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ANATOL ODZIJEWICZ

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The reduction of symplectic structure and Sternberg construction

by

Anatol ODZIJEWICZ
Institute of Physics, Warsaw University Division,
15-424 Bialystok, 41 Lipova

ABSTRACT. — In this paper we show that a reduced G-symplectic manifold $\tilde{Q}$, where Q is a G-invariant submanifold of a strong Hamiltonian symplectic space such that the momentum map image $\Psi(Q)$ is a coadjoint orbit, is isomorphic to the one given by Sternberg construction.

RÉSUMÉ. — Étant donné un espace G-symplectique hamiltonien propre et une sous-variété Q de cet espace, invariante par G et telle que l'image de Q par l'application moment soit une orbite de G, on montre que la variété réduite G-symplectique est isomorphe à celle donnée par la construction de Sternberg.

§ 1. INTRODUCTION

Studying a classical mechanical system one often needs to construct new symplectic manifolds (phase spaces in physical terminology) from the given ones. The most typical examples of such constructions are the Cartesian product of symplectic manifolds and the reduction of the symplectic structure to a submanifold. The first case takes place when one wants to describe a composed system, the second one, e. g. is crucial for the integration of Hamiltonian equations. However, there are further examples of constructions which differ from the mentioned above.

Briefly, our purpose is to present the construction of a G-symplectic manifold (§ 2) which is a discrete version of Sternberg's one (see [8]) and to show that the obtained structure is typical for many physical situations. Namely, in § 3 we will prove a theorem which says that any strong Hamiltonian G-symplectic manifold mapped by the momentum map onto a coadjoint orbit $\mathcal{O} := \mathrm{Ad}^{\#}(\mathrm{G})$ of the group G is symplectomorphic with a « G-symplectic bundle » (see § 2) over $\mathcal{O}$. Finally we shall present some applications of this theorem to physical problems.

At the very end, let us give some assumptions and notational remark. We suppose that all objects considered here, i. e. groups, manifolds, forms, maps are either real or complex analytical. If the symplectic form on manifold P is denoted by a small Greek letter, e. g. ω, then the homomorphism of the Lie group G in the group of symplectomorphisms Mor (P, ω) of P will be denoted by the corresponding capital Greek letter Ω. So, by (P, ω, Ω) we will denote a G-symplectic manifold.

§ 2. SYMPLECTIC BUNDLE OVER A HOMOGENEOUS SYMPLECTIC MANIFOLD

In the present section we shall define a G-symplectic manifold which could be considered as a bundle over some G-homogeneous symplectic manifold (M, γ, Γ).

Let H be the isotropy group of M, M = G/H. We suppose that H acts in a discrete way on a symplectic manifold (K, φ) preserving the symplectic form φ. This means that H acts on the manifold K, only through the canonical projection $\mathrm{H} \to \mathrm{D} := \mathrm{H}/\tilde{\mathrm{H}}$, where $\tilde{\mathrm{H}}$ is the connected component of the neutral element $e \in \mathrm{H}$, composed with an action on K of the discrete group D.

We can define the fiber bunder $\pi_{\mathrm{P}} : \mathrm{P} \to \mathrm{M}$, with base space M, and fiber K, associated with the principal H-bundle $\pi : \mathrm{G} \to \mathrm{M}$, and the above described action of H on K.

In the particular case when H is connected, $\mathrm{H} = \tilde{\mathrm{H}}$, we have simply $\mathrm{P} := \mathrm{M} \times \mathrm{K}$.

In the general case, P can be easily obtained by the following construction, which shows that it is a G-symplectic manifold. The quotient manifold $\hat{\mathrm{M}} := \mathrm{G}/\tilde{\mathrm{H}}$ covers $\mathrm{M} = \hat{\mathrm{M}}/\mathrm{D}$ and thus it is a symplectic G-space, on which the discrete group D acts by a symplectic action. The group D acts also on the product $\hat{\mathrm{M}} \times \mathrm{K}$, by the action

$$(d, (\hat{m}, k)) \to (\hat{m}d^{-1}, \Phi(d)k), \qquad d \in \mathrm{D},$$

and we have $\pi_{\mathrm{D}} : \hat{\mathrm{M}} \times \mathrm{K} \to \mathrm{P} := (\hat{\mathrm{M}} \times \mathrm{K})/\mathrm{D} = \mathrm{G} \times_{\mathrm{H}} \mathrm{K}$.

The product of two symplectic manifolds $(\hat{\mathrm{M}} \times \mathrm{K}, \mathrm{pr}_{\hat{\mathrm{M}}}^{*} \hat{\gamma} + \mathrm{pr}_{\mathrm{K}}^{*} \varphi)$ has a D-invariant symplectic structure, therefore, we obtain the symplectic structure σ on P by quotient. We have

$$\pi_D^*\sigma = \mathrm{pr}_M^* \hat{\gamma} + \mathrm{pr}_K^* \varphi . \tag{1}$$

Moreover, the symplectic action of G on $\hat{M} \times K$

$$(g, (\hat{m}, k)) \to (\hat{\Gamma}(g)\hat{m}, k) \tag{2}$$

commutes with the action of D on $\hat{M} \times K$. Thus, it defines, by quotients the symplectic action $\Sigma : G \to \mathrm{Mor}\,(P, \sigma)$.

For brevity we shall call (P, σ, Σ) a G-*symplectic fibre bundle* over (M, γ, Γ) with (K, φ, Φ) as the typical fibre.

Finally, we notice that the action Σ defines an n-dimensional distribution $\mathscr{H}$ on P, $n = \dim M$, where $\mathscr{H}_p$ is the subspace of $T_p(P)$ tangent to the orbit $\Sigma(G) \cdot p$, $p = (\hat{m}, k) \in P$. $\mathscr{H}$ is transverse to the fibres of the projection $\pi_P : P \to M$ and $\pi_{P*}\mathscr{H} = T(M)$. Thus, we can decompose every two vectors $\xi_p, \eta_p \in T_p(P)$ on horizontal (tangent to $\mathscr{H}$) and vertical (tangent to $\pi_P^{-1}(m)$, $m = \pi_P(p)$) parts: $\xi_p = \xi_p^v + \xi_p^h$, $\eta_p = \eta_p^v + \eta_p^h$. We have the formula which will be used later

$$\sigma(\xi_p, \eta_p) = \gamma(\pi_{P*}\xi_p^h, \pi_{P*}\eta_p^h) + \varphi(\kappa_{\hat{m}}\xi_p^v, \kappa_{\hat{m}}\eta_p^v) \tag{3}$$

where $\kappa_{\hat{m}}$ is the canonical map of $\pi_P^{-1}(m)$ on the typical fibre K, $\kappa_{\hat{m}}(p) := k$. Because of $\kappa_{\hat{m}h} - 1 = \Phi(h) \circ \kappa_{\hat{m}}$, for $h \in H$, and H-invariance of φ the second component of the right side of (3) does not depend on the choice of $\hat{m} \in \hat{\pi}^{-1}(m)$, where $\hat{\pi} : \hat{M} \to M$.

§ 3. THE STRUCTURE OF THE REDUCED SYMPLECTIC MANIFOLD

The G-symplectic manifold (P, ω, Ω) is called a Hamiltonian G-symplectic manifold if and only if there is map $\Psi : P \to \mathscr{G}^*$ such that

$$\xi_X \lrcorner\, \omega = - d\langle \Psi, X \rangle$$

for each $X \in \mathscr{G}$, where $\mathscr{G}$ is the Lie algebra of G, ξ_X is the vector field on P generated by the one-parameter group $\Omega(\exp(tX))$, $t \in R$, and $\mathscr{G}^*$ is the dual of $\mathscr{G}$. According to Souriau (see [*7*]) one calls Ψ the momentum map of (P, ω, Ω). For connected P the momentum map is defined up to a constant $c \in \mathscr{G}^*$. If the map Ψ is G-equivariant, i. e.

$$\Psi \circ \Omega(g) = \mathrm{Ad}^{\#}(g) \circ \Psi \tag{5}$$

for each $g \in G$, one says that (P, ω, Ω) is a strong Hamiltonian G-symplectic manifold. For more exhaustive information on this subject (see e. g. [*1*], [*3*], [*7*]).

Let Q be a G-invariant submanifold of P. By $\mathscr{I} \subset T(Q)$ we denote the distribution of the degeneracy subspaces of the symplectic form ω with respect to T(Q), i. e. $\mathscr{I}_p := \{ \xi_q \in T_q(Q) : \xi_q \lrcorner \omega |_{T_q(Q)} = 0 \}$. We assume that Q is chosen in such a way that $\mathscr{I}$ is of constant rank. Since $d\omega = 0$, $\mathscr{I}$ is locally integrable. If one assumes that the equivalence relation $\mathscr{R} \subset Q \times Q$ given by $\mathscr{I}$ satisfy the assumptions of Godement's theorem (see [*6*]) and that $\mathscr{R}$ is closed in $Q \times Q$, then $(\tilde{Q} := Q/\sim, \tilde{\omega}, \tilde{\Omega})$ is a G-symplectic manifold, where $\tilde{\omega}$ and $\tilde{\Omega}$ denote the reduced symplectic form and the reduced action of G on Q respectively. If the above takes place one says that there is reduction of the symplectic structure of P to submanifold Q. If one additionally assumes that (P, ω, Ω) is a strong Hamiltonian space, then $(\tilde{Q}, \tilde{\omega}, \tilde{\Omega})$ will be a strong Hamiltonian space, too. The existence of the reduced momentum map $\tilde{\Psi} : \tilde{Q} \to \mathscr{G}^*$ follows from the constancy of Ψ on the maximal connected integral submanifolds of $\mathscr{I}$, which is implied by the condition (4).

THEOREM 1. — Let (P, ω, Ω) be a strong Hamiltonian space and let Q be G-invariant submanifold of P such that:

a) there is reduction of the symplectic structure of P to Q,

b) $\Psi(Q) := \mathcal{O}$ is an $\mathrm{Ad}^{\#}(G)$-orbit and a submanifold of $\mathscr{G}^*$.

Then $(\tilde{Q}, \tilde{\omega}, \tilde{\Omega})$ is isomorphic to a G-symplectic bundle $G \times_H K$, where H is the stabilizer of $o \in \mathcal{O}$ and K is the H-symplectic manifold, given by the reduction of the symplectic structure of P to $\Psi^{-1}(o) \cap Q$.

Proof. — Let $\mathrm{Int}_q\, \mathscr{I}$ be the maximal connected integral submanifold of $\mathscr{I}$ which contains $q \in Q$. We have the following inclusion relations.

LEMMA 2. —

a) $\mathrm{Int}_q\, \mathscr{I} \subset \Psi^{-1}(o) \cap Q$,

b) $G_0 \cdot q \subset \Psi^{-1}(o) \cap Q$,

c) $G_0 \cdot q \subset \mathrm{Int}_q\, \mathscr{I}$,

where G_0 is the stabilizer of $o = \Psi(q) \in \mathcal{O}$ and $\tilde{G}_0$ is the connected component of the unit of G_0.

Proof of lemma. — Because $\mathrm{Int}_q\, \mathscr{I}$ is connected, *a*) is a consequence of $\underset{\xi}{£}\Psi = 0$, for $\xi \in \Gamma(\mathscr{I})$.

The point *b*) results from the G-equivalence of the momentum map Ψ.

It is well known that for a given point $p \in P$, the kernel $\mathrm{der}\,(\Psi_*)_p$ of the linear map tangent to the momentum map is the symplectic orthogonal orth $T_p(G \cdot p)$ of the space tangent at p to the G-orbit of p. When $p \in Q$, by the assumption made in theorem 1, that $\Psi(Q)$ is a coadjoint orbit, we see that

$$T_p(Q) \subset \ker(\Psi_*)_p + T_p(G \cdot p)$$

and because Q is G-invariant

$$T_p(Q) \supset T_p(G \cdot p)$$

This leads to

$$T_p(Q) \cap \text{orth } T_p(Q) \supset T_p(G \cdot p) \cap \text{orth}\,(\ker(\Psi_*)_p + T_p(G \cdot p))$$
$$\supset T_p(G \cdot p) \cap \text{orth ker}\,(\Psi_*)_p \cap \text{orth } T_p(G \cdot p) = T_p(G \cdot p) \cap \ker\,(\Psi_*)_p\,.$$

But $T_p(Q) \cap \text{orth } T_p(Q)$ is the space tangent at p to $\text{Int}_p\,\mathcal{I}$, and $\ker(\Psi_*)_p \cap T_p(G \cdot p)$ is the space tangent at p to the orbit $G_0 \cdot p$, $o = \Psi(p)$. This implies *c*) of Lemma 2. ■

The reduced momentum map $\tilde{\Psi} : \tilde{Q} \to \mathcal{O}$ is a submersion because Ψ is submersion, too. Hence, $\tilde{\Psi}^{-1}(o) = (\Psi^{-1}(o) \cap Q)/\sim$, $o \in \mathcal{O}$, is a submanifold of $\tilde{Q}$. It is evidently clear that $\tilde{\omega}\,|_{\Psi-1(0)}$ is a G_0-invariant and closed two-form on $\tilde{\Psi}^{-1}(o)$. We have

$$\tilde{\omega}(\tilde{\xi}^v, \tilde{\xi}_X) = \underset{\tilde{\xi}^v}{\pounds} \langle \Psi, X \rangle = 0 \tag{6}$$

where $\tilde{\xi}^v \in \Gamma(T(\tilde{Q}))$ is such that $\tilde{\Psi}_* \tilde{\xi}^v = 0$ and the vector field $\tilde{\xi}_X$ is generated by $X \in \mathcal{G}$. For each vector $\tilde{\xi}_q \in T_q(\tilde{Q})$ one has the following decomposition $\tilde{\xi}_q = \tilde{\xi}^v_q + (\tilde{\xi}_X)_q$. Using this decomposition, the non-singularity of $\tilde{\omega}$ and the formula (6) we find that the two-form $\varphi := \tilde{\omega}\,|_{\tilde{\Psi}-1(0)}$ is non-singular on $K := \tilde{\Psi}^{-1}(o)$. Hence, $(K, \varphi, \tilde{\Omega}\,|_H)$ is $H := G_0$-symplectic manifold on which, by the point *c* of the Lemma 2, H acts in a discrete way. On the other hand, in virtue of Kirillov-Kostant-Souriau theorem (see [*1*], [*3*], [*7*]) the orbit $(\mathcal{O}, \alpha, \text{Ad}^\#)$ is a G-homogeneous symplectic manifold, the symplectic form α is given by the Kirillov construction (see [*3*]). This allows us to construct the G-symplectic fibre bundle $\pi : G \times_H K \to \mathcal{O}$, which is isomorphic to $\tilde{Q}$ as a G-fibre bundle. The symplectic form $I^*\tilde{\omega}$, where $I : G \times_H K \cong \tilde{Q}$, is identical with the one given by (1). In order to see this it is enough to decompose $\xi_p \in T_p(G \times_H K)$ on the vertical and horizontal (with respect to $\mathcal{H}$) parts and apply the formulas (3) and (6). ■

We define the equivalence relation on $\tilde{Q} : q_1 \sim q_2$, $q_1, q_2 \in \tilde{Q}$, if $\tilde{\Psi}(q_1) = \tilde{\Psi}(q_2)$ and if there is $g \in G$ such that $\tilde{\Omega}(g)q_1 = q_2$. If one assumes that the quotient space $\tilde{Q}/\sim := \tilde{\tilde{Q}}$, is a manifold then, from $\tilde{\Omega}(G)$-invariance of $\tilde{\omega}$, $(\tilde{\tilde{Q}}, \tilde{\tilde{\omega}}, \tilde{\tilde{\Omega}})$ is the strong Hamiltonian symplectic manifold isomorphic to the Cartesian product of the symplectic manifolds K/H and $\mathcal{O}$. K/H is the space of orbits of the group H in K. In order to prove this isomorphism we observe that each orbit of the action $\tilde{\tilde{\Omega}} : G \to \text{Mor}\,(\tilde{\tilde{Q}}, \tilde{\tilde{\omega}})$ has a one element intersection with $\tilde{\Psi}^{-1}(o)$, for any $o \in \mathcal{O}$. This relates Theorem 2 to a similar result given in [*4*].

Now, let us consider some concrete instances of our theorem. We begin with the case when G is semisimple Lie group. Then, (see [*1*], [*7*]) any G-symplectic manifold is a strong Hamiltonian space. Additionally, by the Cartan-Killing form we have the canonical identification $\mathcal{G}^* \cong \mathcal{G}$

and we can consider Ψ as an Ad(G)-equivariant map of P into $\mathcal{G}$. But, there is a theorem (see [9]) which says that any Ad(G)-orbit $\mathcal{O}$ is a submanifold of $\mathcal{G}$. Hence, we could weaken the hypothesis of the theorem. Namely, it is enough to assume that: P is a G-symplectic manifold, there is the reduction of the symplectic structure of P to Q and Ψ maps Q on an Ad(G)-orbit.

Let us assume that P is a G-homogeneous symplectic manifold. Then our theorem reduces to well known statement (see [1], [3]) which says that each G-homogeneous symplectic manifold covers some co-adjoint orbit of G and that the covering map is a symplectomorphism.

At the end of this paragraph let us conclude from Theorem 2 the following.

COROLLARY 3. — Let all assumptions of Theorem 2 be satisfied and let H be a connected Lie group. Then $(\tilde{Q}, \tilde{\omega}, \tilde{\Omega})$ is isomorphic to the Cartesian product of G-symplectic manifolds $(\mathcal{O} \times K, \mathrm{pr}_{\mathcal{O}}^{*} \alpha + \mathrm{pr}_{K}^{*} \tilde{\omega}|_{K}, \mathrm{Ad}^{\#} \times \mathrm{id})$.

Proof. — By the connectednees of H it acts trivially on K. Thus the fibre bundle $G \times_{H} K$ is isomorphic to $\mathcal{O} \times K$.

§ 4. EXAMPLES

A. Relativistic mechanics.

Here, as the symmetry group one has the Poincare group $P_{+}^{\uparrow}$. Lep P be a $P_{+}^{\uparrow}$-symplectic manifold and let the action of $P_{+}^{\uparrow}$ on P be strong Hamiltonian. Thus decomposing the momentum map $\Psi(p) = p_{\mu}(p)\mathcal{P}^{*\mu} + \frac{1}{2}M_{\mu\nu}(p)\mathcal{L}^{*\mu\nu}$, where $\mathcal{P}^{*\mu}$, $\mathcal{L}^{*\mu\nu} = -\mathcal{L}^{*\nu\mu}$ is the basis of $\mathcal{P}_{+}^{\uparrow *}$ given by the space-time translations and the Lorentz transformations respectively, we define the four-momentum $p_{\mu}(p)$ and the relativistic angular-momentum $M_{\mu\nu}(p)$ for the state $p \in P$. The $\mathrm{Ad}^{\#}(P_{+}^{\uparrow})$-orbits $\mathcal{O}_{m^2,s,\varepsilon}$ are given by the algebraic equations (see [5]):

$$
\begin{aligned}
& p^{\mu}p_{\mu} = m^2 \\
& p^{\mu}p_{\mu} = -s^2 w^{\mu}w_{\mu} \quad \text{iff} \quad m^2 \neq 0 \qquad \text{or} \qquad p^{\mu} = -sw^{\mu} \quad \text{iff} \quad m^2 = 0 \\
(7) \quad & \operatorname{sign} p^0 = 1, -1 \\
& w^{\mu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}p_{\nu}M_{\alpha\beta} \qquad \text{(the Pauli-Lubansky vector)}
\end{aligned}
$$

The parameters m, s and ε have the interpretation of mass, spin and the energy signum. They parametrize the $\mathrm{Ad}^{\#}(P_{+}^{\uparrow})$-orbits. Depending on m^2, s and ε the small groups for $\mathcal{O}_{m^2,s,\varepsilon}$ are SO(3), SL(2, R) or the Euclidean group E(2). All of them are connected Lie groups. Hence, if we assume that all states $p \in P$ have the same masses, spins and energy signums we will have from the Corollary 3 that P is symplectically isomorphic to the Cartesian product of $\mathcal{O}_{m^2,s,\varepsilon}$ with the symplectic manifold (K, φ) which describes the internal degrees of freedom of the relativistic mechanical system.

B. Systems with constant square of the angular-momentum.

Let us consider a classical mechanical system such that all states $p \in P$ have defined angular-momentums $\vec{M}(p)$ and $\vec{M}^2 = \text{const}$ on P. Using the symplectic geometry language we assume that P is a connected SO(3)-symplectic manifold. Because of the simplicity of SO(3), P is a strong Hamiltonian space. By the identification $\mathcal{SO}(3) \cong \mathcal{SO}(3)^* \cong \mathbb{R}^3$ we can write $\Psi(p) = M_1(p)e^1 + M_2(p)e^2 + M_3(p)e^3$, where $e^i \in \mathcal{SO}(3)$ generates the rotation with respect to i-axis of the Cartesian system of coordinates in $\mathbb{R}^3$ and $\vec{M}(p) = [M_1(p), M_2(p), M_3(p)]$ is the angular-momentum vector. The condition $\vec{M}^2 = \text{const}$ means that Ψ maps P on an Ad(SO(3))-orbit, i. e. on the sphere $\mathbb{S}^2_M$ in $\mathbb{R}^3$ with radius equal to $\sqrt{\vec{M}^2(p)}$. The small group for $\mathbb{S}^2_M$ is U(1). It follows from Corollary 3 that P factorizes as the Cartesian product of symplectic manifolds $\mathbb{S}^2_M \times K$. Here, K denotes a (dim P $-$ 2)-dimensional symplectic manifold on which SO(3) acts trivially. The symplectic structure of $\mathbb{S}^2_M$ is given by the canonical volume two-form of $\mathbb{S}^2_M$.

C. The dual pair (O(1, 3), SL(2, R).

First we define a dual pair of Lie groups in the relativistic case. For the general notion of a dual pair (see [*2*]). Let us consider the cotangent bundle $T^*(M_4)$ to Minkowski space M_4 with $\omega = dq^0 \wedge do^0 - d\vec{q} \wedge d\vec{p}$ as symplectic form. Here, one can interprete $\bar{q} = (q^0, \vec{q})$ as the position of a scalar relativistic particle in spacetime and $\bar{p} = (p^0, \vec{p})$ as its four-momentum vector. The linear symplectic group $Sp(8)$ of $T^*(M_4) \cong \mathbb{R}^8$ contains the Lorentz group O(1, 3) and the real unimodular group SL(2, R) as subgroups in such a way that each one of them is the centralizer of the other. According to Howe we will call (O(1, 3), SL(2, R)) a dual pair. Thus, consequently, we have the dual pair (Ω_0, Ω_{SL}) of actions:

$$\begin{aligned} &\Omega_0(g)\begin{bmatrix} \bar{q} \\ \bar{p} \end{bmatrix} = \begin{bmatrix} g\bar{q} \\ g\bar{p} \end{bmatrix} \qquad g \in O(1,3) \\ &\Omega_{SL}(g)\begin{bmatrix} \bar{q} \\ \bar{p} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} \bar{q} \\ \bar{p} \end{bmatrix} \qquad g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, R), \end{aligned} \tag{8}$$

where $\begin{bmatrix} \overline{q} \\ \overline{p} \end{bmatrix} \in T^*(M_4)$, and corresponding to them the dual pair of momentum maps:

$$(9) \qquad \begin{aligned} \Psi_0(\overline{q}, \overline{p}) &= q^\mu p^\nu - q^\nu p^\mu = \begin{bmatrix} 0 & \vec{L}^T \\ \vec{L} & M \end{bmatrix} \\ \Psi_{SL}(\overline{q}, \overline{p}) &= \begin{bmatrix} \overline{q}\cdot\overline{p}, & -\overline{q}^2 \\ \overline{p}^2, & -\overline{q}\cdot\overline{p} \end{bmatrix} = \begin{bmatrix} x^2, & x^1 + x^3 \\ x^1 - x^0, & -x^2 \end{bmatrix}. \end{aligned}$$

By the definition $\overline{q}\cdot\overline{p} := q^0p^0 - \vec{q}\cdot\vec{p}$, $M_{kl} := \varepsilon_{klm}M_m$, where $\vec{L}$, $\vec{M} \in \mathbb{R}^3$ and $x^0, x^1, x^2 \in \mathbb{R}$. Because the two groups are simple we made the identifications: $\mathcal{O}(1, 3) \cong \mathcal{O}(1, 3)^*$ and $\mathcal{S}\ell(2, R) \cong \mathcal{S}\ell(2, R)^*$.

Let $Q \subset T^*(M_4)$ be given by the conditions: $\overline{q}^2 + \overline{p}^2 < 0$, $\overline{q} \neq 0$, $\overline{p} \neq 0$, $\overline{q} \not\sim \overline{p}$ and $H(\overline{q}, \overline{p}) := \det \Psi_{SL}(\overline{q}, \overline{p}) = \overline{q}^2\overline{p}^2 - (\overline{q}\cdot\overline{p})^2 = 0$. A simple computation shows that Q is both $SL(2\cdot R)$ and $O(1, 3)$-invariant submanifold of $T^*(M_4)$. The images $\Psi_0(Q) = \{\vec{L}, \vec{M}) : \vec{L}\cdot\vec{M} = 0 = \vec{M}^2 - \vec{L}^2, \vec{L} \neq 0, \vec{M} \neq 0\}$ and $\Psi_{SL}(Q) = \{ (x^0, x^1, x^2) : (x^0)^2 - (x^1)^2 - (x^2)^2 = 0 \text{ and } x^0 > 0 \} =: C^2_+$ are coadjoint orbits, thus, they are G-homogeneous symplectic manifolds isomorphic to $(T^*_0(\mathbb{S}^2), d\gamma)$ and (C^2_+, σ) respectively. Here $T^*_0(\mathbb{S}^2)$ is the cotangent bundle of $\mathbb{S}^2$ with removed null section, γ is the canonical one-form on $T^*_0(\mathbb{S}^2)$ and σ is the SO(1, 2)-invariant volume two-form on the punctured two-dimensional half-cone.

The maximal connected integral submanifold of $\mathcal{I} \subset T(Q)$ which contains the point $(\overline{q}_0, \overline{p}_0) \in Q$ has the following form

$$(10) \qquad \text{Int}_{(q_0,p_0)}\mathcal{I} = \left\{ \exp\left(t\begin{bmatrix} \overline{q}_0\cdot\overline{p}_0, & -\overline{q}_0^2 \\ \overline{p}_0^2, & -\overline{q}_0\cdot\overline{p}_0 \end{bmatrix}\right)\begin{bmatrix} \overline{q}_0 \\ \overline{p}_0 \end{bmatrix} : t \in \mathbb{R} \right\}.$$

Thus, $(\overline{q}_0, \overline{p}_0) \sim (\overline{q}, \overline{p})$ if and only if

$$(11) \qquad \begin{aligned} \overline{q} &= \overline{q}_0 + t[(\overline{q}_0\cdot\overline{p}_0)\overline{q}_0 - \overline{q}_0^2\overline{p}_0 \\ \overline{p} &= \overline{p}_0 + t[\overline{p}_0^2\overline{q}_0 - (\overline{q}_0\cdot\overline{p}_0)p_0] \end{aligned}$$

for some $t \in \mathbb{R}$. Now, after application of Theorem 1 twice (in the SL(2, R) and O(1, 3) cases) we find that the reduced symplectic manifold $(\tilde{Q}, \tilde{\omega}, \tilde{\Omega})$ is isomorphic to symplectic bundle $SL(2, R) \times_{H_{SL}} C^2_0(\mathbb{S}^2) \to C^2_+$ and to the symplectic bundle $O(3, 1) \times_{H_O} \mathbb{R}^2_* \to T^*_0(\mathbb{S}^2)$ respectively. Where $H_{SL} := = \left\{ \begin{bmatrix} \mp 1 & t \\ 0 & \mp 1 \end{bmatrix} : t \in \mathbb{R} \right\} \subset SL(2, R)$ and H_0 is the subgroup of O(1, 3) generated by the subgroups:

$$\tilde{H}_0 = \left\{ \begin{bmatrix} 1 + \frac{1}{2}\vec{b}^2, & \vec{b}^T, & \frac{1}{2}\vec{b}^2 \\ \vec{b}\ , & E\ , & \vec{b} \\ -\frac{1}{2}\vec{b}^2, & -\vec{b}^T, & 1 - \frac{1}{2}\vec{b}^2 \end{bmatrix} : \vec{b} \in \mathbb{R}, E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

and

$$\left\{ \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \\ \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \right\}$$

The connected component of unity of H_0 is $\tilde{H}_0 \cong (\mathbb{R}^2, +)$. The action $\Phi_0 : H_0 \to \operatorname{Mor}(\mathbb{R}^2_*, \varphi_0 = dx \wedge dy)$ is defined by $\Phi(\Lambda)\begin{bmatrix} x \\ y \end{bmatrix} = \operatorname{sign} \Lambda^0_0 \begin{bmatrix} x \\ y \end{bmatrix}$, $\Lambda = [\Lambda^\mu_\nu]_{\mu,\nu=0,1,2,3} \in O(1,3)$. $C^2_0(\mathbb{S}^2) := \{ (\vec{q}, \vec{p}, p^0) \in \mathbb{R}^7 : \vec{q}^2 = 1,\ \vec{q}\cdot\vec{p}=0$ and $0 = (p^0)^2 = \vec{p}^2 \}$ is the bundle of punctured two-dimensional cones with axis normal to $\mathbb{S}^2$ and $\Phi_{SL}\ (g)(\vec{q}, \vec{p}, p^0) := \frac{1}{2} \operatorname{Tr}(g)(\vec{q}, \vec{p}, p^0)$, $g \in H_{SL}$, is the action of H_{SL} on $C^2_0(\mathbb{S}^2)$. The two-form $\varphi_{SL} := \pi^* d\gamma$, where $\pi : C^2_0(\mathbb{S}^2) \to C^2_0(\mathbb{S}^2)/\Phi_{SL}(H_{SL}) \cong T^*_0(\mathbb{S}^2)$ is the two-fold covering of $T^*_0(\mathbb{S}^2)$, γ is the H_{SL}-invariant symplectic form on $C^2_0(\mathbb{S}^2)$. Additionally we proved the isomorphism $SL(2, R) \times_{H_{SL}} C^2_0(\mathbb{S}^2) \cong O(1, 3) \times_{H_O} R^2_*$.

D. Other applications.

We can study situations similar to that in point C for any dual pair of Lie groups. In general we obtain a double fibration of a reduced symplectic manifold over coadjoint orbits which correspond to the dual pair. This could have interesting physical applications, e. g. to the Kepler problem if one considers the dual pair (O(4), SL(2, R)).

It turns out that Theorem 1 is also useful in twistorial mechanics, which will be discussed in a further publication.

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