\| (A^2+1)^{-1}\exp(-ith)\varphi(h)(A^2+1)^{-1} \| \leq K(1+|t|)^{-2+\varepsilon}$ for each $\varepsilon > 0$,

iii) $\| (x^2+1)^{-\mu}\exp(-ith)\varphi(h)(x^2+1)^{-\mu} \| \leq K_{\mu'}(1+|t|)^{-2\mu'}$ for each μ' with $0 < \mu' < \mu \leq 1$

REMARK 2.7. *i*). — The condition on V in Theorem 2.5 implies as in

Remark 2.2 that $(1+x^2)^{\frac{1}{2}\delta}(h_0+1)^{-1}V(h_0+1)^{-1}(1+x^2)^{\frac{1}{2}\delta}$ is bounded. Thus the above condition is really one for $[A, V]$ and roughly speaking for $\nu=3$ it contains potentials V such that $V \in L^{3/2}_{loc}(R^3)$ and $V(x) \sim (1+x^2)^{-\delta}$ as $|x| \to \infty$.

ii) A simple calculation as in Remark 2.2 shows that in the above condition on V, the operator h_0 can be replaced by h. Thus as in [*15*], Lavine's smoothness result [*17*], [*18*] implies that

$$\int_{-\infty}^{\infty} dt \,|\langle \exp(-ith)\varphi_1(h)f, (V+i[A, V])\exp(-ith)\varphi_2(h)g \rangle| \leq K \|f\| \;\|g\| \quad (2.1)$$

where $\varphi_1, \varphi_2 \in C_0^\infty(0, \infty)$. In all this we also have to take note of Theorem 2.1 (*d*).

iii) By the conditions on the potential it is easily seen that

$$(1+x^2)^{-1}(h_0+i)^{-1}(1+A^2) \quad \text{and} \quad (1+x^2)^{-1}[(h+i)^{-1}-(h_0+i)^{-1}](1+A^2)$$

are both bounded. By interpolation we get that $(1+x^2)^{-\mu}(h+i)^{-1}(1+A^2)^{\mu}$ is bounded for μ in $[0, 1]$.

iv) The results of Theorem 2.6 remain true if $V=V_1+V_2$ with V_2 satisfying the hypothesis of the theorem and V_1 such that

$$|V_1(x)| + |x\cdot\nabla V_1(x)| + |x\cdot\nabla(x\cdot\nabla V_1)(x)| = O[(1+x^2)^{-\alpha}] \quad \text{for} \quad \alpha > \frac{1}{2}.$$

Proof of this is given in the Appendix.

Proof of Theorem 2.6. *i*). — As in [*15*] we have as a quadratic form on Dom A $\cap$ Dom h

$$[A, \exp(-ith)] = th\exp(-ith) - \int_0^t ds \exp[-i(t-s)h](V+i[A, V])\exp(-ish) \quad (2.2)$$

An application of (2.1) to the second term on the R. H. S. of (2.2) yields the required result.

ii) Clearly we can find ψ, χ in $C_0^\infty(0, \infty)$ such that $\varphi(h) \equiv h\psi(h)\chi(h)$ and then we have from (2.2) that

$$\begin{aligned} &t(A^2+1)^{-1}\exp(-ith)\varphi(h)(A^2+1)^{-1} \\ &= (A^2+1)^{-1}\psi(h)\{A\exp(-ith)-\exp(-ith)A\}\chi(h)(A^2+1)^{-1} \\ &+ \int_0^t ds(A^2+1)^{-1}\psi(h)\exp[-i(t-s)h](V+i[A, V])\exp(-ish)\chi(h)(A^2+1)^{-1} \\ &\equiv I_1 + I_2 . \quad (2.3) \end{aligned}$$

Since $(A^2+1)^{-1}\psi(h)A(A+i)$ and $(A+i)A\chi(h)(A^2+1)^{-1}$ are both bounded, an application of result (*i*) shows that $\| I_1 \| = O[(1+|t|)^{-1}]$. On the other hand,

$$\| I_2 \| \leq \int_0^t ds \{ \| (A^2+1)^{-1}\psi(h)(h+i) \exp[-i(t-s)h](1+x^2)^{-\frac{1}{2}\delta} \|$$

$$\cdot \| (1+x^2)^{\frac{1}{2}\delta}(h+i)^{-1}(V+i[A,V])(h+i)^{-1}(1+x^2)^{\frac{1}{2}\delta} \|$$

$$\cdot \| (1+x^2)^{-\frac{1}{2}\delta} \exp(-ish)(h+i)\chi(h)(A^2+1)^{-1} \| \}$$

$$\leq K \int_0^t ds(1+|t-s|)^{-1}(1+|s|)^{-1} \leq K_1(1+|t|)^{-1}\log(1+|t|).$$

In the above result we have replaced $(1+x^2)^{-\frac{1}{2}\delta}$ by $(1+A^2)^{-\frac{1}{2}\delta}$ [see Remark 2.7 (*iii*)] and applied the result (*i*). This together with (2.3) proves (*ii*).

iii) This follows from (*ii*), by interpolation and an application of Remark 2.7 (*iii*). Q. E. D.

§ 3. LOCAL DECAY

The existence and inter-twining properties of $\Omega_d^{\pm}$ and $\Omega_0^{\pm}$ imply that $R(\Omega_d^{\pm})$ and $R(\Omega_0^{\pm})$ are subspaces of $\mathscr{H}_c = \mathscr{H}_c(H)$ [the continuous spectral subspace of H] and are pairwise orthogonal. We set

$$\mathscr{H}_c^{\pm} = \mathscr{H}_c \ominus \left\{ \sum_d \oplus R(\Omega_d^{\pm}) \oplus R(\Omega_0^{\pm}) \right\}.$$

By Hunziker's Theorem [6, 18] we get $\sigma_c(H) \subseteq \sigma_{ess}(H) = [\Sigma, \infty)$ where $\Sigma = \min\{\inf \sigma(H^d) : d\}$. We next define the set of thresholds of H and two dense linear manifolds $\mathscr{D}^{\pm}$ in $\mathscr{H}_c^{\pm}$.

DÉFINITION 3.1. — The set of thresholds of H is

$$T = \{ \lambda : \lambda \text{ is an eigenvalue of } h^d \text{ for some } d \} \cup \{ 0 \}$$

and

$$\mathscr{D}^{\pm} = \{ f \in \mathscr{H}_c^{\pm} : \varphi(H)f = f \text{ for some } \varphi \text{ in } C_0^{\infty}((\Sigma,\infty)-T) \text{ with } 0 \leq \varphi \leq 1 \}$$

By Theorem 2.1 (*c*) the set T is a closed countable set. Also since all the four range subspaces as well as $\mathscr{H}_c$ reduce H we get that $\mathscr{H}_c^{\pm}$ reduce H. Hence $\mathscr{D}^{\pm}$ are dense subspaces of $\mathscr{H}_c^{\pm}$ respectively.

For any bounded continuous complex valued function φ on R we shall write $\mathscr{E}_{\pm}(\varphi) = \lim_{S \to \pm\infty} S^{-1} \int_0^S dt\varphi(t)$, whenever the limits exist.

The next theorem states that any state in $\mathscr{H}_c^{\pm}$ asymptotically moves out of a region localised in any of the pair distances x_d.

THEOREM 3.2 (Local decay). — Let V_d be as in Theorem 2.5 for each d. Then

$$\mathscr{E}_\pm(\| F(|x_d| \leq r)V_t f \|) = 0 \quad \text{for } f \text{ in } \mathscr{H}_c^\pm \text{ and } r \text{ in } (0, \infty).$$

We need a few lemmas first.

LEMMA 3.3. — Let V_d be as in Theorem 2.1 for each d and $\varphi \in C_0^\infty(R)$. Then $(1+x_d^2)^{-\mu}[\varphi(H) - \varphi(H^d)]$ is compact for each $\mu > 0$.

Proof. — We can assume that $\mu \leq 1$. As in Theorem 1.1 of [19] it suffices to show the compactness of $(1+x_d^2)^{-\mu}[(H-z)^{-N}-(H^d-z)^{-N}]$ for some $z<\Sigma$ and $N = 1, 2, 3, \ldots$ By induction this is if

$$(i) \quad (1+x_d^2)^{-\mu}(H-z)^{-N}(1+x_d^2)^{\mu}=[(1+x_d^2)^{-\mu}(H-z)^{-1}(1+x_d^2)^{\mu}]^N$$

and

$$(1+x_d^2)^{-\mu}(H^d - z)^{-N}(1 + x_d^2)^{\mu}$$

are bounded for $N = 1, 2, \ldots$ and (ii) $(1+x_d^2)^{-\mu}[(H-z)^{-1}-(H^d-z)^{-1}]$ is compact. It is easy to verify (i). For (ii) we have

$$(1+x_d^2)^{-\mu}[(H^d-z)^{-1}-(H-z)^{-1}]$$
$$= \sum_{a\neq d}(1+x_d^2)^{-\mu}(H^d-z)^{-1}(1+x_a^2)^{-\delta}(1+x_a^2)^{\delta}V_a(H-z)^{-1}$$

The result follows from the compactness of $(1+x_d^2)^{-\mu}(H^d-z)^{-1}(1+x_a^2)^{-\delta}$ for $a \neq d$ and the boundedness of $(1+x_a^2)^{\delta}V_a(H-z)^{-1}$. Q. E. D.

REMARK 3.4. — In the following lemmas we shall use the fact that $V_c(H^d+i)^{-1}(1+x_c^2)^{\delta}$ is bounded whenever $V_c(1+x_c^2)^{\delta}(H^d+i)^{-1}$ is bounded. This can be derived exactly as in Remark 2.2 by using the commutation rules and the hypothesis on V.

LEMMA 3.5. — Let V_d satisfy the hypothesis of Theorem 2.1 for each d. Assume that $\varphi \in C_0^\infty((\Sigma, \infty) - T)$ and $\psi \in C_0^\infty(-\infty, 0)$. Then,

i) $(\Omega_d^\pm - 1)\psi(h^d)\varphi(H^d)(1 + x_d^2)^{-2}F(A_d^\perp \gtrless 0)$ is compact,

ii) $\mathscr{E}_\pm \langle V_t f, \psi(h^d)\varphi(H^d)(1+x_d^2)^{-2}V_t f\rangle = 0$ for f in $\mathscr{H}_c^\pm$.

Proof. — We prove (i) and (ii) for positive sign only.

i) By Theorem 2.1 (a) and (c) there exist eigenvalues $\lambda_1, \ldots, \lambda_n < 0$ of h^d of finite multiplicity such that $\psi(h^d) = \sum_j \psi(\lambda_j)E_{\lambda_j}^d$ where $E_{\lambda_j}^d$ is the projection onto the eigensubspace of eigen-value λ_j. Thus the result trivially follows if $(\Omega_d^+ - 1)E_\lambda^d\varphi(H^d)(1+x_d^2)^{-2}F(A_d^\perp \geq 0)$ is compact where $\lambda < 0$ is an eigenvalue of h^d and E_λ^d is one dimensional projection $\langle e, \rangle e$.

We have,

$$(\Omega_d^+ - 1)E_\lambda^d\varphi(H^d)F(A_d^\perp \geq 0)(1 + x_d^2)^{-2}$$
$$= \sum_{c \neq d} i \int_0^\infty ds V_s^* V_c E_\lambda^d (H^d + i)^{-1} U_s^d E_\lambda^d \varphi_1(H^d) F(A_d^\perp \geq 0)(1 + x_d^2)^{-2}$$
$$= \sum_{c \neq d} \int_0^\infty ds I_{cd}(s) \tag{3.1}$$

where $\varphi_1(H) = (H^d + i)\varphi(H^d)$. It is easily seen that $V_c E_\lambda^d (H^d + i)^{-1}$ is compact so that $I_{cd}(s)$ is compact for each s. It is clear that I_{cd} is norm continuous in s. The result will follow if $\int_0^\infty ds \, \| I_{cd}(s) \| < \infty$.

By Remark 3.4 we get

$$\| I_{cd}(s) \| \leq \| V_c(H^d + i)^{-1}(1 + x_c^2)^\delta \| \; \| (1 + x_c^2)^{-\delta} E_\lambda^d U_s^d \varphi_1(H^d) F(A_d^\perp \geq 0) \|$$
$$\leq K \| (1 + y_d^2)^\delta (1 + x_c^2)^{-\delta} E_\lambda^d \| \; \| (1 + y_d^2)^{-\delta} \exp\left(-\frac{1}{2} isk_d^2 \right) \varphi_1\left(\lambda + \frac{1}{2} k_d^2 \right) F\left(A_d^\perp \geq 0 \right) \| \tag{3.2}$$

Since $(1 + y_d^2)^\delta \leq K_1 (1 + x_c^2)^\delta (1 + x_d^2)^\delta$ and since

$$\| (1 + x_d^2) E_\lambda^d \| \leq \| (1 + x_d^2) e \|_2 \| e \|_2 < \infty$$

by Theorem 2.1 (e) we get that the first factor on the R. H. S. of (3.2) is finite. The second factor is of $O[(1+s)^{-2\delta}]$ by using the results of [9]. Since $\delta > \frac{1}{2}$ we conclude $\int_0^\infty ds \, \| I_{cd}(s) \| < \infty$.

ii) Writing $B = \psi(h^d)\varphi(H^d)(1 + x_d^2)^{-2}$ and using intertwining relations we get that

$$\mathcal{E}_+ \langle V_t f, BV_t f \rangle$$
$$= \mathcal{E}_+ \langle V_t f, (1 - \Omega_d^+) BF(A_d^\perp \geq 0) V_t f \rangle + \mathcal{E}_+ \langle U_t^d (\Omega_d^+)^* f, BF(A_d^\perp \geq 0) V_t f \rangle$$
$$+ \mathcal{E}_+ \langle V_t f, (1 - \Omega_d^-) BF(A_d^\perp \leq 0) V_t f \rangle$$
$$+ \mathcal{E}_+ \langle F(A_d^\perp \leq 0) U_t^d \overline{\psi}(h^d) \overline{\varphi}(H^d) (\Omega_d^-)^* f, (1 + x_d^2)^{-2} V_t f \rangle \tag{3.3}$$

By (i) and an application of Wiener's Theorem [20] the first and third term of R. H. S. of (3.3) are 0. Since $f \in \mathcal{H}_c^+$ we get that $(\Omega_d^+)^* f = 0$. For the fourth term note that $\text{s-}\lim_{t \to \infty} F(A_d^\perp \leq 0) \exp\left(-\frac{1}{2} itk_d^2 \right) = 0$ by [11]. Thus (ii) is proved.

Q. E. D.

Remark. — In the lemma above, the factor $(1 + x_d^2)^{-2}$ plays no role.

However if we do not invoke the boundedness of $(1 + x_d^2)E_\lambda^d$ by Theorem 2.1 (*e*), then this factor is useful. In fact, it is enough to note that

$L \equiv V_c\psi(h^d)(H^d + i)^{-1}(1 + x_d^2)^{-2}$ is compact and $L(1 + x_d^2)^\delta$ is bounded.

LEMMA 3.6. — Let V_d be as in Theorem 2.6 for each *d*. Assume that $\delta > 1$, $\psi \in C_0^\infty(0, \infty)$ and $\varphi \in C_0^\infty((\Sigma, \infty) - T)$. Then

i) $(\tilde{\Omega}_d^\pm - 1)\psi(h^d)\varphi(H^d)(1 + x_d^2)^{-2}$ is compact,

ii) $\mathscr{E}_\pm \langle V_t f, \psi(h^d)\varphi(H^d)(1 + x_d^2)^{-2} V_t f \rangle = 0$ for f in $\mathscr{H}_c^\pm$.

Proof. — As usual we prove only for the positive sign.

i) As in the proof of Lemma 3.5 (*i*) it is enough to show $\int_0^\infty ds \, \| I_{cd}(s) \| < \infty$ for $c \neq d$ where $I_{cd}(s) = V_s^* V_c U_s^d \varphi(H^d)\psi(h^d)(1 + x_d^2)^{-2}$.
For this, writing $\varphi_1(\lambda) = (\lambda + i)\varphi(\lambda)$ we arrive at

$$\| I_{cd}(s) \| \leq \| V_c(H^d + i)^{-1}(1 + x_c^2)^\delta \| \cdot \| (1 + x_c^2)^{-\delta} \exp(-ish^d)\psi(h^d)(1 + x_d^2)^{-2} \| \cdot \| (1 + x_d^2)^2 \varphi_1(H^d)(1 + x_d^2)^{-2} \| .$$

While $V_c(H^d + i)^{-1}(1 + x_c^2)^\delta$ is bounded by Remark 3.4, the boundedness of $(1 + x_d^2)^2\varphi_1(H^d)(1 + x_d^2)^{-2}$ can be inferred by a calculation similar to that in Lemma 2.3. Thus expanding $\exp(-ish^d)$ as in Theorem 2.5 and putting $\psi_1(\lambda) = (\lambda + i)\psi(\lambda)$, we have

$$\begin{aligned} \| I_{cd}(s) \| \leq {} & K \| (1 + x_c^2)^{-\delta} \exp(-ish_0^d)(1 + x_d^2)^{-2} \| \; \| (1 + x_d^2)^2\psi(h^d)(1 + x_d^2)^{-2} \| \\ & + K_1 \int_0^s d\tau \, \{ \| (1 + x_c^2)^{-\delta} \exp[-i(s - \tau)h_0^d](1 + x_d^2)^{-\frac{1}{2}\delta} \| \\ & \| (1 + x_d^2)^{\frac{1}{2}\delta} V_d(h^d + i)^{-1}(1 + x_d^2)^{\frac{1}{2}\delta} \| \; \| (1 + x_d^2)^{-\frac{1}{2}\delta} \exp(-i\tau h^d)\psi_1(h^d)(1 + x_d^2)^{-2} \| \} \end{aligned} \tag{3.4}$$

Proceeding as in the proof of Lemma 16.3 [*6*] and replacing L^2-estimates by L^p-estimates, it follows that for $\mu > 1$ there exists μ' in $(1, \mu)$ such that

$$\| (1 + x_c^2)^{-\mu} \exp(-ish_0^d)(1 + x_d^2)^{-\mu} \| \leq K_{\mu'}(1 + s^2)^{-\mu'} . \tag{3.5}$$

The second factor in the integrand in (3.4) is finite by Remark 2.2. Since $\psi_1 \in C_0^\infty(0, \infty)$ we can use Theorem 2.6 (*iii*) for the third factor in the integrand of (3.4) which together with (3.5) yields that for some δ' in $(1, \delta)$,

$$\begin{aligned} \| I_{cd}(s) \| & \leq K_2(1 + |s|)^{-\delta'} + K_3 \int_0^s d\tau (1 + |\tau|)^{-\delta'}(1 + |s - \tau|)^{-\delta'} \\ & \leq K_5(1 + |s|)^{-\delta'} . \end{aligned}$$

This completes the proof of (*i*).

ii) By Theorem 2.1 (*a*), $\tilde{\Omega}_d^{\pm} = \Omega_0^{\pm}\omega_d^{\pm *} + \Omega_d^{\pm}$ so that $R(\tilde{\Omega}_d^{\pm}) \subseteq R(\Omega_0^{\pm}) \oplus R(\Omega_d^{\pm})$ and hence $\tilde{\Omega}_d^{\pm *} f = 0$ for all $f \in \mathcal{H}_c^{\pm}$ respectively. Thus

$$\mathcal{E}_+ \langle V_t f, \psi(h^d)\varphi(H^d)(1 + x_d^2)^{-2} V_t f \rangle$$
$$= \mathcal{E}_+ \langle V_t f, (1 - \tilde{\Omega}_d^{+})\psi(h^d)\varphi(H^d)(1 + x_d^2)^{-2} V_t f \rangle .$$

Now the result follows by (*i*) and Wiener's Theorem [20]. Q. E. D.

LEMMA 3.7. — Let V_d satisfy conditions of Theorem 2.5 for each d and $\varphi \in C_0^\infty((\Sigma, \infty) - T)$. Then there exists $\psi_1 \in C_0^\infty(R)$ with $\psi_1 = 1$ in a neighbourhood of 0, $0 \leq \psi_1 \leq 1$ and ψ_2 in $C_0^\infty(0, \infty)$ such that

$$i) \qquad \psi_1(h^d)\left[1 - \psi_2\left(\frac{1}{2}k_d^2\right)\right]\varphi(H^d) = 0 ,$$

$$ii) \qquad (\tilde{\Omega}_d^{\pm} - 1)\psi_1(h^d)\psi_2\left(\frac{1}{2}k_d^2\right)(1 + x_d^2)^{-2} F(A_d^{\perp} \gtrless 0) \text{ is compact},$$

$$iii) \qquad \mathcal{E}_{\pm} \langle V_t f, \psi_1(h^d)\varphi(H^d)(1 + x_d^2)^{-2} V_t f \rangle = 0 \text{ for } f \text{ in } \mathcal{H}_c^{\pm} .$$

Proof. i). — Let supp $\varphi \subseteq \left(-\infty, -\frac{1}{2}a^2\right) \cup \left(\frac{1}{2}a^2, b\right)$ for some $b, a > 0$. Choose $a_1, a_2 > 0$ such that $a^2 > a_1^2 + 2a_2^2$. (We shall impose further restrictions on a_1, a_2 in the proof of (*ii*)). Choose ψ_1, ψ_2 in $C_0^\infty(R)$ such that supp $\psi_1 \subseteq \left(-\frac{1}{2}a_1^2, \frac{1}{2}a_1^2\right)$, $\psi_1 = 1$ on

$\left(-\frac{1}{4}a_1^2, \frac{1}{4}a_1^2\right)$, $0 \leq \psi_1 \leq 1$; supp $\psi_2 \subseteq \left[\frac{1}{2}a_2^2, \infty\right)$ and $\psi_2 = 1$ on $[a_2^2, b]$.

Then it is easily verified that $\psi_1(h^d)\left[1 - \psi_2\left(\frac{1}{2}k_d^2\right)\right]\varphi(H^d) = 0$.

ii) (For positive sign only). As in the proof of Lemma 3.6 (*i*) it is enough to show that $\int_0^\infty ds \, \| I_{cd}(s) \| < \infty$ for $c \neq d$, where

$$I_{cd}(s) = V_s^* V_c (H^d + i)^{-1}(1 + x_c^2)^{\delta}(1 + x_c^2)^{-\delta} .$$
$$U_s^d (H^d + i)\psi_1(h^d)\psi_2\left(\frac{1}{2}k_d^2\right)(1 + x_d^2)^{-2} F(A_d^{\perp} \geq 0).$$

We observe that $(H^d + i)\psi_1(h^d)\psi_2\left(\frac{1}{2}k_d^2\right)$ is a linear combination of terms of the type $\psi_3(h^d)\psi_4\left(\frac{1}{2}k_d^2\right)$ where ψ_3 and ψ_4 have same support properties as ψ_1 and ψ_2 respectively.

So using Remark 3.4 the result follows if $\int_0^\infty ds J_{cd}(s) < \infty$ for $c \neq d$ where

$$J_{cd}(s) = \| (1+x_c^2)^{-2} U_s^d \psi_3(h^d) \psi_4\left(\frac{1}{2} k_d^2\right)(1 + x_d^2)^{-2} F(A_d^\perp \geq 0) \| .$$

Now using the resolution of the identity,

$$1 = F(|x_d| \geq c_1 s) + F(|x_d| \leq c_1 s, |y_d| \geq c_2 s) + F(|y_d| \leq c_2 s)$$

we have for all $c_1, c_2 > 0$

$$\begin{aligned} J_{cd}(s) \leq & \| (1+x_c^2)^{-\delta} \| \ \| F(|x_d| > c_1 s) U_s^d \psi_3(h^d)(1+x_d^2)^{-2} \| . \\ & \left\| \psi_4\left(\frac{1}{2} k_d^2\right) F(A_d^\perp \geq 0) \right\| \\ & + \| (1 + x_c^2)^{-\delta} F(|x_d| < c_1 s, |y_d| > c_2 s) \| . \\ & \left\| U_s^d \psi_3(h^d) \psi_4\left(\frac{1}{2} k_d^2\right)(1+x_d^2)^{-2} F(A_d^\perp \geq 0) \right\| \\ & + \| (1+x_c^2)^{-\delta} \psi_3(h^d) \| \ \left\| F(|y_d| \leq c_2 s) U_s^d \psi_4\left(\frac{1}{2} k_d^2\right) F(A_d^\perp \geq 0) \right\| . \\ & \| (1+x_d^2)^{-2} \| \end{aligned} \quad (3.6)$$

Clearly for $c \neq d$ we have $x_c = \rho_{cd} x_d + \sigma_{cd} y_d$ with ρ_{cd}, σ_{cd} real and nonzero. Set $\rho = \max \{ |\rho_{cd}| : c \neq d \}$ and $\sigma = \min \{ |\sigma_{cd}| : c \neq d \}$ so that $\rho, \sigma > 0$. Now choose a_1, a_2, c_1, c_2 such that $a^2 > a_1^2 + 2a_2^2$, $a_1 < c_1$, $\rho c_1 < \sigma c_2$, $c_2 < a_2$. With this choice of constants apply Theorem 2.5 (*i*), and results of [*11*] respectively to the first and third terms of R. H. S. to get

$$\int_0^\infty ds \| J_{cd}(s) \| < \infty \qquad \text{for} \qquad c \neq d .$$

iii) (For positive sign only). Put

$$B = \psi_1(h^d) \psi_2\left(\frac{1}{2} k_d^2\right)(1+x_d^2)^{-2}, \qquad C = (1+x_d^2)^2 \varphi(H^d)(1+x_d^2)^{-2}$$

so that both are bounded operators. We have observed in the proof of Lemma 3.6 (*ii*) that $(\tilde{\Omega}_d^+)^* f = 0$ for f in $\mathscr{H}_c^+$. So we get using (*i*), for f in $\mathscr{H}_c^+$, that

$$\begin{aligned} & \mathscr{E}_+ \langle V_t f, \psi_1(h^d) \varphi(H^d)(1+x_d^2)^{-2} V_t f \rangle \\ & = \mathscr{E}_+ \langle V_t f, (1-\tilde{\Omega}_d^+) B F(A_d^\perp \geq 0) C V_t f \rangle + \mathscr{E}_+ \langle V_t f, (1-\tilde{\Omega}_d^-) B F(A_d^\perp \leq 0) C V_t f \rangle \\ & + \mathscr{E}_+ \langle F(A_d^\perp \leq 0) U_t^d \overline{\psi}_2\left(\frac{1}{2} k_d^2\right)(\tilde{\Omega}_d^-)^* f, \psi_1(h^d) \varphi(H^d)(1+x_d^2)^{-2} V_t f \rangle . \end{aligned}$$

Now the result follows as in the proof of Lemma 3.5 (*ii*). Q. E. D.

Proof of Theorem 3.2 (For positive sign only). — Clearly it suffices to show $\mathscr{E}_+(\|(1+x_d^2)^{-1}V_tf\|^2)=0$ for f in the dense linear subspace $\mathscr{D}^+$ of $\mathscr{H}_c^+$. For f in $\mathscr{D}^+$ choose φ in $C_0((\Sigma,\infty)-T)$ such that $0 \leq \varphi \leq 1$ and $\varphi(H)f = f$. For this φ choose ψ_1 as in Lemma 3.7, ψ_2 in $C_0^\infty(-\infty,0)$, ψ_3 in $C_0^\infty(0,\infty)$ so that $(\psi_1+\psi_2+\psi_3)(h^d)\varphi(H^d)=\varphi(H^d)$. Then we have

$$\|(1+x_d^2)^{-1}V_tf\|^2=\langle V_tf,\ [\varphi(H)-\varphi(H^d)](1+x_d^2)^{-2}V_tf\rangle + \sum_{j=1}^{3}\langle V_tf,\ \psi_j(h^d)\varphi(H^d)(1+x_d^2)^{-2}V_tf\rangle .$$

Now the result follows by Lemma 3.3, Wiener's Theorem [20], Lemma 3.5 (*ii*), Lemma 3.6 (*ii*) and Lemma 3.7 (*iii*). Q. E. D.

§ 4. ASYMPTOTIC EVOLUTION OF SOME OBSERVABLES

The local decay result now allows us to imitate the two body methods in the asymptotic evolution of A and Q^2 in the three body case with one difference viz., all the statements are to be made for vectors in $\mathscr{H}_c^\pm$ (except (*iii*), (*iv*) of Theorem 4.1).

THEOREM 4.1.

i) $\underset{t\to\pm\infty}{s\text{-lim}}\ V_t^* \exp(-iuA/t)V_tf = \exp(-itH)f$ for u in R, f in $\mathscr{H}_c^\pm$,

ii) $\underset{t\to\pm\infty}{s\text{-lim}}\ V_t^* \psi(A/t)V_tf = \psi(H_{\text{cont}})f$ for $\psi \in C_0^\infty(R)$, $f \in \mathscr{H}_c^\pm$,

iii) $\underset{t\to\pm\infty}{w\text{-lim}}\ V_tf = 0$ for $f \in \mathscr{H}_c(H)$,

iv) $\underset{t\to\pm\infty}{s\text{-lim}}\ F(|x_d|<r)V_tf=0$ for r in $(0,\infty)$, $f \in \mathscr{H}_c^\pm \oplus R(\Omega_0^\pm)$,

v) $\underset{t\to\pm\infty}{s\text{-lim}}\ V_t^*U_t \exp(-iuA/t)U_t^*V_tf = f$ for u in R, $f \in \mathscr{H}_c^\pm$,

vi) $\underset{t\to\pm\infty}{s\text{-lim}}\ V_t^*U_t \exp(-uQ^2t^{-2})U_t^*V_tf=f$ for $u \geq 0$, $f \in \mathscr{H}_c^\pm$,

vii) $\underset{t\to\pm\infty}{s\text{-lim}}\ V_t^*U_t\psi(|Q|/|t|)U_t^*V_tf = \psi(0)f$
for each bounded continuous function ψ on R and f in $\mathscr{H}_c^\pm$,

viii) $\underset{t\to\pm\infty}{s\text{-lim}}\ F(|Q| \geq a|t|)U_t^*V_tf = 0$ for each $a > 0$ and f in $\mathscr{H}_c^\pm$.

Proof. — We prove the results only for the positive sign.

i) Since $\mathscr{D}^+$ is dense in $\mathscr{H}_c^+$ we can assume $f \in \mathscr{D}^+$. As in the proof of Theorem 2.5 [*21*] it suffices to verify the conditions (*i*), (*ii*), (*iii*) of Lemma 2.10 [21]. The verification of the first two are obvious. For the third one we easily get

$$\| (H+i)^{-1} \{ (V_{t+s}^{*}AV_{t+s}-A)t^{-1}-H \} f \| \leq st^{-1} \| f \| + (t+s)t^{-1}.$$

$$\sum_{d} (t+s)^{-1} \int_{0}^{t+s} d\tau \, \| (H+i)^{-1} \{ V_d + i[A_d, V_d] \} V_\tau f \|$$

For each fixed s, the R. H. S. of the above inequality goes to zero as $t \to \infty$ since $(H+i)^{-1}\{ V_d + i[A_d, V_d] \}(H+i)^{-1}(1+x_d^2)^{\frac{1}{2}\delta}$ is bounded and since by Theorem 3.2 $\mathscr{E}_+(\| (1+x_d^2)^{-\frac{1}{2}\delta} V_t(H+i)f \|) = 0$.

ii) Follows from (*i*) as in [21].

iii) If $f \in R(\Omega_0^+) \oplus \sum_d \oplus R(\Omega_d^+)$ the result is obvious. So we can assume $f \in \mathscr{H}_c^+$ and by density $f \in \mathscr{D}^+$. Then choose φ in $C_0^\infty(R - \{0\})$ such that $0 \leq \varphi \leq 1$ and $\varphi(H)f = f$. Then we see that $s\text{-}\lim_{t\to\infty} \varphi(A/t)g = 0$ for every g and by (*ii*) that $s\text{-}\lim_{t\to\infty} [1-\varphi(A/t)]V_t f = 0$. So we get $w\text{-}\lim_{t\to\infty} V_t f = 0$.

iv) On $R(\Omega_0^+)$ the proof is easy. On $\mathscr{H}_c^+$ we can go back to the whole set of computations in the proof of Theorem 3.2 and prove the required result using (*iii*).

v), (*vi*), (*vii*) are proved similarly as in Theorem 2.11 of [21].

viii) Follows easily from (*vii*). Q. E. D

§ 5. A LOW ENERGY RESULT

In the next theorem we shall prove that the evolved states in $\mathscr{D}^\pm$ with small enough kinetic energies of any pair, asymptotically vanish in the remote past or distant future.

THEOREM 5.1. — Let V_d be as in Theorem 2.6 for each d. Let $f^\pm \in \mathscr{D}^\pm$. Then there exist constants $\alpha_d(f^\pm) = \alpha_d^\pm > 0$ such that

$$\lim_{t\to\pm\infty} \| F(|p_d| \leq \alpha_d^\pm)V_t f^\pm \| = 0.$$

For the proof we need a few Lemmas.

LEMMA 5.2. — Let $f \in \mathscr{H}_c^\pm$ and $\varphi \in C_0^\infty(R)$. Then for each d

i) $$s\text{-}\lim_{t\to\pm\infty} [\varphi(H) - \varphi(H^d)]V_t f^\pm = 0,$$

ii) $$s\text{-}\lim_{t\to\pm\infty} [\varphi(H) - \varphi(H_0)]V_t f^\pm = 0,$$

iii) $$s\text{-}\lim_{t\to\pm\infty} [\varphi(h_0^d) - \varphi(h^d)]V_t f^\pm = 0.$$

Proof. i). — By Theorem 4.1 (*iv*) it suffices to show that

$$\lim_{r\to\infty} \| [\varphi(\mathrm{H}) - \varphi(\mathrm{H}^d)]\mathrm{F}(|x_c| > r, c = 1, 2, 3) \| = 0 .$$

This is done by using Theorem 1.1 of [*19*].

ii) The proof is similar to that of (*i*).

iii) Follows from Theorem 4.1 (*iv*) and Lemma 2.3 (*ii*). Q. E. D.

LEMMA 5.3. — Suppose V_d satisfies the hypothesis of Theorem 2.5 for each d. If $f^{\pm} \in \mathscr{D}^{\pm}$ then there exists $\psi_{d,\pm}$ in $\mathrm{C}_0^\infty(\mathrm{R})$, depending on f, with $\psi_{d,\pm} = 1$ in a neighbourhood of 0, $0 \le \psi_{d,\pm} \le 1$, such that, for each d,

$$\lim_{t\to\pm\infty} \| (\tilde{\Omega}_d^{\pm} - 1)\psi_{d,\pm}(h_0^d)\mathrm{V}_t f^{\pm} \| = 0 .$$

Proof. — (For positive sign and for a fixed d only). For f in $\mathscr{D}^+$ choose φ in $\mathrm{C}_0^\infty((\Sigma, \infty) - \mathrm{T})$ such that $0 \le \varphi \le 1$ and $\varphi(\mathrm{H})f = f$. For this φ choose a, a_1, a_2 and ψ_1, ψ_2 as in the proof of Lemma 3.7 (*i*) so that [after writing $\mathrm{H}^d = h^d + \frac{1}{2}k_d^2$] we get $\psi_1(h^d)\left[1-\psi_2\left(\frac{1}{2}k_d^2\right)\right]\varphi(\mathrm{H}^d) = 0$. Then using this relation and Lemma 5.2 (*i*), (*iii*) we get,

$$\begin{aligned}\eta &= \lim_{t\to\infty} \| (\tilde{\Omega}_d^+ - 1)\psi_1(h_0^d)\mathrm{V}_t f \| \\ &\le \lim_{t\to\infty} \left\| (\tilde{\Omega}_d^+ - 1)\psi_1(h^d)\psi_2\left(\frac{1}{2}k_d^2\right)\mathrm{V}_t f \right\| \\ &\le \lim_{t\to\infty} \left\| (\tilde{\Omega}_d^+ - 1)\psi_1(h^d)\psi_2\left(\frac{1}{2}k_d^2\right)\mathrm{U}_t\mathrm{F}(|\mathrm{Q}| < c_1 t) \right\| \quad \text{for each } c_1 > 0 .\end{aligned}$$

In the last step we have used Lemma 4.1 (*viii*). Expanding $\tilde{\Omega}_d^+ - 1$, it is clear that the result follows if $\lim_{t\to\infty} \int_0^\infty ds \mathrm{I}_{cd}(s, t) = 0$ for $c \ne d$ where

$$\mathrm{I}_{cd}(s, t) = \left\| \mathrm{V}_c \mathrm{U}_s^d \psi_1(h^d)\psi_2\left(\frac{1}{2}k_d^2\right)\mathrm{U}_t\mathrm{F}(|\mathrm{Q}| \le c_1 t) \right\|$$

Since $\mathrm{Q}^2 = x_d^2 + y_d^2$ we have,

$$\mathrm{F}(|\mathrm{Q}| \le c_1 t) = \mathrm{F}(|\mathrm{Q}| \le c_1 t)\mathrm{F}(|x_d| \le c_1 t, |y_d| \le c_1 t) \tag{5.1}$$

Let ψ_3, ψ_4 be as in the proof of Lemma 3.7 (*ii*). Then using (5.1) and Remark 3.4 the result follows if $\lim_{t\to\infty} \int_0^\infty ds \mathrm{J}_{cd}(s, t) = 0$ for $c \ne d$ where

$$\mathrm{J}_{cd}(s, t) = \left\| (1 + x_c^2)^{-\delta} \mathrm{U}_s^d \psi_3(h^d)\psi_4\left(\frac{1}{2}k_d^2\right)\mathrm{U}_t\mathrm{F}(|x_d| < c_1 t, |y_d| < c_1 t) \right\| .$$

As in the proof of Lemma 3.7 (*ii*) we have for each $e_1, e_2 > 0$ that

$$J_{cd}(s,t) \leq K \| F(|x_d| > e_1(t+s)) \exp(-ish^d)\psi_3(h^d) \exp(-ith_0^d) F(|x_d| < c_1(t+s)) \| + K \| (1+x_c^2)^{-\delta} F(|x_d| < e_1(t+s), |y_d| > e_2(t+s)) \| + K \| F(|y_d| < e_2(t+s)) \exp\left[-\frac{1}{2} i(s+t)k_d^2\right] \psi_4\left(\frac{1}{2}k_d^2\right) F(|y_d| < c_1(t+s)) \| . \quad (5.2)$$

Now let ρ, σ be as in the proof of Lemma 3.7 (*ii*). Now choose the constants c_1, a_1, a_2, e_1, e_2 such that $c_1 + a_1 < e_1$, $\rho e_1 < \sigma e_2$, $e_2 + c_1 < a_2$. Then apply Theorem 2.5 (*i*), the relation $\rho e_1 < \sigma e_2$ and Lemma 2.4 (*ii*) to the first, second and third terms of R. H. S. of (5.2) to get $\lim_{t\to\infty} \int_0^\infty ds J_{cd}(s,t) = 0$.
Q. E. D.

Proof of Theorem 5.1. (For the positive sign only). — For f in $\mathscr{D}^+$ let $\psi_{d,+}$ be as in Lemma 5.3. Then using $(\tilde{\Omega}_d^+)^* f = 0$ and Lemma 5.3 we have

$$\lim_{t\to\infty} \| \sqrt{\psi_{d,+}}(h_0^d) V_t f \|^2 = \lim_{t\to\infty} \langle V_t f, (1 - \tilde{\Omega}_d^+)\psi_{d,+}(h_0^d) V_t f \rangle = 0 .$$

Now the result follows since $\psi_{d,+} = 1$ in a neighbourhood of 0. Q. E. D.

§ 6. COMPLETENESS OF THREE BODY SCATTERING

Here we shall show that $\mathscr{D}^{\pm} = \{0\}$ leading to $\mathscr{H}_c^{\pm} = \{0\}$ by density.

THEOREM 6.1. — Let the two body potentials V_d satisfy the conditions of Theorem 2.6 for each d with $\delta > 1$. Then

i) the singular continuous spectral subspace $\mathscr{H}_{sc}(H) = \{0\}$,

ii) $\mathscr{H}_{ac}(H) = \sum_d \oplus R(\Omega_d^+) \oplus R(\Omega_0^+) = \sum_d \oplus R(\Omega_d^-) \oplus R(\Omega_0^-)$.

First we need a lemma.

LEMMA 6.2. — For each d let $(1+x_d^2)^{\delta} V_d (h_0^d + 1)^{-1}$ be bounded for some $\delta > \frac{1}{2}$. Let $\psi_d \in C_0^\infty(0,\infty)$, $0 \leq \psi_d \leq 1$. Then there exists some $\beta > 0$ such that.

$$\lim_{t\to\pm\infty} \left\| (\Omega_0^{\pm} - 1) \prod_d \psi_d(h_0^d) U_t F(|Q| < \beta|t|) \right\| = 0$$

Proof. (For positive sign only). — We first note that for any $\alpha > 0$ there is some $\beta > 0$ such that

$$\prod_c \mathrm{F}(|x_c| \leq \alpha t) \geq \mathrm{F}(|\mathrm{Q}| \leq \beta t).$$

Then we note that the result will follow if $\lim_{t\to\infty} \int_0^\infty ds \mathrm{I}_{\alpha,a}(s, t) = 0$ for each a and some $\alpha > 0$ where

$$\mathrm{I}_{\alpha,a}(s, t) = \left\| \mathrm{V}_a \prod_d \psi_d(h_0^d)\mathrm{U}_{s+t} \prod_c \mathrm{F}(|x_c| < \alpha t) \right\|.$$

Since $\psi_a \in \mathrm{C}_0^\infty(0, \infty)$ by Lemma 2.4 (*ii*) there exists $\alpha_a > 0$ such that

$$\| (1+x_a^2)^{-\delta} \exp [-i(t+s)h_0^a](h_0^a+i)\psi_a(h_0^a)\mathrm{F}(|x_a| \leq \alpha_a(t+s)) \| = \mathrm{O}[(t+s)^{-2\delta}] \tag{6.1}$$

Now choose $\alpha = \min \{ \alpha_a : a \}$. Then

$$\mathrm{I}_{\alpha,a}(s, t) \leq \| \mathrm{V}_a(h_0^a+i)^{-1}(1+x_a^2)^\delta \| \cdot \left\| (1+x_a^2)^{-\delta} \prod_{d\neq a} \psi_d(h_0^d)(1+x_a^2)^\delta \right\| .$$
$$\| (1+x_a^2)^{-\delta} \exp [-i(t+s)h_0^a](h_0^a+i)\psi_a(h_0^a)\mathrm{F}(|x_a| \leq \alpha(t+s)) \| .$$

The first factor of R. H. S. is bounded by hypothesis whereas the second is easily verified to be bounded. Now using (6.1) and noting $\delta > \frac{1}{2}$ we get that $\lim_{t\to\infty} \int_0^\infty ds \mathrm{I}_{\alpha,a}(s, t) = 0$. Q. E. D.

Proof of Theorem 6.1 *i*), *ii*). (For positive sign only). — By (1.1) the Theorem will follow if $\mathscr{D}^+ = \{0\}$.

Let $f \in \mathscr{D}^+$ with $\| f \| \leq 1$. Choose φ in $\mathrm{C}_0^\infty((\Sigma, \infty) - \mathrm{T})$ such that $0 \leq \varphi \leq 1$ and $\varphi(\mathrm{H})f = f$. For this f let α_d^+ be as in Theorem 5.1. Now choose ψ_d in $\mathrm{C}_0^\infty(0, \infty)$ such that

$$\varphi(\mathrm{H}_0) \leq \left\{ \sum_d \mathrm{F}(|p_d| \leq \alpha_d^+) \right\} + \prod_d \psi_d(h_0^d). \tag{6.2}$$

By Theorem 4.1 (*viii*) Lemma 6.2 and intertwining relation we get

$$\lim_{t\to\infty} \left\langle \mathrm{V}_t f, \prod_d \psi_d(h_0^d)\mathrm{V}_t f \right\rangle = 0. \tag{6.3}$$

Now by (6.2) we get

$$\| f \|^2 \leq \lim_{t\to\infty} \left\{ \langle V_t f, [\varphi(H) - \varphi(H_0)] V_t f \rangle + \sum_d \langle V_t f, F(|p_d| \leqslant \alpha_d^+) V_t f \rangle + \left\langle V_t f, \prod_d \psi_d(h_0^d) V_t f \right\rangle \right\}$$

$= 0$ by Lemma 5.2 (*ii*), Theorem 5.1 and (6.3).

Thus $\mathscr{D}^+ = \{ 0 \}$ proving the result. Q. E. D.

Remark. — The hypothesis $\delta > 1$ is necessary only in our proof of Lemma 3.6 and not explicitly in others. However, since Lemma 3.6 is essential for proving the chain of results starting with local decay (Theorem 3.2), we state the final result (Theorem 6.1) for $\delta > 1$ only.

APPENDIX

We shall use the notations of section 2 here.

THEOREM A.1. — Let $V = V_1 + V_2$ with V_2 as in Theorem 2.6 and V_1 in $C^3(R^\nu)$ such that $|D^\alpha V_1(x)| \leq K_\alpha(1 + |x|)^{-\beta-|\alpha|}$ for some $\beta > 1$ and for all multi-indices of length $|\alpha| \leq 3$. Then all the conclusions of Theorem 2.6 are true.

Proof. — We shell write $\tilde{V}$, $\tilde{V}_1$, $\tilde{V}_2$, W_s for $-\tilde{V} = V + i[A, V]$, $-\tilde{V}_j = V_j + i[A, V_j]$ and $W_s = \exp(-ish)$.

i) It is clear from the hypotheses that $\tilde{V}$ satisfies the smoothness result (2.1) and thus (*i*) follows easily.

ii) Choose ψ, χ in $C_0^\infty(0, \infty)$ such that $h^2\psi(h)\chi(h) = \varphi(h)$ and compute:

$$\begin{aligned}
[A, \psi(h)[A, W_t]\chi(h)] &= t[A, h\psi(h)]W_t\chi(h) + th\psi(h)W_t[A, \chi(h)] \\
&+ \left[A, \psi(h)\int_0^t dsW_{t-s}\tilde{V}_2W_s\chi(h)\right] + \psi(h)\int_0^t dsW_{t-s}[A, \tilde{V}_1]W_s\chi(h) \\
&+ \left[A, \psi(h)\right]\int_0^t dsW_{t-s}\tilde{V}_1W_s\chi(h) + \psi(h)\int_0^t dsW_{t-s}\tilde{V}_1W_s[A, \chi(h)] \\
&+ th\psi(h)[A, W_t]\chi(h) + \psi(h)\int_0^t ds[A, W_{t-s}]\tilde{V}_1W_s\chi(h) \\
&+ \psi(h)\int_0^t dsW_{t-s}\tilde{V}_1[A, W_s]\chi(h),
\end{aligned} \tag{A.1}$$

and

$$\begin{aligned}
&(A^2+1)^{-1}\{\text{L. H. S. of (A.1)}\}(A^2+1)^{-1} \\
&\qquad = \sum_{j=1}^{9}(A^2+1)^{-1}\{j\text{th term of R. H. S. of (A.1)}\}(A^2+1)^{-1}.
\end{aligned} \tag{A.2}$$

Norm of L. H. S. of (A.2) is clearly bounded in t and by (*i*) so are the norms of the first and second terms of R. H. S. of (A.2). As in [*15*] the norms of the third and fourth terms of R. H. S. of (A.2) are bounded by (2.1) whereas using (*i*), the norms of the fifth and the sixth terms are $O\,[\log(1 + |t|)]$. Now we have,
Sum of the seventh, eighth and ninth terms of R. H. S. of (A.1)

$$\begin{aligned}
&= t^2h^2\psi(h)W_t\chi(h) + th\psi(h)\int_0^t dsW_{t-s}\tilde{V}_2W_s\chi(h) \\
&+ th\psi(h)\int_0^t dsW_{t-s}\tilde{V}_1W_s\chi(h) + \psi(h)\int_0^t ds(t-s)hW_{t-s}\tilde{V}_1W_s\chi(h) \\
&+ \psi(h)\int_0^t ds\int_0^{t-s} d\tau W_{t-s-\tau}\tilde{V}W_\tau\tilde{V}_1W_s\chi(h) + \psi(h)\int_0^t dsW_{t-s}\tilde{V}_1shW_s\chi(h) \\
&+ \psi(h)\int_0^t dsW_{t-s}\tilde{V}_1\int_0^s d\tau W_{s-\tau}\tilde{V}W_\tau\chi(h).
\end{aligned} \tag{A.3}$$

Sum of third, fourth and sixth terms of R. H. S. of (A.3)

$$
\begin{aligned}
&= 2th\psi(h)\int_0^t ds \mathrm{W}_{t-s}\tilde{\mathrm{V}}_1\mathrm{W}_s\chi(h) + i\psi(h)\mathrm{W}_t\int_0^t ds\, s\frac{d}{ds}(\mathrm{W}_s^*\tilde{\mathrm{V}}_1\mathrm{W}_s)\chi(h) \\
&= 2th\psi(h)\int_0^t ds \mathrm{W}_{t-s}\tilde{\mathrm{V}}\mathrm{W}_s\chi(h) - 2th\psi(h)\int_0^t ds \mathrm{W}_{t-s}\tilde{\mathrm{V}}_2\mathrm{W}_s\chi(h) \\
&+ it\psi(h)\tilde{\mathrm{V}}_1\mathrm{W}_t\chi(h) - i\psi(h)\int_0^t ds\mathrm{W}_{t-s}\tilde{\mathrm{V}}_1\mathrm{W}_s\chi(h) \\
&= 2th\psi(h)[\mathrm{A}, \mathrm{W}_t]\chi(h) - 2t^2h^2\psi(h)\mathrm{W}_t\chi(h) - 2th\psi(h)\int_0^t ds\mathrm{W}_{t-s}\tilde{\mathrm{V}}_2\mathrm{W}_s\chi(h) \\
&+ it\psi(h)\tilde{\mathrm{V}}_1\mathrm{W}_t\chi(h) - i\psi(h)\int_0^t ds\mathrm{W}_{t-s}\tilde{\mathrm{V}}_1\mathrm{W}_s\chi(h). \qquad (\mathrm{A}.4)
\end{aligned}
$$

Substitute (A.4) in (A.3) to get,

Sum of seventh, eighth and ninth terms of R. H. S. of (A.1)

$$
\begin{aligned}
&= -t^2\varphi(h)\mathrm{W}_t + 2th\psi(h)[\mathrm{A}, \mathrm{W}_t]\chi(h) + it\psi(h)\tilde{\mathrm{V}}_1\mathrm{W}_t\chi(h) \\
&- i\psi(h)\int_0^t ds\mathrm{W}_{t-s}\tilde{\mathrm{V}}_1\mathrm{W}_s\chi(h) - th\psi(h)\int_0^t ds\mathrm{W}_{t-s}\tilde{\mathrm{V}}_2\mathrm{W}_s\chi(h) \\
&+ \psi(h)\int_0^t ds\int_0^{t-s} d\tau\mathrm{W}_{t-s-\tau}\tilde{\mathrm{V}}(h+i)^{-1}\mathrm{W}_\tau(h+i)\tilde{\mathrm{V}}_1\mathrm{W}_s\chi(h) \\
&+ \psi(h)\int_0^t ds\mathrm{W}_{t-s}\tilde{\mathrm{V}}_1(h+i)\int_0^s d\tau\mathrm{W}_{s-\tau}(h+i)^{-1}\tilde{\mathrm{V}}\mathrm{W}_\tau\chi(h), \qquad (\mathrm{A}.5)
\end{aligned}
$$

and

$(\mathrm{A}^2+1)^{-1}\{$ L. H. S. of (A.5) $\}(\mathrm{A}^2+1)^{-1}$

$$
= \sum_{j=1}^{7}(\mathrm{A}^2+1)^{-1}\{j\text{th term of R. H. S. of (A.5)}\}(\mathrm{A}^2+1)^{-1}. \qquad (\mathrm{A}.6)
$$

By (i) the norms of the second and third terms of R. H. S. of (A.6) are bounded in t while the same is true for the fourth term by (2.1). The norm of the fifth term is of $\bigcirc\ [\log(1+|t|)]$ by the boundedness of $(1+|x|)(h+i)^{-1}\tilde{\mathrm{V}}_2(h+i)^{-1}(1+|x|)$ and by (i). An easy calculation using the commutation rules and the properties of V_1 show that $(h+i)\tilde{\mathrm{V}}_1(h+i)^{-1}(1+|x|)$ is bounded. This together with (i) shows that the norms of the last two terms are $\bigcirc\ [\{\log(1+|t|)\}^2]$. So we get,

$$
t^2\,\|(\mathrm{A}^2+1)^{-1}\mathrm{W}_t\varphi(h)(\mathrm{A}^2+1)^{-1}\| = \bigcirc\ [1+\{\log(1+|t|)\}^2]
$$

leading to the desired result.

iii) Follows as in Theorem 2.6 (*iii*). Q. E. D.

THEOREM A.2. — Let h_0, h, V be as in Theorem 2.1 with $\delta > \frac{1}{2}$ and let φ be as in Theorem 2.5. Then the conclusion (*ii*) of Theorem 2.5 is valid.

Proof. — We set $\mathrm{W}_t^0 \equiv \exp(-ith_0)$ and $\mathrm{W}_t \equiv \exp(-ith)$. First we note that by Lemma 2.3 (*i*) and the hypotheses on V, one has

$$
\|\mathrm{F}(|x| > \alpha r)\mathrm{V}[\varphi(h) - \varphi(h_0)]\| \leq \mathrm{K}(1+r)^{-4\delta}. \qquad (\mathrm{A}.7)
$$

Let $\varphi_1 \in C_0^\infty(\mathbb{R})$, $0 \le \varphi_1 \le 1$, $\varphi_1 = 1$ on the support of φ and sup supp $\varphi_1 = \frac{1}{2} b_1^2$ such that $b_1 + c + \varepsilon < a$ for some $\varepsilon > 0$. We also set

$$F_j \equiv F\left(|x| > \left(c + \left(\frac{N-j}{N}\right)\varepsilon \right)(s-t_j) + \left(a - \frac{j\varepsilon}{N} \right)(t+t_j) \right), \qquad j = 1, 2, \dots, N-1,$$

N being a positive integer to be determined later.

Then by (A.7) we have for some $\varepsilon' > 0$,

$$\begin{aligned} \mathscr{A}(s,t) &\equiv \| F(|x| > a(t+s)) W_s \varphi(h) W_t^0 F(|x| < c(t+s)) \| \\ &\le O((t+s)^{-1-\varepsilon'}) + \| F(|x| > a(t+s)) \varphi_1(h_0) W_s \varphi(h) W_t^0 F(|x| < c(t+s)) \| \\ &\le O((t+s)^{-1-\varepsilon'}) + K \int_0^s \| F(|x| > a(t+s)) W_{s-t_1}^0 \varphi_1(h_0)(1-F_1) \| \, dt_1 \\ &+ \int_0^s dt_1 \| F(|x| > a(t+s), W_{s-t_1}^0 \varphi_1(h_0) V F_1 W_{t_1} \varphi(h) W_t^0 F(|x| < c(t+s)) \|. \end{aligned}$$

Repeating this operation N − 1 times and using Lemma 2.4 (*i*) at each stage we get

$$\begin{aligned} \mathscr{A}(s,t) \le\ & O((t+s)^{-1-\varepsilon'}) \\ &+ \int_{0 \le t_{N-1} < \dots \le t_1 < s} dt_1 \dots dt_{N-1} \| F(|x| > a(t+s)) W_{s-t_1}^0 \varphi_1(h_0) V F_1 W_{t_1-t_2}^0 \varphi_1(h_0) \\ & \qquad V F_2 \dots V F_{N-1} W_{t_{N-1}} \varphi(h) W_t^0 F(|x| < c(t+s)) \|. \end{aligned}$$

It can be easily seen that the integrand above can be bounded by $(1 + |t+s|)^{-2\delta(N-1)}$ so that if we choose $N > 1 + (2\delta - 1)^{-1}$, we have the estimate

$$\mathscr{A}(s,t) \le O((1 + |t+s|)^{-1-\varepsilon_1}) \qquad \text{for some } \varepsilon_1 > 0. \qquad \text{Q. E. D.}$$

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