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A Generalization of Dixon's Description of Extended Bodies (*)

by

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ABSTRACT. — Dixon's work on the description of extended bodies in General Relativity is extended to include non-metric theories of gravity and a special class of distributional energy momentum tensors. Several splitting theorems are proven and are used to define momentum, angular momentum and the « skeleton », as well as to investigate their properties. We then define reduced moments and prove a reconstruction theorem : A momentum, an angular momentum and a « skeleton » obeying Dixon's laws of motion determine an (admissible) mass tensor.

RÉSUMÉ. — Les résultats de Dixon sur la description des corps étendus en relativité générale sont généralisés pour qu'ils s'appliquent aux théories non métriques de gravitation et à une classe spéciale de tenseurs impulsion-énergie distributionnels. On démontre plusieurs théorèmes de décomposition et on s'en sert pour définir et étudier l'impulsion, le moment cinétique et le « squelette ». On définit ensuite des moments réduits et on démontre le théorème : Une impulsion, un moment cinétique et un « squelette » qui satisfont les lois de mouvement de Dixon déterminent un tenseur de masse admissible.

1. INTRODUCTION

Since the comparison of general relativity theory with astrophysical observation is based mainly on measurements performed on practically

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isolated systems consisting of a finite number of discrete bodies, the study of such systems merits considerable attention.

At present we are still far from exact quantitative statements about the motion of the constituents of such systems. So far the research on this subjects has proceeded along three major lines:

The first on deals with various plausible, but not rigorously justified approximation schemes. For slow motion, weak-field situations one recovers Newton's equations as a first approximation:

The CM-motion of a single body appears to be largely independent of its detailed internal structure and can be calculated to high accuracy from a knowledge of a few parameters; furthermore, the motion is independent of the « self-field » of the body and completely determined by the field produced by the other bodies in the system (see Dixon's Varenna Lectures [*8*] for a detailed exposition). One expects, that a similar behaviour should also persist in some higher orders of the approximation scheme. Unfortunately, at present all existing approximation methods are plagued by mathematical or conceptual difficulties.

The other two lines treat certain aspects of the problem in an exact, covariant manner: The (expected) asymptotic structure of spacetime far away from material sources, and the local structure and motion of the bodies relative to an unspecified gravitational field (i. e. not the full field equations, but only their consequence $\nabla \cdot T = 0$ are used). The local theory has been developed by W. G. Dixon ([*3*]-[*8*]) and continued by J. Ehlers and E. Rudolph ([*9*]), and R. Schattner ([*15*]). Their work shows that—even in the general relativistic context—one can introduce concepts such as CM-line, (reduced) moments, force, and torque, and a mass-constant, having certain desirable properties.

Each of these three different approaches has its merits and drawback; there remains the challenging problem of combining them in a satisfactory manner. One wishes to connect the local quantities (mass, momentum, angular momentum, ...) with the corresponding asymptotic quantities (some attempts in this direction have been made by R. Schattner and M. Streubel [*18*] [*21*]) and to link Dixon's theory to (controlled) approximation methods incorporating some nice features of the Newtonian description (e. g. separation of a « self-field », motion dominated by a few parameters).

In this paper the main subject of research is the relation between the « skeleton », a mathematical object representing the structure of the body, and the energy-momentum tensor. Dixon has proven that in a metric theory of gravity the following holds: Suppose one has a smooth symmetric $\binom{2}{0}$ tensor field T whose support is contained in a spatially bounded, timelike world tube, and which satisfies the local law of motion $\nabla \cdot T = 0$; a timelike worldline l; and a timelike unit vector field u along l. Then there exist a momentum P, an angular momentum S and a skeleton $\hat{T}$ satisfying Dixon's integral laws of motion (and some further conditions in order to ensure the uniqueness of $\hat{T}$). P, S and $\hat{T}$ together (satisfying Dixon's integral laws of motion) contain the same information as the energy-momentum tensor T (satisfying the local law of motion). The representation of a body by (P, S, $\hat{T}$) provides the decisive advantage that the complicated system of partial differential equations $\nabla \cdot T = 0$ has been replaced by finitely many ordinary differential equations for globally defined quantities which closely resemble the corresponding Newtonian and special-relativistic equations.

The momentum P is a vector field along l, the angular momentum S is a bivector field along l, and the skeleton $\hat{T}$ is a one-parameter family of distributions on the tangent spaces along l acting on symmetric $\binom{0}{2}$-tensor fields over the natural projection.

Dixon's proof is constructive, and Dixon gives a formula that indicates how to reconstruct T from (P, S, $\hat{T}$). But inserting an « arbitrary » triple (P, S, $\hat{T}$) in the reconstruction formula, one obtains in general, not a smooth tensor field T, but only a linear functional. So the question arises of what spaces of T'*s* and $\hat{T}$'*s* one has to choose in order to make the relation between T and (P, S, $\hat{T}$) symmetric. We will give an—at least partial—answer to this question: We will give definitions of suitable spaces which allow one to establish symmetry between the two descriptions of the body. Our T'*s* will in general be not smooth functions, but rather special distributions. (Hence, admittedly our result is not completely satisfactory). Thus we will have to pay due attention to the functional analysis of such objects. Furthermore, in view of the aforementioned interesting questions concerning the relation between the Newtonian and the relativistic description of bodies, we will formulate our theory without using any metric structure on spacetime. In fact, all of our constructions will depend only on a few assumptions which hold true for a wide class of theories of gravity: That spacetime be a 4-manifold endowed with a symmetric linear connexion ∇ and that matter be described by a symmetric $\binom{2}{0}$ tensor field whose covariant divergence vanishes.

We have tried to keep paper self-contained even though many ideas and proofs have been adapted from Dixon's work.

After the introduction of our notation ana convections, in section 3 we motivate and formulate our basic assumptions. We define function and distribution spaces which are well suited for our purposes (in appendix 1 we summarize some statements on test field spaces and distributions, while in appendix 2 we briefly discuss some special bitensor fields, cf. [*16*] [*17*] for details).

Section 4 is devoted to decomposition theorems for test field spaces. These splitting theorems are in their original form (without consideration of topological and support properties) due to Dixon and play the key role in the construction of the skeleton. In the fifth section we define momentum P, angular momentum S and a « skeleton » $\hat{T}$ for our mass tensor distribution T. We investigate their properties and show that the set (P, S, $\hat{T}$) completely determines the action of T on the test field space, and furthermore, that Dixon's integral laws of motion are satisfied. In section 6, we ask a converse question: How to construct a mass tensor T from a given triple (P, S, $\hat{T}$) which has the appropriate properties? Reduced moments are defined and are used to establish statements about the structure of $\hat{T}$. These allow one to prove a reconstruction theorem which symmetrizes completely the relation between T and (P, S, $\hat{T}$): a set (P, S, $\hat{T}$) satisfying Dixon's laws of motions determines an admissible tensor T^{ab} obeying $\nabla \cdot T = 0$.

2. NOTATION AND CONVENTIONS

For a manifold M, M_z denotes the tangent space $z \in M$, $TM \overset{\pi}{\to} M$ the tangent bundle, $T^r_sM \overset{\pi^r_s}{\to} M$ the tensor bundle of type (r, s). The subspaces of T^r_sM satisfying certain symmetry conditions are denoted by an indication of the symmetry class in brackets, e. g. $T^0_{[2]}M$ is the space of twice covariant symmetric tensor fields (irreducible symmetry [2]).

For a set $L \subseteq M$, $(T^r_s, T)L$ is the space of C^∞-maps $\psi : \pi^{-1}(L) \to T^r_sM$ such that $\pi^r_s \circ \psi = \pi$ (« tensor fields of type (r, s) on $\pi^{-1}(L)$ over π » [6]). $(.)_{,a}$ means partial differentiation. Covariant differentiation is denoted by ∇_a; $\nabla_{a_1 \ldots a_k} := \nabla_{a_1} \ldots \nabla_{a_k}$, $(\$\lambda)_{ab} := -\nabla_{(a}\lambda_{b)}$, $\nabla_{bc}A_a - \nabla_{cb}A_a = R^d{}_{abc}A_d$. Absolute differentiation along a curve $x(u)$ is denoted by D_u or $\dfrac{D}{du}$. $T_{\{abc\}} := T_{abc} - T_{bca} + T_{cab}$.

The space of two-point-tensor fields (bitensor fields) on M is denoted by $T^{r,s}_{p,q}(M)$. If the arguments z, x are restricted to subsets $Z \subseteq M$, $X \subseteq M$, we write $T^{r,s}_{p,q}(Z \times X)$. We use $i, k, l, \ldots$ for indices at z, $a, b, c, \ldots$ for indices at x. $\langle \alpha \rangle$ is the coincidence limit of the bitensor field α.

With the aid of the relative position $X^k := -\sigma^k(z, x)$ (see Appendix 2) we can treat a two-point tensor field $t(z, x)$ with scalar character at x as function of z and X rather than of z and x.

Let A, B be vector fields on TM over π, $\psi \in (T^r_s, T)M$. We have a covariant directional derivative $\nabla_{(A,B)}\psi := A^k\nabla_{k*}\psi + B^k\nabla_{*k}\psi$ with ∇_{k*}, ∇_{*k} as in [6] [8]. (In natural coordinates on T_zM ∇_{*k} is just $\dfrac{\partial}{\partial X^k}$; $\nabla_{*k}\psi = H^a{}_k\nabla_a\psi$).

3. BASIC ASSUMPTIONS. RESTRICTABLE DISTRIBUTIONS

It is well known (see e. g. [*11*] [*22*] [*13*] [*7*] [*10*]) that—when formulated properly in the language of differential geometry—the Newtonian theory of gravity has several structures in common with Einstein's theory (which has been expressed in that language from the beginning). In fact, both theories can be described as special cases within a whole family of theories of spacetime and gravity for which there exists a common framework (see e. g. [*10*]).

In the subsequent sections we will actually use only some rudimentary aspects of these theories: The basic objects in our presentation are

i) a 4-dimensional C^∞-manifold M,

ii) a symmetric linear connexion ∇ on M (describing « inertia » and « gravity »), and

iii) a twice contravariant, symmetric tensor field T^{ab}, the covariant divergence of which vanishes:

$$\nabla_a T^{ab} = 0 . \tag{3.1}$$

T^{ab} represents the distribution of mass, momentum and stress of the matter, and will be frequently referred to as the « mass tensor ».

Note that we use neither any metric structure on M nor field equations nor the notion of timelike/spacelike vectors nor any energy condition, etc.

We want to describe the behaviour of an isolated single body with respect to a suitable observer. Hence we require the following:

$$\text{There exists a closed set W such that} \qquad \operatorname{supp}(T^{ab}) \subseteq W , \tag{3.2}$$

a worldline $l \equiv z(s)$ contained in W (representing the observer) and a covector field $u_k(s)$ along l (determining the local rest space of the observer), $u_k(s)\dot{z}^k(s) \neq 0$ for all s.

$$\text{We choose a parametrization such that} \qquad u_k(s)\dot{z}^k(s) = 1 \qquad \text{for all } s . \tag{3.3}$$

3.1. Remarks. — Using more structure on M and much stronger conditions than above, one can single out a unique pair (l, u_k) ; l is then considered to be the centre-of-mass worldline and u_k the surface element determining the local rest space of the system (cf. [*8*] [*15*]). This choice is appealing but not necessary for our treatment. It is, however, a « natural » prescription in order to obtain well-defined, *uniquely* determined laws of motion.

For technical reasons, we impose further conditions on W, l, u_k:

There exists an open submanifold $N \subseteq M$ with the following properties:

i) $W \subseteq N$;

ii) For any $x \in \mathrm{N}$ there exists a uniquely determined point $z(s) \in l$ and a unique vector ξ^k at $z(s)$ with $\xi^k u_k(s) = 0$ such that the geodesic $x(u)$ with $\dot{x}^k(0) = \xi^k$ is completely contained in N, connects x and $z(s)$, and $x(1) = x$. Then a (Fermi-) coordinate system which covers N can be constructed as follows: Let $e^k_\alpha(s)$, $\alpha = 1, 2, 3$, be three linearly independent smooth vector fields along l, $u_k e^k = 0$. Let $x^k(0) = \zeta^\alpha e^k_\alpha(s)$. Then x has coordinates (s, ξ^α). Let $\Sigma(s_0) \subseteq \mathrm{N}$ denote the hypersurface $s = s_0$. $\Sigma(s_0)$ is star-shaped with respect to $z(s_0)$. We have a smooth « time function » t on N:

$$t(x) = s \Leftrightarrow x \in \Sigma(s)\,.$$

Furthermore we assume:

iii) For any $s_1 \leqslant s_2$: $\left(\bigcup_{s_1 \leqslant s \leqslant s_2} \Sigma(s) \right) \cap \mathrm{W}$ is compact.

iv) There exists an open subset $\mathrm{V} \subseteq \pi^{-1}(l)$ with the following properties: $\exp_{z(s)}$ is well defined on $\mathrm{V}_s := \mathrm{V} \cap \mathrm{M}_{z(s)}$, $\mathrm{U}_s := \exp_{z(s)}(\mathrm{V}_s)$ is a normal neighbourhood of $z(s)$, star-shaped w. r. t. $z(s)$, $\Sigma(s) \subseteq \mathrm{U}_s \subseteq \mathrm{N}$. For $x \in \mathrm{U}_s$ $\sigma^k(z(s), x)$ is well-defined.

We have a diffeomorphism from V into an open subset U of $l \times \mathrm{M}$, defined (in natural coordinates) by $(s, \mathrm{X}) \mapsto (s, \exp_{z(s)} \mathrm{X})$.

We assume the existence of a continuous linear map $\mathrm{E}: \mathscr{E}(\mathrm{V}) \to \mathscr{E}(\pi^{-1}(l))$ and of an open neighbourhood $\mathrm{V}_1 \subseteq \mathrm{V}$ of $\bigcup_{s \in \mathbb{R}} \exp^{-1}_{z(s)} (\Sigma(s) \cap \mathrm{W})$ such that for all Φ $\mathrm{E}[\Phi] \mid \mathrm{V}_1 = \Phi \mid \mathrm{V}_1$. (Obviously this assumption is implied by some additional geometrical restrictions).

Without restriction of generality we shall assume $\mathrm{M} = \mathrm{N}$, since all relevant constructions depend on quantities defined on N.

Even if T^{ab} was introduced as a differentiable tensor field it will prove to be useful for the following to work with a more general class of mass tensors, with tensor distributions (cf. Appendix 1):

Suppose we have a 4-form η on M which vanishes nowhere. It is well known that then the map $\mathrm{T}^{ab} \mapsto \left(\varphi_{ab} \mapsto \int \mathrm{T}^{ab}(x) \varphi_{ab}(x) \eta(x) \right)$ provides an inclusion of the space of locally integrable $\binom{2}{0}$-tensor fields into the space $(\mathscr{D}^0_{[2]})'$ of tensor distributions. The matter tensor will from now on be considered as a tensor distribution in $(\mathscr{D}^0_{[2]})'$. As the support is contained in the closed set W, we may extend the range of definition of T^{ab} to the larger test field space $\mathscr{F}^0_{[2]}$ (cf. (A1.9)). The law of motion (3.1) is now understood to hold in the sense of distributions, i. e.

$$\langle \mathrm{T}^{ab}, \nabla_{(a} \lambda_{b)} \rangle = 0 \qquad \text{for all} \quad \lambda_a \in \mathscr{D}^0_1 (\text{resp. } \lambda_a \in \mathscr{F}^0_1)\,. \qquad (3.4)$$

Finally we impose a further restriction on the mass distribution:
We want that it makes sense to speak about a mass distribution at a given

time. Clearly, for a general distribution in $(\mathscr{F}^0_{[2]})'$ there is no well behaved « restriction to a hypersurface $\Sigma(s)$ »; hence momentum, angular momentum, etc., defined along the lines of Dixon's approach, would become tensor distributions along l. In order to avoid this unattractive possibility (and a lot of other complications) we will focus our attention to distributions in $(\mathscr{F}^0_{[2]})'$ which can be restricted w. r. t. $(\Sigma(s))_{s\in\mathbb{R}}$ in the sense of [*1*]:

3.2. DEFINITION. — $L \in (\mathscr{F}^0_{[2]})'$ is restrictable (with respect to $(\Sigma(s))_{s\in\mathbb{R}}$) iff for any $\varphi \in \mathscr{F}^0_{[2]}$ there exists a smooth function $l \to \mathbb{R}$, $s \mapsto \int_{\Sigma(s)} L^{ab}\varphi_{ab}$, such that the following conditions are satisfied:

(R1) For any $f \in \mathscr{E}(\mathbb{R})$,

$$\langle L^{ab}, \varphi_{ab}\cdot(f\circ t)\rangle = \int ds f(s)\int_{\Sigma(s)} L^{ab}\varphi_{ab}. \tag{3.5}$$

(Note that the l. h. s. is well defined!).

(R2) If $\varphi_n \overset{\mathscr{F}}{\to} 0$ then $\int_{\Sigma(s)} L\varphi_n \overset{\mathscr{E}}{\to} 0$

$\left(\text{i. e. for any } m \in \mathbb{N}_0, \left(\left(\frac{d}{ds}\right)^m \int_{\Sigma(s)} L\varphi_n\right) \text{goes to zero, uniformly on any compact}\right.$ subset of real line).

3.3. REMARK. — Clearly, for $L \in (\mathscr{F}^0_{[2]})'$, $\varphi \in \mathscr{F}^0_{[2]}$, $\int_{\Sigma(s)} T\varphi$ (if it exists) is uniquely determined by (R1).

3.4. EXAMPLES. — *i*) The « monopole » distribution

$$\varphi \mapsto M\int \dot{z}^a(s)\dot{z}^b(s)\varphi_{ab}(z(s))dx$$

is restrictable.

ii) If L is defined by the smooth tensor field T (see above), then L is restrictable.

We quote a few results on restrictable distributions (the easy proofs can be found in [*16*]):

3.5. PROPOSITION. — Let $L \in (\mathscr{F}^0_{[2]})$ be restrictable.

i) For any $m \in \mathbb{N}_0$, $s \in \mathbb{R}$, $\varphi \mapsto \left(\frac{d}{ds}\right)^m \int_{\Sigma(s)} L^{ab}\varphi_{ab}$ is a distribution with compact support contained in $\Sigma(s) \cap \operatorname{supp}(L)$ (Hence it can be extended to a distribution on $\mathscr{E}^0_{[2]}$).

ii)
$$\langle L^{ab}, \varphi_{ab}\rangle = \int ds \int_{\Sigma(s)} L^{ab}\varphi_{ab} \quad \text{for all} \quad \varphi_{ab} \in \mathscr{F}^0_{[2]}. \tag{3.6}$$

iii) For any $\varphi \in \mathscr{F}^0_{[2]}$, $s \mapsto \int_{\Sigma(s)} \mathrm{L}^{ab}\varphi_{ab}$ has compact support, hence

$$\int ds \frac{d}{ds} \int_{\Sigma(s)} \mathrm{L}^{ab}\varphi_{ab} = 0 . \tag{3.7}$$

iv) If $\varphi = \psi \cdot (h \circ t) \quad (h \in \mathscr{E}(\mathbb{R}), \psi \in \mathscr{F}^0_{[2]}) \Rightarrow \int_{\Sigma(s)} \mathrm{L}\varphi = h(s) \int_{\Sigma(s)} \mathrm{L}\psi.$ (3.8)

v) If $\chi \in \mathscr{F}^0_1$, $\nabla_a \mathrm{L}^{ab} = 0$, then

$$\frac{d}{ds} \int_{\Sigma(s)} \mathrm{L}^{ab}\chi_{(a}\nabla_{b)}t = \int_{\Sigma(s)} \mathrm{L}^{ab}\nabla_{(a}\chi_{b)} \tag{3.9}$$

Let us summarize our requirements on the mass tensor T^{ab}:

(M1) $\mathrm{T}^{ab} \in (\mathscr{D}^0_{[2]})'$, supp $\mathrm{T} \subseteq \mathrm{W} \quad (\Rightarrow \mathrm{T}^{ab} \in (\mathscr{F}^0_{[2]})')$
(M2) $\forall \lambda_a \in \mathscr{F}^0_1 : \quad \langle \mathrm{T}^{ab}, \nabla_{(a}\lambda_{b)} \rangle = 0$
(M3) T^{ab} is restrictable w. r. t. $(\Sigma(s))_{s\in\mathbb{R}}$.

A tensor distribution which obeys (M1), (M2), (M3) is called « admissible ».

3.6. Remark. — Admissible mass tensors will be well suited for all questions arising from the law of motion (3.1), especially for the problem of finding reduced moments in the sense of Dixon and of establishing equivalence between various descriptions of the body. On the other hand, in general, tensors of this type will not be useful as sources in field equations for the gravitational field.

4. THE SPLITTING OF THE TEST FIELD SPACE

In this section we show that for each symmetric φ_{ab} there exists—in a neighbourhood of a point z—a unique symmetric β_{ab} such that

$$\beta_{ab}(x)\sigma^b(z, x) = 0 \quad \text{and} \quad \varphi_{ab} = \beta_{ab} + \nabla_{(a}\lambda_{b)} \qquad \text{for some 1-form } \lambda_a .$$

Furthermore we give conditions that make λ_a unique and find explicit expressions for β_{ab} and λ_a in terms of φ_{ab}.

Finally we introduce a related splitting which does not refer to a point $z \in \mathrm{M}$ but refers to the pair (l, u_k) defined in the previous section. (These splittings have been introduced by Dixon [*6*], [*8*]).

4.1. Proposition. — Let $z \in \mathrm{M}$, $\beta_{ab} \in \mathscr{E}^0_{[2]}$. In a normal neignbourhood of z the following two statements are equivalent:

i) $$\sigma^a(z, x)\beta_{ab}(x) = 0 \tag{4.1}$$

ii) $$\sigma^b(z, x)\sigma^c(z, x)\nabla_{\{b}\beta_{ac\}}(x) = 0 \tag{4.2}$$

and $$\beta_{ab}(z) = 0\,. \tag{4.3}$$

Proof. — *i*) ⇒ *ii*): Differentiation of (4.1) yields

$$\sigma^a{}_c\beta_{ab} + \sigma^a\nabla_c\beta_{ab} = 0\,. \tag{4.4}$$

Transvection with σ^b and σ^c respectively yields—using $\sigma^c\sigma^a{}_c = \sigma^a$ (4.2). Passing to the coincidence limit in (4.4) we obtain (4.3).

ii) ⇒ *i*): Suppose (4.2) and (4.3) hold. Let $x(u)$ be an affinely parametrized geodesic with $x(0) = z$. We multiply (4.2) with σ^a. This gives

$$\sigma^a\sigma^b\sigma^c\nabla_{\{b}\beta_{ac\}} = 0\,. \tag{4.5}$$

Using $$\sigma^a(z, x(u)) = u\dot{x}^a(u) \tag{4.6}$$

we get $$\mathrm{D}_u(\dot{x}^a(u)\dot{x}^b(u)\beta_{ab}(x(u))) = 0\,. \tag{4.7}$$

This may be integrated inferring the initial condition from (4.3):

$$\dot{x}^a(u)\dot{x}^b(u)\beta_{ab}(x(u)) = 0 \tag{4.8}$$

or $$\sigma^a\sigma^b\beta_{ab} = 0 \tag{4.9}$$

(4.9) implies $$2\sigma^b{}_a\sigma^c\beta_{bc} + \sigma^b\sigma^c\nabla_a\beta_{bc} = 0\,. \tag{4.10}$$

We insert (4.10) into (4.2) and get

$$\sigma^b{}_a\sigma^c\beta_{bc} = -\,\sigma^b\sigma^c\nabla_b\beta_{ac}\,; \tag{4.11}$$

hence along $x(u)$):

$$\mathrm{D}_u\left(\frac{1}{u}\beta_{ab}(x(u))\sigma^b(z, x(u))\right) + \frac{1}{u^2}\sigma^b{}_a(z, x(u))\sigma^c(z, x(u))\beta_{bc}(x(u)) = 0 \tag{4.12}$$

or

$$u\mathrm{D}_u(\beta_{ab}(x(u))\sigma^b(z, x(u))) + (\sigma^c{}_a(z, x(u)) - \delta^c{}_a)\beta_{cb}(x(u))\sigma^b(z, x(u)) = 0\,. \tag{4.13}$$

Using (4.3) we see that the singular initial value problem (4.13), (4.3) admits the unique regular solution

$$\beta_{ab}\sigma^b = 0\,. \tag{4.14}$$

4.2. Lemma. — Let $\lambda_a \in \mathscr{E}^0_1$ and $\varphi_{ab} \in \mathscr{E}^0_{[2]}$. Then in a normal neighbourhood of z

$$(\varphi_{ab} - \nabla_{(a}\lambda_{b)})\sigma^b = 0 \tag{4.15}$$

is equivalent with the inhomogeneous adjoint Jacobi equation [17]

$$\mathrm{D}^2_u\lambda_a(x(u)) + \mathrm{R}^d{}_{\dot{x}\dot{x}a}(x(u))\lambda_d(x(u)) = \dot{x}^b(u)\dot{x}^c(u)\nabla_{\{b}\varphi_{ac\}}(x(u)) \tag{4.16}$$

along all geodesics $x(u)$ emanating from $z = x(0)$ together with the initial condition

$$\nabla_{(k}\lambda_{l)}(z) - \varphi_{kl}(z) = 0\,. \tag{4.17}$$

Proof. — This is an easy consequence of prop. 4.1 and of the symmetries of the Riemann tensor.

Trivially the following statement holds:

4.3. LEMMA. — Let $\lambda_a \in \mathscr{E}_1^0$ and $\varphi_{ab} \in \mathscr{E}_{[2]}^0$ be such that $\nabla_{(k}\lambda_{l)}(z) = \varphi_{kl}(z)$ (4.18)

Then $$\lambda_k(z) = 0, \; \nabla_{[k}\lambda_{l]}(z) = 0 \tag{4.19}$$

is equivalent with
$$\left\{ \begin{array}{l} \lambda_k(z) = 0 \qquad (4.20) \\ \nabla_k \lambda_l(z) = \varphi_{kl}(z) \qquad (4.21) \end{array} \right\}$$

Now we are ready to prove existence and uniqueness for our splitting problem:

4.4. PROPOSITION. — Given $\varphi \in \mathscr{E}_{[2]}^0$ and $z \in \mathrm{M}$ there is (locally) a unique choice of $\lambda_a \in \mathscr{E}_1^0$ such that (4.15) and (4.19) hold: We have the splitting $\varphi_{ab} = \nabla_{(a}\lambda_{b)} + \beta_{ab}$

with $$\sigma^a \beta_{ab} = 0, \quad \lambda_k(z) = 0, \quad \nabla_{[k}\lambda_{l]}(z) = 0 . \tag{4.22}$$

Proof. — In view of Lemma (4.2), (4.3), (4.15) and (4.19) are equivalent with (4.16), (4.20), (4.21). But (4.16) can be integrated along all geodesics through z using the initial conditions (4.20), (4.21), i. e.

$$\lambda_k(x(0)) = 0, \qquad \mathrm{D}_u \lambda_k(x(0)) = \dot{x}^l(0)\varphi_{kl}(x(0)) . \tag{4.23}$$

The (unique) 1-form λ_a obtained in such a way satisfies (4.15) and (4.19). Using Appendix 2 we can give an explicit representation of λ_a:

4.5. PROPOSITION. — λ_a as defined in Prop. 4.4 is given (locally) by

$$\begin{aligned} \lambda_a(x) = & - h_a{}^k(z, x)\sigma^l(z, x)\varphi_{kl}(z) \\ & + \int_0^1 \frac{(1-u)}{u^2} h_a{}^q(y_x(u), x)\sigma^p(z, y_x(u))\sigma^r(z, y_x(u))\nabla_{\{p}\varphi_{qr\}}(y_x(u))du , \end{aligned} \tag{4.24}$$

where $y_x(u)$ is defined by

$$y_x(u) := \exp_z(-u\sigma^k(z, x)) . \tag{4.25}$$

In *flat* space we can sharpen our results

4.6. PROPOSITION. — Let (X^k) be standard coordinates on $\mathbb{R}^4$. Let ψ_{kl} be a symmetric C^2-tensor field on $\mathbb{R}^4$. Then the following three statements are equivalent:

i) $$\mathrm{X}^k \psi_{kl}(\mathrm{X}) = 0 \qquad \text{for all X} \tag{4.26}$$

ii) There exists a tensor field H_{klmn} with symmetry [2, 2] such that

$$\psi_{kl}(\mathrm{X}) = \mathrm{X}^m \mathrm{X}^n \mathrm{H}_{kmln}(\mathrm{X}) \qquad \text{for all X} . \tag{4.27}$$

iii) $$\mathrm{X}^m \mathrm{X}^n \partial_{\{m}\psi_{ln\}}(\mathrm{X}) = 0 \qquad \text{for all X} \qquad \text{and} \qquad \psi_{kl}(0) = 0 . \tag{4.28}$$

Proof. — *i)* ⇔ *iii)*: Prop. 4.1.
ii) ⇒ *i)*: Obvious from the symmetry properties of H.
i) ⇒ *ii)*: We differentiate (4.26). This yields

$$\psi_{km} + X^n \partial_m \psi_{kn} = 0 . \tag{4.29}$$

Antisymmetrization gives

$$X^n \partial_{[m} \psi_{k]n} = 0 . \tag{4.30}$$

Differentiating again and multiplying with X^m gives

$$X^m \partial_{[m} \psi_{k]l} = X^m X^n \partial_{l[k} \psi_{m]n} . \tag{4.31}$$

Now, let us put

$$f_{kl}(u) := u \psi_{kl}(uX) . \tag{4.32}$$

Then

$$f_{kl}(0) = 0, \qquad f_{kl}(1) = \psi_{kl}(X) , \tag{4.33}$$

$$f'_{kl}(u) = \psi_{kl}(uX) + uX^n(\partial_n \psi_{kl})(uX) . \tag{4.34}$$

Substitution of X by uX in (4.29) gives, when inserted in (4.34)

$$f'_{kl}(u) = 2uX^m \partial_{[m} \psi_{k]l}(uX) , \tag{4.35}$$

whence

$$f'_{kl}(0) = 0 . \tag{4.36}$$

Differentiating (4.35) we find

$$f''_{kl}(u) = 2X^m \partial_{[m} \psi_{k]l}(uX) + 2uX^m X^n \partial_{n[m} \psi_{k]l}(uX) . \tag{4.37}$$

Finally we substitute X → uX in (4.31), insert the result in (4.37) and find:

$$f''_{kl}(u) = 4uX^m X^n \partial_{[k[l} \psi_{m]n]}(uX) . \tag{4.38}$$

(Antisymmetrization is taken over (k, m), (l, n) separately). Now we put (4.33), (4.36), (4.38) into Taylor's formula

$$f_{kl}(1) = f_{kl}(0) + f'_{kl}(0) + \int_0^1 (1 - u) f''_{kl}(u) du \tag{4.39}$$

and find $\psi_{kl}(X) = f_{kl}(1) = X^m X^n H_{kmln}(X)$ with

$$H_{klmn}(X) := 4 \int_0^1 u(1 - u) \partial_{[k[m} \psi_{l]n]}(uX) du . \tag{4.40}$$

4.7. PROPOSITION. — Let ψ be a symmetric C^2-tensor field on $\mathbb{R}^4$, let

$$\psi_{kl} = B_{kl} + \partial_{(k} \Lambda_{l)} \tag{4.41}$$

be the uniquely determinated splitting such that

$$X^k B_{kl}(X) = 0 \qquad \text{for all X} , \tag{4.42}$$

$$\Lambda_k(0) = 0, \qquad \partial_{[k} \Lambda_{l]}(0) = 0 . \tag{4.43}$$

We have the explicit representations

$$\Lambda_k(X) = X^l \psi_{kl}(0) + X^m X^n \int_0^1 (1-u)\partial_{\{m}\psi_{kn\}}(uX)du \tag{4.44}$$

$$\partial_{(l}\Lambda_{k)}(X) = \psi_{kl}(0) + 2\int_0^1 (1-u)X^m \partial_m \psi_{kl}(uX)du$$
$$+ X^m X^n \int_0^1 u(1-u)(\partial_{km}\psi_{ln} - \partial_{kl}\psi_{mn} + \partial_{lm}\psi_{kn})(uX)du \tag{4.45}$$

$$B_{kl}(X) = 4X^m X^n \int_0^1 u(1-u)\partial_{[k[l}\psi_{m]n]}(uX)du. \tag{4.46}$$

Proof. — (4.44), (4.45) follow from (4.24). B_{kl} satisfies (4.42), hence it has the form (4.27). Explicit evaluation of (4.40) gives (4.46).

Clearly Prop. 4.7. provides a splitting of $(T^0_{[2]}, T)l$: We only have to replace ∂_k in the formulae above by ∇_{*k}.

4.8. PROPOSITION. — For any $\psi \in (T^0_{[2]}, T)l$ there exists a unique $\Lambda \in (T^0_1, T)l$ such that (in a natural coordinate system (x, X) on $\pi^{-1}(l)$) for all $s \in \mathbb{R}$:

$$\Lambda_k(s, 0) = 0 \tag{4.47}$$

$$\nabla_{*[k}\Lambda_{l]}(s, 0) = 0, \tag{4.48}$$

and, for all $X \in M_{z(s)}$:

$$X^k(\psi_{kl}(s, X) - \nabla_{*(k}\Lambda_{l)}(s, X)) = 0. \tag{4.49}$$

Prop. 4.8. allows to define two maps from $(T^0_{[2]}, T)l$ into itself:

$$\mathfrak{p} : \psi_{kl} \mapsto B_{kl} := \psi_{kl} - \nabla_{*(k}\Lambda_{l)} \tag{4.50}$$

$$\mathfrak{q} : \psi_{kl} \mapsto \nabla_{*(k}\Lambda_{l)}. \tag{4.51}$$

From the explicit expressions in Prop. 4.7. one can easily read off the following properties of the decomposition op's $\mathfrak{p}$, $\mathfrak{q}$:

4.9. PROPOSITION. — $\mathfrak{p}$ and $\mathfrak{q}$ are continuous linear maps from $(T^0_{[2]}, T)l$ into itself with the following properties:

$$\mathfrak{p} + \mathfrak{q} = \mathrm{id} \tag{4.52}$$

$$\mathfrak{p} \circ \mathfrak{p} = \mathfrak{p}, \qquad \mathfrak{q} \circ \mathfrak{q} = \mathfrak{q} \tag{4.53}$$

$$\mathfrak{p} \circ \mathfrak{q} = \mathfrak{q} \circ \mathfrak{p} = 0 \tag{4.54}$$

$$\mathfrak{p}[\psi] = \psi \Leftrightarrow X^k \psi_{kl}(s, X) = 0 \quad \text{for all} \quad s, X \in M_{z(s)} \tag{4.55}$$

$$\mathfrak{q}[\psi] = \psi \Leftrightarrow \exists \Omega_k \in (T^0_1, T)l : \psi_{kl} = \nabla_{*(k}\Omega_{l)} \tag{4.56}$$

Let $S_{z(s)} \subseteq M_{z(s)}$ be closed and starshaped w. r. t. $0 \in M_{z(s)}$. Furthermore, let $\psi \in (T^0_{[2]}, T)l$ be such that

$$\mathrm{supp}\,(\psi \mid M_{z(s)}) \cap S_{z(s)} = \emptyset \tag{4.57}$$

Then
$$\operatorname{supp}(\mathfrak{p}[\psi] \mid M_{z(s)}) \cap S_{z(s)} = \emptyset \tag{4.58}$$
$$\operatorname{supp}(\mathfrak{q}[\psi] \mid M_{z(s)}) \cap S_{z(s)} = \emptyset . \tag{4.59}$$

In the subsequent sections we want to apply splitting theorems within our framework for describing bodies. Clearly we search for a characterization of the state of the system at any time (together with « evaluation equations ») rather than an overall description of the system. Technically, while the latter procedure would involve a global splitting with respect to a fixed point z (which clearly causes difficulties), the former needs only local (in time) decompositions with respect to $z(s)$ for all $s \in \mathbb{R}$. Therefore we must consider what happens if we move the reference point z along l.

4.10. PROPOSITION. — For any $\varphi_{ab} \in \mathscr{E}^0_{[2]}$ the decomposition according to prop. 4.4. defines fields $\lambda_a \in \mathscr{E}^{0,0}_{0,1}$ and $\beta_{ab} \in \mathscr{E}^{0,0}_{0,[2]}$ in a neighbourhood of the diagonal set of M × M:

$$\varphi_{ab}(x) = \beta_{ab}(z, x) + \nabla_{(a}\lambda_{b)}(z, x) \tag{4.60}$$
$$\sigma^a(z, x)\beta_{ab}(z, x) = 0, \ \lambda_a(z, z) = 0, \ \nabla_{[a}\lambda_{b]}(z, z) = 0 . \tag{4.61}$$

Proof. — In the neighbourhood of any point $z \in M$ one gets a decomposition (4.60), (4.61). The differentiability of β_{ab}, λ_a with respect to z is a consequence of the smooth dependence of solutions of ordinary differential equations on the data and the coefficients.

4.11. LEMMA. — $\partial_s\lambda_a(z(s), x) := \dot{z}^k(s)(\nabla_k\lambda_a)(z(s), x)$ (4.62)

obeys the equation

$$D_u^2(\partial_s\lambda_a) + R^d{}_{\dot{x}\dot{x}a}(\partial_s\lambda_d) = 2u^{-2}\sigma^b\sigma^c{}_k\nabla_{\{b}\partial_{|s|}\beta_{ac\}} \tag{4.63}$$

(locally) along any affinely parametrized geodesic $x(u)$ with $x(0) = z(s)$.

Proof. — Differentiation of

$$\varphi_{ab}(x) = \beta_{ab}(z(s), x) + \nabla_{(a}\lambda_{b)}(z(s), x) \tag{4.64}$$

with respect to s gives

$$\partial_s\beta_{ab}(z(s), x) + \nabla_{(a}\partial_{|s|}\lambda_{b)}(z(s), x) = 0 . \tag{4.65}$$

For fixed s, we trivially have

$$(-\partial_s\beta_{ab}(z(s), x) - \nabla_{(a}\partial_{|s|}\lambda_{b)}(z(s), x))\sigma^b(z(s), x) = 0 , \tag{4.66}$$

hence Lemma 4.2 implies

$$D_u^2\partial_s\lambda_a + R^d{}_{\dot{x}\dot{x}a}\partial_s\lambda_a = -\dot{x}^b\dot{x}^c\nabla_{\{b}\partial_{|s|}\beta_{ac\}} \tag{4.67}$$

along all geodesics $x(u)$ emanating from $x(0) = z(s)$.

According to Prop. 4.1. we have

$$\sigma^b(z(s), x)\sigma^c(z(s), x)\nabla_{\{b}\beta_{ac\}}(z(s), x) = 0 . \tag{4.68}$$

Differentiating with respect to s we get

$$\sigma^b\sigma^c\nabla_{\{b}\partial_{|s|}\beta_{ac\}} + 2\sigma^b\sigma^c{}_k\dot{z}^k\nabla_{\{b}\beta_{ac\}} = 0\,. \tag{4.69}$$

Using $\sigma^a(z(s), x(u)) = u\dot{x}^a(u)$ and substituting (4.69) into (4.67) we find (4.63).

4.12. LEMMA. — The initial conditions to determine $\partial_s\lambda_a$ from (4.63) are

$$\partial_s\lambda_m(z(s), x(0)) = -\,\dot{z}^n(s)\varphi_{nm}(z(s))\,, \tag{4.70}$$

$$\nabla_m\partial_s\lambda_n(z(s), x(0)) = -\,2\dot{z}^k(s)\nabla_{[m}\varphi_{n]k}(z(s))\,, \tag{4.71}$$

where the limit $x \to z$ is taken after the differentiation has been performed, i. e. $\nabla_m\partial_s\lambda_n$ is the coincidence limit of $\dot{z}^k\nabla_k\nabla_a\lambda_b$.

Proof. — We differentiate (4.20). This yields

$$0 = \partial_s\lambda_m(z(s), z(s)) = \delta^a_m \langle \partial_s\lambda_a \rangle + \delta^a_m\dot{z}^b \langle \nabla_b\lambda_a \rangle \tag{4.72}$$

along l. Using (4.21) we get (4.70).

Differentiation of (4.21) gives

$$\delta^a_k\delta^b_l \langle \nabla_a\partial_s\lambda_b \rangle + \dot{z}^a\delta^b_k\delta^c_l \langle \nabla_{ab}\lambda_c \rangle = \dot{z}^m\nabla_m\varphi_{kl}\,. \tag{4.73}$$

It is a consequence of (4.20) that, at $z(s)$,

$$\nabla_{[kl]}\lambda_m = 0 \tag{4.74}$$

or

$$\nabla_{kl}\lambda_m = \nabla_{(kl)}\lambda_m\,. \tag{4.75}$$

In virtue of this property we get in the limit $u \to 0$ from (4.16):

$$\nabla_{kl}\lambda_m = \nabla_{\{k}\varphi_{lm\}}\,. \tag{4.76}$$

We insert (4.76) into (4.73) and get

$$\nabla_m\partial_s\lambda_n + \dot{z}^l\nabla_{\{l}\varphi_{nm\}} = \dot{z}^l\nabla_l\varphi_{mn} \tag{4.77}$$

which is (4.71).

4.13. LEMMA. — Let ψ_a be the solution of (4.63) with zero initial data. Then

$$\psi_a(z(s), x) = 2\int_0^1 \frac{(1-u)}{u^2}\, h_a{}^q\sigma^p\sigma^r{}_k\dot{z}^k\nabla_{\{p}\beta_{qr\}}du \tag{4.78}$$

where the index a refers to the point x, k to $z(s)$, and p, q, r to

$$y_{z(s),x}(u) := \exp_{z(s)}(-\,u\sigma^k(z(s), x))\,. \tag{4.79}$$

The solution of (4.63) with initial data (4.70), (4.71) is then

$$\begin{aligned}\partial_s\lambda_a(z(s), x) = \psi_a(z(s), x) - k_a{}^k(z(s), x)\dot{z}^l(s)\varphi_{kl}(z(s))\\ - 2h_a{}^{[k}(z(s), x)\sigma^{l]}(z(s), x)\dot{z}^m(s)\nabla_k\varphi_{ml}(z(s))\end{aligned} \tag{4.80}$$

4.14. REMARK. — The singularity at $u = 0$ in the integral (4.78) is only apparent: This may be seen as follows: Let $\mathrm{B} := \exp^*_{z(s)}\beta$, i. e.

$$\mathrm{B}_{kl} = \mathrm{H}^a{}_k\mathrm{H}^b{}_l\beta_{ab}\,.$$

Then $\sigma^b\beta_{ab} = 0$ implies $X^kB_{kl} = 0$ in a starshaped neighbourhood of $0 \in M_{z(s)}$. Hence, according to Prop. 4.6., $B_{kl}=X^mX^nH_{kmln}$ and $\beta_{ab}=\sigma^d\sigma^e h_{adbc}$ with $h := \exp^{*-1} H$. Therefore all terms in $\sigma^p\nabla_{\{p}\beta_{qr\}}$ contain at least two times $\sigma^s(z(s), x(u))$ which is proportional to u.

4.15. DEFINITION. — For $\varphi_{ab} \in \mathscr{E}^0_{[2]}$, let β_{ab}, λ_a be defined as in Prop. 4.10. We define

$$(\mathfrak{a}[\varphi])_a(s, x) := \lambda_a(z(s), x) \tag{4.81}$$

$$(\mathfrak{b}[\varphi])_{ab}(s, x) := \nabla_{(a}\lambda_{b)}(z(s), x) \tag{4.82}$$

$$(\mathfrak{c}[\varphi])_{ab}(s, x) := \varphi_{ab}(x) - \nabla_{(a}\lambda_{b)}(z(s), x) = \beta_{ab}(z(s), x) \tag{4.83}$$

$$(\mathfrak{d}[\varphi])_a(s, x) := (\partial_s\lambda_a)(z(s), x)\,. \tag{4.84}$$

Clearly $\mathfrak{a}[\varphi], \ldots, \mathfrak{d}[\varphi]$ are defined on the open subset $U \subseteq l \times M$, and are elements of $\mathscr{E}^{0;0}_{0,s}U$, $s = 1$ or $[2]$.
(4.78) defines a bitensor

$$(\mathfrak{e}[\beta])_a(s, x) := \psi_a(z(s), x) \tag{4.85}$$

for any $\beta_{ab} \in \mathscr{E}^{0,0}_{0,[2]}(U)$ with satisfies

$$\sigma^a(z(s), x)\beta_{ab}(z(s), x) = 0 \qquad \text{for all} \qquad (z(s), x) \in U\,. \tag{4.86}$$

4.16. DEFINITION AND LEMMA. — (4.86) defines a closed linear subspace $\mathscr{H}$ of $\mathscr{E}^{0,0}_{0,[2]}(U)$.

4.17. PROPOSITION. — *i*) Let $\mathfrak{T} \in \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}\}$. The map $\mathfrak{T}: \mathscr{E}^0_{[2]}(M) \to \mathscr{E}^{0;0}_{0,s}(U)$, $s = 1$ or $[2]$, is linear and continuous. Furthermore

$$\mathfrak{c}[\varphi] \in \mathscr{H} \tag{4.87}$$

$$\mathfrak{b}[\varphi] = -\,\$_x\mathfrak{a}[\varphi] \tag{4.88}$$

$$\mathfrak{b} + \mathfrak{c} = \mathrm{id} \tag{4.89}$$

$$\mathfrak{d}[\varphi] = \partial_s\mathfrak{a}[\varphi] \tag{4.90}$$

ii) $\mathfrak{n}$ is a continuous linear map $\mathscr{H} \to \mathscr{E}^{0;0}_{0,1}(U)$.

iii)
$$\mathfrak{d}_a[\varphi] = (\mathfrak{e} \circ \mathfrak{c})_a[\varphi] - k_a{}^k\dot{z}^l\varphi_{kl} - 2h_a{}^{[k}\sigma^{l]}\dot{z}^m\nabla_k\varphi_{lm}\,. \tag{4.91}$$

4.18. DEFINITION. — Let

$$v : M \to \mathbb{R} \times M, \qquad x \mapsto (t(x), x)\,, \tag{4.92}$$

let $\quad \mathfrak{T} \in \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}\}$, $\varphi \in \mathscr{E}^0_{[2]}$. $\quad$ Then $\quad \underline{\mathfrak{T}}[\varphi] := \mathfrak{T}[\varphi] \circ v\,.$ (4.93)

(This is well-defined because $\Sigma(s) \subseteq U_s$, hence $(t(x), x) \in U$). Furthermore,

$$\text{for} \quad \beta \in \mathscr{H}, \qquad \underline{\mathfrak{e}}[\beta] := \mathfrak{e}[\beta] \circ v\,. \tag{4.94}$$

4.19. LEMMA. — *i*) $\underline{\mathfrak{T}}$ is a linear map $\mathscr{E}^0_{[2]} \to \mathscr{E}^0_s$, $s = 1$ or $[2]$.

ii) Let S be a closed set such that $S \cap \Sigma(s)$ is starshaped with respect to $z(s)$ for all $s \in \mathbb{R}$. Suppose $\varphi \in \mathscr{E}^0_{[2]}$ with $\operatorname{supp}(\varphi) \cap S = \emptyset$. Then $\operatorname{supp}(\mathfrak{T}[\varphi]) \cap S = \emptyset$.

Proof. — *i)* is obvious from the explicit representations (cf. (4.24), (4,80)).

ii) Let $\gamma(z, x)$ denote the geodesic segment joining z and x. Then it follows that $\mathfrak{T}[\varphi](x)=0$ whenever $\varphi=0$ in a neighbourhood of $\gamma(z(t(x)), x)$. Now let $x \in S$. Then, by hypothesis, $\gamma(z(t(x)), x) \subseteq S$ and

$$\operatorname{supp}(\varphi) \cap \gamma(z(t(x)), x) = \emptyset .$$

4.20. PROPOSITION. — Let $F \in \mathscr{J}$. There exists a $\hat{F} \in \mathscr{J}$ such that $\mathfrak{T}[\varphi] \in \mathscr{F}^0_s(\hat{F})$ for all $\varphi \in \mathscr{F}^0_{[2]}(F)$. In fact, $\mathfrak{T}$ is a continuous linear map $\mathscr{F}^0_{[2]}(F) \to \mathscr{F}^0_s(\hat{F})$ and, *a fortiori*, a continuous linear map $\mathscr{F}^0_{[2]} \to \mathscr{F}^0_s$ ($s = 1$ or $[2]$).

Proof. — From Lemma 4.19. it is obvious that, for given F, the following set $\hat{F}$ is a possible candidate:

$$\hat{F} := \{ y \in M \mid y = \exp_{z(t(x))} [- v\sigma^k(z(t(x)), x)],\ v \geq 1,\ x \in F \} \quad (4.95)$$

i. e. $\hat{F}$ consists of all geodesic rays emerging from points $x \in F$ and passing through $z(t(x))$ if continued beyond x.

$\hat{F}$ is closed, and we have to prove that $\hat{F} \cap W$ is compact:

Since $\Sigma(t(x)) \cap W$ is starshaped with respect to $z(t(x))$ the ray through $z(t(x))$ and x does not intersect W beyond x if $x \notin W$. Therefore

$$\hat{F} \cap W = \{ y \mid y = \exp_{z(t(x))} [- v\sigma^k(z(t(x)), x)],\ v \geq 1,\ x \in F \cap W \} \cap W . \quad (4.96)$$

$$F \cap W \text{ is compact, hence } F \cap W \subseteq \bigcup_{s_1 \leq s \leq s_2} \Sigma(s) \cap W \text{ for some } s_1 < s_2 . \quad (4.97)$$

Therefore

$$\hat{F} \cap W \subseteq \Big\{ y \mid y = \exp_{z(t(x))} [- v\sigma^k(z(t(x)), x],\ v \geq 1,\ x \in \bigcup_{s_1 \leq s \leq s_2} \Sigma(s) \cap W \Big\} \cap W = \bigcup_{s_1 \leq s \leq s_2} \Sigma(s) \cap W . \quad (4.98)$$

Hence $\hat{F} \cap W$, a closed subset of the compact set $\bigcup_{s_1 \leq s \leq s_2} \Sigma(s) \cap W$, is itself compact. $\mathfrak{T}$ maps $\mathscr{F}^0_{[2]}(F)$ into $\mathscr{F}^0_s(\hat{F})$ linearly. Continuity can be seen as follows:

Let $D \in \mathscr{J}^{\perp}$, $n \in \mathbb{N}$. We define

$$\tilde{D} := \bigcup_{x \in D \cap F} \gamma(z(t(x)), x) , \quad (4.99)$$

i. e. $\tilde{D}$ consists of all geodesic segments with one endpoint in $D \cap F$ and the other one on 1 which are contained in some $\Sigma(s)$. $D \cap F$ is compact which implies that $\tilde{D}$ is also compact. Now it is an immediate consequence of the explicit representations of $\underline{\mathfrak{T}}$ that for all $\varphi \in \mathscr{F}^0_{[2]}(F)$:

$$P_{D,n}(\underline{\mathfrak{T}}[\varphi]) \leqslant C \cdot P_{\tilde{D},n+2}(\varphi) \tag{4.100}$$

where C is a positive constant independent of φ.

4.21. PROPOSITION. — *i*) $\underline{\mathfrak{n}}$ is a continuous linear map $\mathscr{H} \to \mathscr{E}^0_1$ such that for all $\beta \in \mathscr{H}$ satisfying $\beta \circ \nu \in \mathscr{F}^0_{[2]}(F)$, $\underline{\mathfrak{e}}[\beta] \in \mathscr{F}^0_1(\hat{F})$ where $\hat{F}$ is defined as in (4.95).

ii) If S is closed and such that $S \cap \Sigma(s)$ is starshaped with respect to $z(s)$ for all s, then $S \cap \operatorname{supp}(\beta \circ \nu) = \emptyset$ implies $\operatorname{supp}(\underline{\mathfrak{e}}[\beta]) \cap S = \emptyset$.

iii) Let $g : l \times M \to \mathbb{R}$ be constant on $\{ z(s_1) \} \times \Sigma(s_2)$ for all $s_1 s_2 \in \mathbb{R}$, i. e. there exists a $\tilde{g} : l \times \mathbb{R} \to \mathbb{R}$ such that $g(z(s), x) = \tilde{g}(z(s), t(x))$. For any $\beta \in \mathscr{H}$ also $\psi := g \cdot \beta \in \mathscr{H}$, and one has:

$$\underline{\mathfrak{e}}[\psi](x) = \tilde{g}(z(t(x)), t(x)) \cdot \underline{\mathfrak{e}}[\beta](x) \qquad \text{for all} \quad x \in M \tag{4.101}$$

iv) Let $\beta \in \mathscr{H}$, $\psi(z(s), x) := (-u_k(s)\sigma^k(z(s), x)) \cdot \beta(z(s), x)$. Then

$$\underline{\mathfrak{e}}[\psi] = 0 . \tag{4.102}$$

Proof. — *i*), *ii*) can be shown as above.

iii) Let $x \in M$, $z := z(t(x))$, $y := \exp_z(-u\sigma^k(z, x))$. We will show

$$\sigma^p(z, y)\nabla_{\{p}\psi_{qr\}}(z, y) = \tilde{g}(z, t(y))\sigma^p(z, y)\nabla_{\{p}\beta_{qr\}}(z, y),$$

then (4.78) immediately gives (4.101) because $\tilde{g}(z, t(y)) = \tilde{g}(z, t(x))$. The Leibnitz rule yields

$$\sigma^p\nabla_{\{p}\psi_{qr\}} = \tilde{g}(z, t(y))\sigma^p\nabla_{\{p}\beta_{qr\}} + (\sigma^p\nabla_p g) \cdot \beta_{qr} - (\nabla_q g)(\sigma^p\beta_{pr}) + (\nabla_r g)(\sigma^p\beta_{pq}) . \tag{4.103}$$

The second term on the r. h. s. vanishes by assumption, the third and fourth term are zero because $\beta \in \mathscr{H}$.

iv) can be shown similarly.

4.22. LEMMA. — $\omega_a \in \mathscr{E}^0_1(M)$. Define

$$\xi_a[\omega](s, x) := k_a{}^k(z(s), x)\omega_k(z(s)) + h_a{}^k(z(s), x)\sigma^l(z(s), x)\nabla_{[k}\omega_{l]}(z(s)) . \tag{4.104}$$

Then
$$\mathfrak{a}[-\$\omega] = \omega - \xi[\omega], \tag{4.105}$$

hence (cf. Prop. 4.17)
$$\mathfrak{b}[\$\omega] = \$(\omega - \xi[\omega]) \tag{4.106}$$

$$\mathfrak{c}[\$\omega] = \$\xi[\omega] . \tag{4.107}$$

Proof. — For fixed s, $\mathfrak{a}[-\$\omega]$ is the solution of the equation

$$D^2_u\lambda_a + R^d{}_{\dot{x}\dot{x}a}\lambda_d = \dot{x}^b\dot{x}^c[\nabla_{(bc)}\omega_a + \nabla_{[ba]}\omega_c + \nabla_{[ca]}\omega_b] \tag{4.108}$$

along the geodesic $x(u)$ with $x(0) = z(s)$, $x(l) = x$, with the initial data

$$\lambda(0) = 0, \qquad \nabla\lambda(0) = -\$\omega(z(s)). \tag{4.109}$$

Using the definition of the Riemann tensor (4.108) is transformed into

$$D_u^2(\lambda_a - \omega_a) + R^d{}_{\dot{x}\dot{x}a}(\lambda_d - \omega_d) = 0, \tag{4.110}$$

hence
$$\lambda_a - \omega_a = k_a{}^k(\lambda_k - \omega_k) - \sigma^l h_a{}^k(\nabla_l\lambda_k - \nabla_l\omega_k) \tag{4.111}$$

which is (4.105).

5. DEFINITION AND BASIC PROPERTIES OF THE SKELETON. DIXON'S INTEGRAL LAWS OF MOTION

In this section we define momentum, angular momentum and skeleton of an admissible distribution T and investigate some of their properties.

5.1. DEFINITION. — Let T be an admissible distribution. Then we define

i) the *momentum* of T (w. r. t. (l, u_k))

$$P^k(s) := \int_{\Sigma(s)} T^{ab}\nabla_a t(\cdot)k_b{}^k((z\circ t)(\cdot), \cdot), \tag{5.1}$$

ii) the *angular momentum* of T (w. r. t. (l, u_k))

$$S^{kl}(s) := \int_{\Sigma(s)} T^{ab}\nabla_a t(\cdot)h_b^{[k}((z\circ t)(\cdot), \cdot)\sigma^{l]}((z\circ t)(\cdot), \cdot) \tag{5.2}$$

5.2. LEMMA. — P^k, S^{kl} are smooth vector (bivector) fields along l.

Proof. — T is restrictable.

Let $\beta \in \mathscr{H}$, then $(\beta\circ v)_{ab}$, $\nabla_a t\cdot(\underset{\sim}{e}[\beta])_b$ are smooth tensor fields. Since T is restrictable and $\varphi \mapsto \int_{\Sigma(s)} T\varphi$ is a distribution with compact support, the map

$$s \mapsto t_s[\beta] := \int_{\Sigma(s)} T^{ab}(\beta\circ v)_{ab} - \int_{\Sigma(s)} T^{ab}\nabla_a t\cdot(\underset{\sim}{e}[\beta])_b \tag{5.3}$$

is well-defined and smooth. In fact, t_s has the following properties:

5.3. LEMMA. — *i*) For any $s \in \mathbb{R}$,

$$t_s \in \mathscr{H}' \quad \text{and} \quad \operatorname{supp}(t_s) \subseteq \{ z(s) \}\, x(W(s)) \tag{5.4}$$

ii) For any $s \in \mathbb{R}$, there exists a distribution $\bar{t}_s$ on $\{ z(s) \} \times \Sigma(s)$ with $\operatorname{supp}(\bar{t}_s) \subseteq \{ z(s) \} \times (W \cap \Sigma(s))$ such that for all $\beta \in \mathcal{H}$

$$t_s[\beta] = \bar{t}_s[\beta|_{\{z(s)\} \times \Sigma(s)}] \tag{5.5}$$

iii) If $\beta_n \overset{\mathcal{H}}{\to} 0$, then $\left(\frac{d}{ds}\right)^m t_s[\beta_n] \overset{\mathcal{E}}{\to} 0$ for all $m \in \mathbb{N}_0$.

Proof. — *i*) Trivially $\beta \mapsto t_s[\beta]$ is linear. Furthermore $\beta \mapsto \beta \circ v$, $\beta \mapsto dt \otimes \underline{e}[\beta]$ are continuous (cf. Prop. 4.21). Since $\varphi \mapsto \int_{\Sigma(s)} T\varphi$ is a distribution (Prop. 3.5), $\beta \mapsto t_s[\beta]$ is in $\mathcal{H}'$.

Prop. 3.5. implies the vanishing of t_s for all $\beta \in \mathcal{H}$ such that $\beta \circ v = 0$ and $\underline{e}[\beta] = 0$ in a neighbourhood of $\Sigma(s) \cap W$. But such β's can be constructed as follows: Choose $\beta \in \mathcal{H}$ such that $\beta = 0$ in a neighbourhood of $\{ z(s) \} \times (W \cap \Sigma(s))$. The set $S := W \cap \Sigma(s)$ is closed and starshaped w. r. t. $z(s)$, $\beta \circ v$ vanishes in a neighbourhood of S, hence $\operatorname{supp}(\underline{e}[\beta]) \cap S = \emptyset$. (Prop. 4.21.).

ii) Fermi-coordinates (t, x^α), $\alpha = 1, 2, 3$, on M according to section 3 define a coordinate system $(s; t, x^\alpha)$ on $l \times M$. For any s, $\beta \mapsto t_s[\beta]$ is a distribution with compact support on the 3-surface $(s; s, x^\alpha)$ which has the following property:

If $g : l \times M \to \mathbb{R}$ is smooth,

$$g(s; t, x^\alpha) = \tilde{g}(s, t) \qquad \text{for all } s, t, x^\alpha, \tag{5.6}$$

then

$$t_s[g \cdot \beta] = \tilde{g}(s, s) t_s[\beta]. \tag{5.7}$$

(This is a consequence of the admissibility of T of Prop. 4.21 (*iii*). But this implies (*ii*).

iii) Let $\beta_n \overset{\mathcal{H}}{\to} 0$, then $\beta_n \circ v \overset{\mathcal{E}}{\to} 0$ and $dt \otimes \underline{e}[\beta_n] \overset{\mathcal{E}}{\to} 0$. (Prop. 4.21). Hence (R2) implies (*iii*).

Let (s, X^m) be a natural coordinate system on $\pi^{-1}(l)$, and let $(T^0_{[2]}, T)l$ denote the space of smooth symmmetric $\binom{0}{2}$-tensor fields over π along l (together with the $\mathcal{E}$-topology with respect to the derivatives ∇_{*k}, $\partial_s := \dot{z}^l \nabla_{l*}$). If $\psi_{kl} \in (T^0_{[2]}, T)l$, then $\mathfrak{p}[\psi]$ satisfies

$$X^k(\mathfrak{p}[\psi])_{kl}(s, X^m) = 0 \qquad \text{for all} \qquad X \in M_{z(s)}. \tag{5.8}$$

Hence $\mathfrak{p}[\psi]$ defines a field $\beta \in \mathcal{H}$ by

$$\beta_{ab}(z(s), x) := \sigma^k{}_a(z(s), x) \sigma^l{}_b(z(s), x) (\mathfrak{p}[\psi])_{kl}(s, \exp^{-1}_{z(s)}(x)) \tag{5.9}$$

to which we can apply t_s.

5.4. Definition. — Let, for given $\psi \in (T^0_{[2]}, T)l$, β be as in (5.9). For any $s \in \mathbb{R}$, we define the *skeleton* $\hat{T}_{z(s)}$ of T (w. r. t. (l, u_k)) by

$$\langle \hat{T}_{z(s)}, \psi \rangle := t_s[\beta]. \tag{5.10}$$

5.5. PROPOSITION. — *i)* For any $s \in \mathbb{R}$, there exists a distribution $\overline{\hat{T}}_{z(s)}$ on $M_{z(s)}$ such that for all $\psi \in (T^0_{[2]}, T)l$

$$\langle \hat{T}_{z(s)}, \psi \rangle = \langle \overline{\hat{T}}_{z(s)}, \psi |_{M_{z(s)}} \rangle . \tag{5.11}$$

In the following, we will use the symbol $\hat{T}_{z(s)}$ for both distributions. $\hat{T}_{z(s)}$ is a distribution of compact support

$$\operatorname{supp}(\hat{T}_{z(s)}) \subseteq \exp^{-1}_{z(s)}(W \cap \Sigma(s)) . \tag{5.12}$$

ii) For any ψ, $s \mapsto \langle \hat{T}_{z(s)}, \psi \rangle$ is a smooth function.
iii) Let $\psi_n \overset{\mathscr{E}}{\to} 0$. Then

$$\left(\frac{d}{ds}\right)^m \langle \hat{T}_{z(s)}, \psi_n \rangle \overset{\mathscr{E}}{\to} 0 \qquad \text{for any} \quad m \in \mathbb{N}_0 . \tag{5.13}$$

Proof. — The statements follow from the properties of t_s (Lemma 5.3) and of $\mathfrak{p}$ (Prop. 4.9.). (5.11) can be seen as follows:
From (5.5) we have

$$\langle \hat{T}_{z(s)}, \psi \rangle = t_s[\beta] = \bar{t}_s[\beta |_{\{z(s)\} \times \Sigma(s)}] . \tag{5.14}$$

But $\beta |_{\{z(s)\} \times \Sigma(s)}$ is completely (and continuously) determined by $\psi \mid M_{z(s)}$.

5.6. PROPOSITION. — $\nabla_* \cdot \hat{T}_{z(s)} = 0 \qquad \text{for all } s .$ (5.15)

Proof. — Let $\Omega \in (T^0_1, T)l$. Then $\langle \nabla_* \cdot \hat{T}_{z(s)}, \Omega \rangle = \langle \hat{T}_{z(s)}, \$_*\Omega \rangle$. But from Prop. 4.9. $\mathfrak{q}[\$_*\Omega] = \$_*\Omega$ and therefore $\mathfrak{p}[\$_*\Omega] = 0$, whence

$$\langle \hat{T}_{z(s)}, \$_*\Omega \rangle = t_s[\exp^{*-1} \mathfrak{p}[\$_*\Omega]] = 0 .$$

5.7. PROPOSITION. —

$$(u_m X^m) X^{[i} \hat{T}^{j][k}_{z(s)} X^{l]} = 0 \qquad \text{for all } s,\ X \in M_{z(s)} . \tag{5.16}$$

Proof. — We have to show $\langle (u_m X^m) X^{[i} \hat{T}^{j][k}_{z(s)} X^{l]}, H_{ijkl} \rangle = 0$ (5.17)

for all H with symmetry [2, 2]. But

$$\langle (u_m X^m) X^{[i} \hat{T}^{j][k}_{z(s)} X^{l]}, H_{ijkl} \rangle = \langle \hat{T}_{z(s)}, \Psi \rangle \tag{5.18}$$

where

$$\Psi_{kl} := -(u_n X^n) X^i X^j H_{kilj} . \tag{5.19}$$

Obviously $\Psi_{kl} X^l = 0$, whence (Prop. 4.9)

$$\mathfrak{p}[\Psi] = \Psi . \tag{5.20}$$

We define

$$\psi := \exp^{*-1}_{z(s)} \Psi, \text{ i. e.} \tag{5.21}$$

$$\psi_{ab} := (u_n \sigma^n) \sigma^i \sigma^j \sigma^k{}_a \sigma^l{}_b H_{kilj} \circ \exp^{-1} := (-u_n \sigma^n) \beta_{ab} . \tag{5.22}$$

But $\beta \in \mathscr{H}$ as a consequence of the symmetries of H.
For all $x \in M$:

$$(\psi \circ v)_{ab}(x) = [-u_n(z(t(x))) \cdot \sigma^n(z(t(x)), x)] \cdot (\beta \circ v)_{ab}(x) = 0 \tag{5.23}$$

and, from Prop. 4.21. (*iv*):

$$\underline{\mathfrak{e}}[\psi](x) = 0 . \tag{5.24}$$

Hence (cf. (5.21))

$$\langle \hat{T}_{z(s)}, \Psi \rangle = \int_{\Sigma(s)} T\psi \circ v - \int_{\Sigma(s)} Tdt \otimes \underline{\mathfrak{e}}[\psi] = 0 . \tag{5.25}$$

5.8. LEMMA. — Let $(\lambda_s)_{s\in\mathbb{R}}$ be a smooth 1-parameter family of smooth 1-form fields. Furthermore let $\lambda \circ v$, $(\partial_s\lambda) \circ v \in \mathscr{F}^0_1$, $(\$_x\lambda) \circ v \in \mathscr{F}^0_{[2]}$. Then

$$\langle T, (\$\lambda) \circ v \rangle = \int ds \int_{\Sigma(s)} Tdt \otimes (\partial_s\lambda) \circ v . \tag{5.26}$$

Proof. — From Prop. 3.5.(*v*) we find

$$\frac{d}{ds}\int_{\Sigma(s)} Tdt \otimes (\lambda \circ v) = \int_{\Sigma(s)} Tdt \otimes (\partial_s\lambda) \circ v - \int_{\Sigma(s)} T(\$_x\lambda) \circ v . \tag{5.27}$$

Prop. 3.5. (*ii*), (*iii*) imply (5.26).

5.9. LEMMA. — Let $\varphi \in \mathscr{F}^0_{[2]}$. Then

$$\langle T, \varphi \rangle = \int ds \int_{\Sigma(s)} T\underline{\mathfrak{c}}[\varphi] - \int ds \int_{\Sigma(s)} Tdt \otimes \underline{\mathfrak{d}}[\varphi] . \tag{5.28}$$

Proof. — From (4.89)

$$\varphi = \underline{\mathfrak{b}}[\varphi] + \underline{\mathfrak{c}}[\varphi] . \tag{5.29}$$

We infer from Prop. 4.5., Lemma 4.13 and Prop. 4.21 that

$$\lambda_s(x) := \lambda(z(s), x) := \mathfrak{a}[\varphi](s, x) \tag{5.30}$$

satisfies the hypotheses of the proceeding Lemma. Hence

$$\langle T, \varphi \rangle = \int ds \int_{\Sigma(s)} T\varphi = \int ds \int_{\Sigma(s)} T(\underline{\mathfrak{b}}[\varphi] + \underline{\mathfrak{c}}[\varphi])$$

$$= \int ds \int_{\Sigma(s)} T\underline{\mathfrak{c}}[\varphi] - \int ds \int_{\Sigma(s)} Tdt \otimes (\partial_s\mathfrak{a}[\varphi] \circ v) \qquad \text{which is } (5.28)$$

5.10. LEMMA. — Let $\varphi \in \mathscr{F}^0_{[2]}$. Then

$$\langle T, \varphi \rangle = \int ds \{ P^k(s)\dot{z}^l(s)\varphi_{kl}(z(s)) + S^{kl}(s)\dot{z}^m(s)\nabla_k\varphi_{lm}(z(s)) + t_s[\mathfrak{c}[\varphi]] \} . \tag{5.31}$$

Proof. — We insert (4.91) into (5.28) and find (using the restrictability of T and Def. 5.1)

$$\langle T, \varphi \rangle = \int ds \Big\{ P^k\dot{z}^l\varphi_{kl} + S^{kl}\dot{z}^m\nabla_k\varphi_{lm} + \int_{\Sigma(s)} T\underline{\mathfrak{c}}[\varphi] - \int_{\Sigma(s)} Tdt \otimes \underline{\mathfrak{e}}[\mathfrak{c}[\varphi]] \Big\} . \tag{5.32}$$

Use of (5.3) yields (5.31).

5.11. LEMMA. — Let $\varphi \in \mathscr{F}^0_{[2]}$. Then

$$\exp^* \mathfrak{c}[\varphi] = \mathfrak{p}[\exp^* \varphi + G \cdot \exp^* \mathfrak{a}[\varphi]], \tag{5.33}$$

where, for $\Lambda \in (T^0_1, T)M$, $G \cdot \Lambda$ is defined by $G^m_{kl}\Lambda_m$.

Proof. — Using the properties $H^a{}_k$, $\sigma^l{}_b$ (cf. [17]) and the definition of G (Appendix 2), one finds

$$H^c{}_n \nabla_c H^a{}_m = H^a{}_k G^k_{mn}. \tag{5.34}$$

For a given λ_a define

$$\Lambda_k := (\exp^* \lambda)_k := H^a{}_k \lambda_a. \tag{5.35}$$

From (5.34) one deduces

$$\$_* \Lambda = - G \cdot \Lambda + \exp^* \$_x \lambda. \tag{5.36}$$

Now we put

$$\lambda_a := (\mathfrak{a}[\varphi])_a. \tag{5.37}$$

Addosubtraction of $\exp^* \varphi$ on the r. h. s. of (5.36) yields, using Prop. 4.17.,

$$\exp^* \mathfrak{c}[\varphi] = \exp^* \varphi + G \cdot \exp^* \mathfrak{a}[\varphi] + \$_* \exp^* \mathfrak{a}[\varphi]. \tag{5.38}$$

But $\sigma^b(\mathfrak{c}[\varphi])_{ab} = 0$ implies

$$X \cdot \exp^* \mathfrak{c}[\varphi](z, X) = 0, \qquad \forall X \in M_z. \tag{5.39}$$

Hence, from (4.55),

$$\mathfrak{p}[\exp^* \mathfrak{c}[\varphi]] = \exp^* \mathfrak{c}[\varphi]. \tag{5.40}$$

Furthermore, from (4.56), (4.54) we find that

$$\mathfrak{p}[\$_* \exp^* \mathfrak{a}[\varphi]] = 0. \tag{5.41}$$

Application of $\mathfrak{p}$ to (5.38) and use of (5.40), (5.41) yield (5.33). Combining the results of the previous Lemmata 5.10 and 5.11 we now deduce:

5.12. — Let $\varphi \in \mathscr{F}^0_{[2]}$. Then

$$\langle T, \varphi \rangle = \int ds[P^k(s)\dot{z}^l(s)\varphi_{kl}(z(s)) + S^{kl}(s)\dot{z}^m(s)\nabla_k\varphi_{lm}(z(s)) + \langle \hat{T}_{z(s)}, \exp^*_{z(s)}\varphi + G \cdot \exp^*_{z(s)}\mathfrak{a}[\varphi] \rangle]. \tag{5.42}$$

Prop. 5.12. tells us that $\langle T, \varphi \rangle$ can be calculated from the knowledge of φ, the momentum P^k, the angular momentum S^{kl}, and the skeleton $\hat{T}_{z(s)}$.

As a consequence of a local law of motion $\nabla \cdot T = 0$, P and S satisfy evolution equations along l, Dixon's integral laws of motion. We use the following strategy to derive these equations (cf. Dixon [6], [8]):

In (5.42) we let φ be of the special form $\varphi_{ab} := - \nabla_{(a}\omega_{b)}$, $\omega_a \in \mathscr{F}^0_1$. Let

$$A_k(s) := \omega_k(z(s)), \qquad B_{kl}(s) := \nabla_{[k}\omega_{l]}(z(s)). \tag{5.43}$$

As in Lemma 4.22 we define a two-point-tensor field $\xi_a := \xi_a[\omega] \in T^{0,0}_{0,1}(l \times M)$ by

$$\xi_a(s, x) := k_a{}^k(z(s), x)A_k(s) + h_a{}^k(z(s), x)\sigma^l(z(s)), x)B_{kl}(s). \tag{5.44}$$

We have

$$2(\exp^*(-\$_x\xi))_{mn} = 2H^a{}_m H^b{}_n \nabla_{(a}\xi_{b)} = 2H^a{}_m H^b{}_n [A_k \nabla_{(a} k^k_{b)} + B_{kl}\nabla_{(a}(h^k_{b)}\sigma^l)] \quad (5.45)$$

whence (using Prop. 5.5. (*i*))

$$\langle \hat{T}_{z(s)}, 2\exp^*(-\$_x\xi)\rangle = 2A_k(s)F^k(s) + B_{kl}L^{kl}(s), \quad (5.46)$$

where we have introduced the abbreviations F^k, L^{kl}.

5.13. DEFINITION. — We define the gravitational *force* F and *torque* L relative to (l, u_k) acting on the body whose skeleton is $\hat{T}$, by

$$F^k(s) := \langle \hat{T}^{mn}_{z(s)}, H^a{}_m H^b{}_n \nabla_{(a} k^k_{b)} \rangle, \quad (5.47)$$

$$L^{kl}(s) := \langle \hat{T}^{mn}_{z(s)}, H^a{}_m H^b{}_n \nabla_{(a}(h^{[k}_{b)}\sigma^{l]}) \rangle. \quad (5.48)$$

Clearly F^k and L^{kl} depend smoothly on s.

5.14. THEOREM. — Let P, S, T, F, L be as defined above. Then as a consequence of the local law of motion $\nabla \cdot T = 0$, P, S have to satisfy Dixon's integral laws of motion:

$$F^k(s) = D_s P^k(s) - \frac{1}{2} S^{ml}(s)\dot{z}^n(s) R^k{}_{nlm}(z(s)), \quad (5.49)$$

$$L^{kl}(s) = D_s S^{kl}(s) - 2P^{[k}(s)\dot{z}^{l]}(s). \quad (5.50)$$

Proof. — Let ω, A, B, ξ be as defined above. The local law of motion and Prop. 5.12..imply

$$0 = \langle T, -(\$\omega)\rangle = \int ds (P^k \dot{z}^l \nabla_{(k}\omega_{l)}$$

$$+ S^{kl}\dot{z}^m \nabla_{k(l}\omega_{m)} + \langle \hat{T}_{z(s)}, \exp^*(-\$\omega) + G \cdot \exp^* \mathfrak{a}(-\$\omega)\rangle) \quad (5.51)$$

We try to evaluate all items in (5.51) in terms of A and B:
From (4.53) and the definition of $\hat{T}$ we know

$$\langle \hat{T}_{z(s)}, \mathfrak{p}[\Psi]\rangle = \langle \hat{T}_{z(s)}, \Psi\rangle \qquad \forall \psi \in (T^0_{[2]}, T)l. \quad (5.52)$$

Therefore, by 5.11,.4.22 and (5.44)

$$\begin{aligned}&\langle \hat{T}_{z(s)}, \exp^*(-\$\omega) + G\cdot\exp^*\mathfrak{a}(-\$\omega)\rangle \\ &\quad = \langle \hat{T}_{z(s)}, \mathfrak{p}(\exp^*(-\$\omega) + G\cdot\exp^*\mathfrak{a}(-\$\omega))\rangle \\ &\quad = \langle \hat{T}_{z(s)}, \exp^*\mathfrak{c}(-\$\omega)\rangle = \langle \hat{T}_{z(s)}, \exp^*(-\$\xi)\rangle = A_k F^k + \frac{1}{2} B_{kl} L^{kl}. \end{aligned} \quad (5.53)$$

We have $$\nabla_{(k}\omega_{l)} = \nabla_l \omega_k + B_{kl}, \quad (5.54)$$

hence $$P^k\dot{z}^l\nabla_{(k}\omega_{l)} = P^k\dot{z}^l\nabla_l\omega_k + P^k\dot{z}^l B_{kl} = P^k(D_s A_k) + P^{[k}\dot{z}^{l]}B_{kl}. \quad (5.55)$$

Furthermore $B_{ab} := \nabla_{[a}\omega_{b]}$ satisfies

$$\nabla_a B_{bc} + \nabla_b B_{ca} + \nabla_c B_{ab} = 0, \quad (5.56)$$

and therefore $$\nabla_{[a}B_{b]c} = -\frac{1}{2}\nabla_c B_{ab}. \quad (5.57)$$

Using (5.54) and (5.57) we find

$$S^{kl}\dot{z}^m\nabla_{k(l}\omega_{m)} = S^{kl}\dot{z}^m\nabla_{[kl]}\omega_m - S^{kl}\dot{z}^m\nabla_{[k}B_{l]m} = -\frac{1}{2}S^{kl}\dot{z}^m R^n{}_{mlk}A_n + S^{kl}D_sB_{kl}. \quad (5.58)$$

We insert (5.53), (5.55) and (5.58) into (5.51) and find

$$0 = \int ds\left\{A_k\left(-D_sP^k - \frac{1}{2}S^{nl}\dot{z}^m R^k{}_{mln} + F^k\right)\right.$$
$$\left. + B_{kl}\left(P^{[k}\dot{z}^{l]} - \frac{1}{2}D_sS^{kl} + \frac{1}{2}L^{kl}\right)\right\}. \quad (5.59)$$

(Note that A, B have compact support!). Since A, B are arbitrary, (5.59) implies (5.49) and (5.50).

6. REDUCED MOMENTS

In the preceeding sections (contained in Paper I) we have shown that an admissible mass tensor distribution T allows one to define a vector field $P^k(s)$ and a bivector field $S^{kl}(s)$ along l and furthermore a family $(\hat{T}_{z(s)})$ of distributions on the spaces $(T^0_{[2]}, T)_{z(s)}$ which have the following properties:

(M1) $P^k(s)$, $S^{kl}(s)$ depend smoothly on s.

(M2) $\hat{T}_{z(s)}$ has compact support in the hyperplane $H_{z(s)} \subseteq M_{z(s)}$ orthogonal to $u_k(s)$, in fact

$$\operatorname{supp}(\hat{T}_{z(s)}) \subseteq \exp^{-1}_{z(s)}(\Sigma(s) \cap W).$$

(M3) $\nabla_* \cdot \hat{T}_{z(s)} = 0$.

(M4) $(u_i(s)X^i)X^{[k}\hat{T}^{l]}_{z(s)}{}^{[m}X^{n]} = 0$ for all $X \in M_{z(s)}$.

(M5) If $\psi \in (T^0_{[2]}, T)l$, then $s \mapsto \langle \hat{T}_{z(s)}, \psi \mid M_{z(s)} \rangle$ is smooth.

(M6) If $\psi_n \in (T^0_{[2]}, T)l$, $\psi_n \xrightarrow{\mathscr{E}} 0$, then $\left(\frac{d}{ds}\right)^m \langle \hat{T}_{z(s)}, \psi_n \rangle \xrightarrow{\mathscr{E}} 0$ for any $m \in \mathbb{N}_0$.

(M7) Define a smooth vector field $F^k(s) := \langle \hat{T}^{mn}_{z(s)}, H^a{}_m H^b{}_n \nabla_{(a}k^k_{b)} \rangle$ and a smooth bivector field $L^{kl}(s) := \langle \hat{T}^{mn}_{z(s)}, H^a{}_m H^b{}_n \nabla_{(a}(h^{[k}_{b)}\sigma^{l]}) \rangle$ along l.

Then Dixon's integral laws of motion hold:

$$D_sP^k(s) = F^k(s) + \frac{1}{2}S^{ml}(s)\dot{z}^n(s)R^k{}_{nlm}(z(s)),$$
$$D_sS^{kl}(s) = L^{kl}(s) + 2P^{[k}(s)\dot{z}^{l]}(s).$$

In fact, T is completely determined by P^k, S^{kl}, $\hat{T}$: For $\varphi \in \mathscr{F}^0_{[2]}$,

$$\langle T, \varphi \rangle = \int ds\,[P^k\dot{z}^l\varphi_{kl} + S^{kl}\dot{z}^m\nabla_k\varphi_{lm}$$
$$+ \langle \hat{T}_{z(s)}, \exp^*\varphi + G\cdot\exp^*\alpha[\varphi] \rangle\,] \quad (6.1a)$$

or equivalently (using (5.38) and (M3))

$$\langle \mathrm{T}, \varphi \rangle = \int ds\,[\mathrm{P}^k \dot{z}^l \varphi_{kl} + \mathrm{S}^{kl}\dot{z}^m \nabla_k \varphi_{lm} + \langle \hat{\mathrm{T}}_{z(s)}, \exp^* \mathfrak{c}[\varphi] \rangle\,] \quad (6.1b)$$

Conversely, we now *assume* that we have a set (P, S, $\hat{\mathrm{T}}$) which obeys (M1-7). In the subsequent sections we will show that such a triple then determines an admissible T.

We start with the definition of reduced moments and an investigation of their properties. Apart from Proposition 6.16 everything in this section refers to a fixed tangent space $\mathrm{M}_{z(s)}$. We therefore omit the argument $z(s)$ whenever there is no risk of confusion.

6.1. DEFINITION. — The *extended skeleton* $\hat{\mathrm{T}}_{ex}$ is defined by

$$\hat{\mathrm{T}}^{kl}_{ex} := \mathrm{P}^{(k}\dot{z}^{l)}\delta - \mathrm{S}^{m(k}\dot{z}^{l)}\nabla_{*m}\delta + \hat{\mathrm{T}}^{kl}, \quad (6.2)$$

where δ denotes the distribution given by $\delta(f) = f(0)$ for all smooth functions f. Obviously $\hat{\mathrm{T}}_{ex}$ has the same support properties as $\hat{\mathrm{T}}$.

6.2. LEMMA. —

$$\langle \hat{\mathrm{T}}_{ex}, \exp^* \varphi + \mathrm{G}\cdot\exp^* \mathfrak{a}[\varphi] \rangle = \mathrm{P}^k\dot{z}^l\varphi_{kl} + \mathrm{S}^{kl}\dot{z}^m\nabla_k\varphi_{lm} + \langle \hat{\mathrm{T}}, \exp^* \varphi + \mathrm{G}\cdot\exp^* \mathfrak{a}[\varphi] \rangle. \quad (6.3)$$

(Hence, by (6.1*a*), T is completely determined by the extended skeleton).

Proof. — (6.3) follows immediately from G(0) = 0 (see Appendix 2, Paper I)

$$(\exp^*_{z(s)} \varphi)(0) = \varphi(z(s)),\ (\nabla_* \exp^*_{z(s)} \varphi)(0) = (\nabla\varphi)(z(s)),$$
$$(\exp^*_{z(s)} \mathfrak{a}[\varphi])(0) = (\mathfrak{a}[\varphi])(z(s), z(s)) = 0.$$

Let us choose a basis $(e^k{}_\alpha, \mathrm{V}^k)$, $\alpha = 1, 2, 3$, in the tangent space $\mathrm{M}_{z(s)}$ such that $u_k\mathrm{V}^k = 1$, $u_k e^k{}_\alpha = 0$. This defines a projection operator

$$\gamma^i{}_k := \delta^i{}_k - \mathrm{V}^i u_k. \quad (6.4)$$

Any vector $\mathrm{X}^k \in \mathrm{M}_{z(s)}$ has a representation $\mathrm{X}^k = \mathrm{X}^v\cdot\mathrm{V}^k + \hat{\mathrm{X}}^k$ where

$$\mathrm{X}^v = u_k\mathrm{X}^k,\ \hat{\mathrm{X}}^k = \gamma^k{}_i\mathrm{X}^i. \quad (6.5)$$

In the dual space we have the dual basis (e^α_k, u_k), hence for any K_i a representation $\mathrm{K}_i = \mathrm{K}_u\cdot u_i + \hat{\mathrm{K}}_i$ $\quad(\mathrm{K}_u = \mathrm{K}_i\mathrm{V}^i,\ \hat{\mathrm{K}}_i = \gamma^k{}_i\mathrm{K}_k)$.

Clearly such a choice provides measures DX on $\mathrm{M}_{z(s)}$ and DK on its dual space. Furthermore using (M2) we now can define distributions $f \mapsto \langle \hat{\mathrm{T}}^{kl}_{ex}, f \rangle$ for $k, l = 1, \ldots, 4$ on $\mathscr{E}^{\mathbb{C}}(\mathrm{M}_{z(s)})$, the space of $\mathbb{C}$-valued smooth functions on $\mathrm{M}_{z(s)}$.

6.3. DEFINITION. — The *moment generating function* $\hat{\mathrm{I}}^{kl}$ is defined to be the Fourier transform of $\hat{\mathrm{T}}^{kl}_{ex}$, i. e.

$$\hat{\mathrm{I}}^{kl}(\mathrm{K}) := \langle \hat{\mathrm{T}}^{kl}_{ex}, e^{-i\mathrm{K}.\mathrm{X}} \rangle \tag{6.6}$$

6.4. LEMMA. — $\hat{\mathrm{I}}$ is an entire analytic function which increases for real $\mathrm{K} \to \infty$ (i. e. $(\mathrm{K}_u^2 + \Sigma \hat{\mathrm{K}}_i^2)^{1/2} \to \infty$) not faster than a polynomial.

Proof. — (M2) and the theorem of Paley-Wiener.

6.5. DEFINITION. — The *reduced moments of the first kind* (« the I's") are defined by

$$\mathrm{I}^{m_1\ldots m_\nu kl} := i^\nu \partial_*^{m_1\ldots m_\nu} \hat{\mathrm{I}}^{kl}(0), \qquad \text{where} \qquad \partial_*^m := \frac{\partial}{\partial \mathrm{K}_m}. \tag{6.7}$$

6.6. LEMMA. —

$$\mathrm{I}^{kl} = \mathrm{P}^{(k}z^{l)} \tag{6.8}$$

$$\mathrm{I}^{mkl} = \mathrm{S}^{m(k}z^{l)} \tag{6.9}$$

$$\mathrm{I}^{m_1\ldots m_\nu kl} = \langle \hat{\mathrm{T}}^{kl}, \mathrm{X}^{m_1} \ldots \mathrm{X}^{m_\nu} \rangle, \qquad \nu \geq 2. \tag{6.10}$$

Proof. — One has $\mathrm{I}^{m_1\ldots m_\nu kl} = \langle \hat{\mathrm{T}}^{kl}_{ex}, \mathrm{X}^{m_1} \ldots \mathrm{X}^{m_\nu} \rangle$ for $\nu \geq 0$. Obviously $\langle \hat{\mathrm{T}}^{kl}_{ex}, \mathrm{X}^{m_1} \ldots \mathrm{X}^{m_\nu} \rangle = \langle \hat{\mathrm{T}}^{kl}, \mathrm{X}^{m_1} \ldots \mathrm{X}^{m_\nu} \rangle$ for $\nu \geqslant 2$. For $\nu = 0$ (M3) shows $\nabla_{*i}(\mathrm{X}^l \hat{\mathrm{T}}^{ik}) = \hat{\mathrm{T}}^{lk}$, therefore

$$\langle \hat{\mathrm{T}}^{kl}, 1 \rangle = \langle \nabla_i(\mathrm{X}^l \hat{\mathrm{T}}^{ik}), 1 \rangle = 0, \tag{6.11}$$

which implies (6.8). Similarly one proves (6.9).

6.7. LEMMA. —

$$\mathrm{I}^{m_1\ldots m_\nu kl} = \mathrm{I}^{(m_1\ldots m_\nu)(kl)} \qquad (\nu \geq 2) \tag{6.12}$$

$$\mathrm{I}^{(m_1\ldots m_\nu k)l} = 0. \tag{6.13}$$

Proof. — (6.12) can be seen directly from (6.10). (M3) implies

$$\begin{aligned}\mathrm{I}^{(m_1\ldots m_\nu k)l} &= \langle \hat{\mathrm{T}}^{(k|l|}, \mathrm{X}^{m_1} \ldots \mathrm{X}^{m_\nu)} \rangle = \langle \mathrm{X}^{(m_1} \ldots \mathrm{X}^{m_\nu} \hat{\mathrm{T}}^{k)l}, 1 \rangle \\ &= \frac{1}{\nu+1} \langle \nabla_{*n}(\mathrm{X}^{m_1} \ldots \mathrm{X}^{m_\nu} \mathrm{X}^k \hat{\mathrm{T}}^{nl}), 1 \rangle = 0.\end{aligned} \tag{6.14}$$

6.8. DEFINITION. — The *reduced moments of the second kind* (« the J's") are defined by

$$\mathrm{J}^{m_1\ldots m_\nu ijkl} := \mathrm{I}^{m_1\ldots m_\nu[i[kj]l]}, \qquad \nu \geq 0, \tag{6.15}$$

where antisymmetrization is taken separately over (i, j) and (k, l).

6.9. LEMMA. —

$$\mathrm{J}^{m_1\ldots m_\nu ijkl} = \mathrm{J}^{(m_1\ldots m_\nu)[ij][kl]} \qquad (\nu \geq 0) \tag{6.16}$$

$$\mathrm{J}^{m_1\ldots m_\nu i[jkl]} = 0 \qquad (\nu \geq 0) \tag{6.17}$$

$$\mathrm{J}^{m_1\ldots m_{\nu-1}[m_\nu ij]kl} = 0 \qquad (\nu \geq 1) \tag{6.18}$$

6.10. COROLLARY. — For $v \geqslant 2$, the $J's$ and $I's$ are equivalent:

$$I^{m_1 \dots m_v kl} = 4\frac{v-1}{v+1} J^{(m_1 \dots m_{v-1}|k|m_v)l} \tag{6.19}$$

6.11. LEMMA. —

$$u_{m^1} J^{m_1 \dots m_v ijkl} = 0 \qquad (v \geq 1) \tag{6.20}$$

Proof. —

$$\begin{aligned} u_{m_1} J^{m_1 \dots m_v ijkl} &= u_{m_1} \langle \hat{T}^{[j[l}, X^{|m_1} \dots X^{m_v|} X^{i]} X^{k]} \rangle \\ &= \langle (u_{m_1} X^{m_1}) X^{[i} \hat{T}^{j][l} X^{k]}, X^{m_2} \dots X^{m_v} \rangle = 0 \end{aligned} \tag{6.21}$$

from (M4).

Using the above results one can establish further symmetry properties of the moments:

6.12. LEMMA. —

$$J^{m_1 \dots m_v (i[jk]l)} = 0 \tag{6.22}$$

$$J^{m_1 \dots m_{v-1}[m_v i]ukl} = 0 \tag{6.23}$$

$$J^{um_2 \dots m_{v-1} u m_v u} = 0 \tag{6.24}$$

$$J^{m_2 \dots m_{v-1} uu m_v u} = 0 \tag{6.25}$$

$$J^{m_2 \dots m_v uuu} = 0 . \tag{6.26}$$

6.13. LEMMA. —

$$u_{m_1} u_l I^{m_1 \dots [m_v k]l} = 0 \qquad (v \geq 2) \tag{6.27}$$

$$u_{m_1} u_k u_l I^{m_1 \dots m_v kl} = 0 \qquad (v \geq 2) \tag{6.28}$$

$$u_{m_1} u_{m_2} u_k I^{m_1 \dots m_v kl} = 0 \qquad (v \geq 2) \tag{6.29}$$

$$u_{m_1} u_{m_2} u_{m_3} I^{m_1 \dots m_v kl} = 0 \qquad (v \geq 3) . \tag{6.30}$$

6.14. LEMMA. — The above symmetry and orthogonality properties of the (reduced) moments correspond to the following structure of the moment generating function:

$$\hat{I}^{kl}(K) = A^{kl}(\hat{K}) + K_u \partial_*^{(k} B^{l)}(\hat{K}) + K_u^2 \partial_*^{kl} C(\hat{K}) \tag{6.31}$$

(i. e. $\partial_*^u A^{kl} := u_m \partial_*^m A^{kl} = 0$, etc.).

Proof. — In virtue of Lemma 6.4 and Definition 6.5 $\hat{I}^{kl}$ has the following power series expansion around 0:

$$\hat{I}^{kl}(K) = \sum_{v=0}^{\infty} \frac{(-i)^v}{v!} K_{m_1} \dots K_{m_v} I^{m_1 \dots m_v kl} . \tag{6.32}$$

Inserting $K_j = \hat{K}_j + K_u \cdot u_j$ and using the symmetry and orthogonality properties of the $I's$ we find the decomposition

$$\hat{I}^{kl}(K) = A^{kl}_{(0)}(\hat{K}) + K_u A^{kl}_{(1)}(\hat{K}) + K_u^2 A^{kl}_{(2)}(\hat{K}) , \tag{6.33}$$

where
$$A^{kl}_{(0)}(\hat{K}) = \sum_{\nu=0}^{\infty} \frac{(-i)^{\nu}}{\nu!} \hat{K}_{m_1} \dots \hat{K}_{m_\nu} I^{m_1 \dots m_\nu kl} \tag{6.34}$$
$$A^{kl}_{(1)}(\hat{K}) = \sum_{\nu=0}^{\infty} \frac{(-i)^{\nu+1}}{\nu!} \hat{K}_{m_1} \dots \hat{K}_{m_\nu} I^{m_1 \dots m_\nu ukl}, \tag{6.35}$$
$$A^{kl}_{(2)}(\hat{K}) = \sum_{\nu=0}^{\infty} \frac{(-i)^{\nu+2}}{\nu!} \hat{K}_{m_1} \dots \hat{K}_{m_\nu} I^{m_1 \dots m_\nu uukl}. \tag{6.36}$$

From the definition of $\hat{I}^{kl}$ we deduce that
$$\partial_*^{imn} \hat{I}^{kl}(K) = i \langle X^i X^m X^n \hat{T}^{kl}_{ex}, e^{-iK.X} \rangle . \tag{6.37}$$
Applying (M4) we obtain
$$\partial_*^u \partial_*^{[k[l} \hat{I}^{m]n]} = 0, \tag{6.38}$$
i. e. $\partial_*^{[k[l} \hat{I}^{m]n]}$ does not depend on K_u.

Differentiation of (6.33) yields ($\hat{\partial}^i_* := \gamma^i{}_j \partial^j_*$)
$$\begin{aligned} \partial_*^{kl} \hat{I}^{mn} = {} & \hat{\partial}_*^{kl} A^{mn}_{(0)} + 2V^{(k} \hat{\partial}_*^{l)} A^{mn}_{(1)} + 2V^k V^l A^{mn}_{(2)} \\ & + K_u (\hat{\partial}_*^{kl} A^{mn}_{(1)} + 4V^{(l} \hat{\partial}_*^{k)} A^{mn}_{(2)}) + K_u^2 \hat{\partial}_*^{kl} A^{mn}_{(2)} . \end{aligned} \tag{6.39}$$
Applying (6.38) and using the independence of $A^{mn}_{(.)}$ of K_u we get
$$\hat{\partial}_*^{[k[l} A^{m]n]}_{(2)} = 0, \tag{6.40}$$
$$\hat{\partial}_*^{[k[l} A^{m]n]}_{(1)} + 2V^{[k} \hat{\partial}_*^{[l} A^{m]n]}_{(2)} + 2V^{[l} \hat{\partial}_*^{[k} A^{n]m]}_{(2)} = 0 . \tag{6.41}$$
In (6.40) $\hat{\partial}_*$ may be replaced by ∂_*. But now (6.40) and Prop. 4.7 imply
$$A^{kl}_{(2)} = \partial_*^{(k} c^{l)}, \tag{6.42}$$
where c^m is given by
$$c^m = a^m + K_k b^{km} + K_k A^{km}_{(2)}(0) + \int_0^1 (1-s) K_k K_l \partial_*^{\{k} A^{ml\}}_{(2)}(sK) ds ; \tag{6.43}$$
here a^m and $b^{km} = b^{[km]}$ are arbitrary constants. Furthermore $A^{km}_{(2)}(0) = -I^{uukm}$. We choose
$$a = 0, \qquad b = 0 . \tag{6.44}$$
From Lemma 6.13 we find
$$u_m c^m = 0 \tag{6.45}$$
and
$$\partial_*^u c^m = 0 . \tag{6.46}$$
Let
$$D^{kl} := \partial_*^{[k} c^{l]} . \tag{6.47}$$
By (6.45), (6.46) D^{kl} satisfies
$$u_l D^{lk} = 0 \tag{6.48}$$
and
$$\partial_*^u D^{kl} = 0 . \tag{6.49}$$

In virtue of (6.42) and (6.46) we find from (6.41)

$$\partial_*^{[k[l} A_{(1)}^{m]n]} + V^{[k} \partial_*^{m]} D^{ln} + V^{[l} \partial_*^{n]} D^{km} = 0 . \tag{6.50}$$

Since $A_{(1)}$ and D do not depend on K_u transvection of (6.50) with u_m gives:

$$\partial_*^k [\partial_*^{[l} A_{(1)}^{n]u} - D^{ln}] = 0 . \tag{6.51}$$

The first item in (6.51) vanishes as a consequence of (6.27) and (6.35). Hence D is constant, in fact

$$D^{kl}(K) = D^{kl}(0) = \partial_*^{[k} c^{l]}(0) = - I^{uu[kl]} = 0 . \tag{6.52}$$

Hence, by (6.47), there exists a scalar function c such that

$$c^k = \partial_*^k c . \tag{6.53}$$

From (6.46)

$$\partial_*^u c = u_l c^l = 0 , \qquad \text{hence } c \text{ does not depend on } K_u . \tag{6.54}$$

We plug (6.52) into (6.50). This yields

$$\partial_*^{[k[l} A_{(1)}^{m]n]} = 0 . \tag{6.55}$$

Again by Prop. 4.7

$$A_{(1)}^{mn} = \partial_*^{(m} B^{n)} \tag{6.56}$$

with $$B^m = e^m + K_k f^{km} - i K_k I^{ukm} + \int_0^1 (1-s) K_k K_l \partial_*^{\{k} A_{(1)}^{ml\}}(sK) ds , \tag{6.57}$$

where e^m and $f^{km} = f^{[km]}$ are constants. We get

$$\partial_*^u B^m = f^{um} - i I^{uum} . \tag{6.58}$$

If we choose the free constant f^{km} such that

$$f^{um} = i I^{uum} , \tag{6.59}$$

we find $\partial_*^u B^m = 0$.

6.15. COROLLARY. — $A^{kl}(\hat{K})$, $B^m(\hat{K})$, $C(\hat{K})$ and their derivatives are polynomially bounded as $\hat{K} \to \infty$.

Proof. — The functions $\partial_*^{m_1 \dots m_\nu} \hat{I}^{kl}(K) = \langle \hat{T}^{kl}, (-i)^\nu X^{m_1} \dots X^{m_\nu} e^{-iK.X} \rangle$ are polynomially bounded as $K \to \infty$. But in our coordinates ($K = 1, 2, 3$) we have

$$A_{(0)}^{kl}(\hat{K}) = \hat{I}^{kl}(\hat{K}, 0), \ \partial_*^k A_{(1)}^{mn}(\hat{K}) = \partial_*^{k0} \hat{I}^{mn}(\hat{K}, 0)$$

and

$$\partial_*^k A_{(2)}^{mn}(\hat{K}) = \frac{1}{2} (\partial_*^{k0} \hat{I}^{mn}(\hat{K}, 1) - \partial_*^{k0} \hat{I}^{mn}(\hat{K}, 0)) ,$$

and B^m, C are constructed from these quantities by integration (cf. (6.43), (6.53), (6.57)).

6.16. PROPOSITION. — The distributions

$$\overline{Q}^{(2)} := \frac{1}{2}(u_k X^k)^2 \hat{T}_{ex}, \tag{6.60}$$

$$\overline{Q}^{(1)} := -2V^k \nabla_{*k} \overline{Q}^{(2)} - (u_k X^k)\hat{T}_{ex}, \tag{6.61}$$

$$\overline{Q}^{(0)} := \hat{T}_{ex} - V^k \nabla_{*k} \overline{Q}^{(1)} - V^k V^l \nabla_{*kl} \overline{Q}^{(2)} \tag{6.62}$$

have the following properties:

i) The families $(\overline{Q}^{(q)}_{z(s)})_{s \in \mathbb{R}}$, $q = 0, 1, 2$ have the properties (M2), (M5), (M6).

ii) There exist distributions $Q^{(q)}_{z(s)}$ on $H_{z(s)}$ such that for all $\Psi \in (T^0_{[2]}, T)_{z(s)}$, $q = 0, 1, 2$

$$\langle \overline{Q}^{(q)}, \Psi \rangle = \langle Q^{(q)}, \Psi \circ \gamma \rangle \tag{6.63}$$

(i. e. the $\overline{Q}^{(q)}$'s are the extensions of the $Q^{(q)}$'s to $M_{z(s)}$).

iii)
$$\hat{T}_{ex} = \overline{Q}^{(0)} + V^k \nabla_{*k} \overline{Q}^{(1)} + V^k V^l \nabla_{*kl} \overline{Q}^{(2)}. \tag{6.64}$$

Proof. — *i)* is obvious from the definition and the properties of $\hat{T}_{ex}$. Let (without loss of generality) Ψ have compact support, let $\tilde{\Psi}$ denote its Fourier transform. Using the Parseval identity one finds

$$\langle \overline{Q}^{(2)}, \Psi \rangle = -\frac{1}{(2\pi)^4} \int \partial_*^{mn} C(\hat{K}) \tilde{\Psi}_{mn}(K) DK, \tag{6.65}$$

$$\langle \overline{Q}^{(1)}, \Psi \rangle = \frac{i}{(2\pi)^4} \int \partial_*{}^{(m} B^{n)}(\hat{K}) \tilde{\Psi}_{mn}(K) DK, \tag{6.66}$$

$$\langle \overline{Q}^{(0)}, \Psi \rangle = \frac{1}{(2\pi)^4} \int A^{mn}(\hat{K}) \tilde{\Psi}_{mn}(K) DK, \tag{6.67}$$

From Lemma 6.14, Corollary 6.15 and the standard formulae for Fourier transforms (shuffling derivatives) one deduces (*ii*) and (*iii*).

7. RECONSTRUCTION OF A DISTRIBUTION FROM A SKELETON

Now we are ready to prove a « reconstruction theorem » : We will show that a set (P, S, $\hat{T}$) which obeys (M1), . . ., (M7) determines an admissible mass distribution by (6.1).

7.1. THEOREM. — Let $l \equiv z(s)$, u_k, W be as in Sect. 3, P a smooth vector field along l, S a smooth bivector field along l and, for any s, $\hat{T}_{z(s)}$ a distribution on $(T^0_{[2]}, T)_{z(s)}$ such that (M1) – (M7) are satisfied. Then equation (6.1) defines an admissible mass tensor distribution T.

Remark. — P is the momentum, S the angular momentum and $\hat{T}_{z(s)}$ the skeleton of T in the sense of Def. 5.1 and 5.4. This can be shown using techniques similar to those applied by Dixon [*6*] in his « Uniqueness » proof.

Proof. — For $\varphi \in \mathscr{D}^0_{[2]}$ the map $\varphi \mapsto \langle T, \varphi \rangle$,

$$\langle T, \varphi \rangle := \int ds \,[P^k \dot{z}^l \varphi_{kl} + S^{kl} \dot{z}^m \nabla_k \varphi_{lm} + \langle \hat{T}_{z(s)}, \exp^* \varphi + G \cdot \exp^* \mathfrak{a}[\varphi] \rangle] \quad (7.1)$$

or equivalently

$$\varphi \mapsto \langle T, \varphi \rangle = \int ds \,[P^k \dot{z}^l \varphi_{kl} + S^{kl} \dot{z}^m \nabla_k \varphi_{lm} + \langle \hat{T}_{z(s)}, \exp^* \mathfrak{c}[\varphi] \rangle]$$

$$= \int ds \,[P^k \dot{z}^l \varphi_{kl} + S^{kl} \dot{z}^m \nabla_k \varphi_{lm} + \langle \hat{T}_{ex,z(s)}, \exp^* \mathfrak{c}[\varphi] \rangle] \quad (7.2)$$

(the last line follows from the definition of $\hat{T}_{ex}$ and Remark 4.14) is well defined and linear.

i) The support of the linear functional (7.2) is contained in W: Let $\varphi \in \mathscr{D}^0_{[2]}$ with supp $\varphi \cap W = \emptyset$. We have to show $\langle T, \varphi \rangle = 0$. Obviously $l \cap$ supp $\varphi = \emptyset$ whence the first two terms in (7.2) vanish. Furthermore, for any $s \in \mathbb{R}$, $\hat{T}_{z(s)} \circ \exp^*_{z(s)}$ (well defined on U_s!) is a distribution which by hypothesis has compact support contained in $\Sigma(s) \cap W$. From the explicit expressions for cf. (4.84), (4.24) and our geometrical assumptions in Sect. 3 one can read off $\mathfrak{c}[\varphi](s, \cdot) = 0$ in a neighbourhood of $\Sigma(s) \cap W$. Hence, for all s:

$$\langle \hat{T}_{z(s)}, \exp^*_{z(s)} \mathfrak{c}[\varphi] \rangle = 0 . \quad (7.3)$$

ii) The map $\varphi \mapsto \langle T, \varphi \rangle$ is continuous:
Let $\varphi_n \overset{\mathscr{D}}{\to} 0$. We have to show $\lim_{n \to \infty} \langle T, \varphi_n \rangle = 0$ (cf. Appendix 1). By assumption, all φ_n have their support in a fixed compact set K where $(\nabla^\alpha \varphi_n)$ converges uniformly to zero for any $\alpha \in \mathbb{N}_0$.

We have $\quad s_1 < t(K) < s_2 \quad$ for suitable s_1, s_2. $\quad$ (7.4)

Hence
$$\int ds \,[P^k \dot{z}^l (\varphi_n)_{kl} + S^{kl} \dot{z}^m \nabla_k (\varphi_n)_{lm}] \to 0 , \quad (7.5)$$

and, by arguments similar to those in (*i*), if $s \leq s_1$ or $s \geq s_2$

$$\exp^*_{z(s)} \mathfrak{c}[\varphi_n] = 0, \quad n \in \mathbb{N}_0 \quad \text{in} \quad V \subseteq \pi^{-1}(l) . \quad (7.6)$$

Therefore, the maps

$$s \mapsto \check{\varphi}_n(s) := \langle \hat{T}_{z(s)}, \exp^*_{z(s)} \mathfrak{c}[\varphi_n] \rangle \quad (7.7)$$

have their support contained in the compact set $[s_1, s_2] \subseteq \mathbb{R}$. Prop. 4.17 implies

$$\mathfrak{c}[\varphi_n] \overset{\mathscr{E}}{\to} 0 \quad \text{on } U \quad (7.8)$$

whence
$$\exp^* \mathfrak{c}[\varphi_n] \overset{\mathscr{E}}{\to} 0 \quad \text{on } V . \quad (7.9)$$

Using the support properties of $\hat{T}_{z(s)}$ (M2), our assumptions in Section 3 and (M5, 6) we see that all $\check{\varphi}_n$ are smooth, and that $(\check{\varphi}_n)$ tends to zero, uniformly on $[s_1, s_2]$.

Hence
$$\int ds \langle \hat{T}_{z(s)}, \exp^*_{z(s)} \mathfrak{c}[\varphi] \rangle = \int_{s_1}^{s_2} ds \check{\varphi}_n(s) \to 0 . \tag{7.10}$$

iii) $\nabla \cdot T = 0$: The proof consists in reading the proof of Theorem 5.14 the other way round: Let ω, A, B be defined as in Section 5. Then by (M7)

$$\langle T, -\$\omega \rangle = \int ds [P^k \dot{z}^l \nabla_{(k}\omega_{l)} + S^{kl}\dot{z}^m \nabla_{k(l}\omega_{m)} + \langle \hat{T}_{z(s)}, \exp^* \mathfrak{c}(-\$\omega) \rangle]$$
$$= \int ds \left[A_k\left(-D_s P^k - \frac{1}{2} S^{nl}\dot{z}^m R^k_{mln} + F^k \right) + B_{kl}\left(P^{[k}\dot{z}^{l]} - \frac{1}{2} D_s S^{kl} + \frac{1}{2} L^{kl} \right) \right] = 0 \tag{7.11}$$

iv) T is restrictable:
Let $\varphi \in \mathscr{F}^0_{[2]}$, $f \in \mathscr{E}(\mathbb{R})$. Then

$$\langle T, \varphi \cdot (f \circ t) \rangle = \int ds [P^k \dot{z}^l \varphi_{kl} f(s) + S^{kl}\dot{z}^m \nabla_k(\varphi_{lm} \cdot (f \circ t))(z(s))$$
$$+ \langle \hat{T}_{ex,z(s)}, \exp^*_{z(s)} \mathfrak{c}[\varphi \cdot (f \circ t)] \rangle] . \tag{7.12}$$

We observe

$$\nabla_k(\varphi_{lm} \cdot (f \circ t))(z(s)) = (\nabla_k \varphi_{lm})(z(s)) \cdot f(s) + \varphi_{lm}(z(s)) u_k(s) D f(s) . \tag{7.13}$$

where we have used

$$(\nabla_k t)(z(s)) = u_k(s) . \tag{7.14}$$

Next we consider

$$\mathfrak{c}[\varphi \cdot (f \circ t)] :$$

From the explicit representation (cf. (4.24), (4.83)) we have with

$$y := \exp_{z(s)} (- u\sigma^k(z(s), x)) :$$

$$(\mathfrak{c}[\varphi \cdot (f \circ t)])_{ab}(s, x) = \varphi_{ab}(x) \cdot f(t(x)) + \nabla_{(a}(h^k_{b)}\sigma^l)\varphi_{kl}(z(s)) \cdot f(s)$$
$$- \nabla_{(a} \int_0^1 \frac{(1-u)}{u^2} h^q_{b)} \sigma^p \sigma^r \{ {}_{\{p}\varphi_{qr\}} \cdot (f \circ t)(y) + \nabla_{\{p} t(y) \varphi_{qr\}} \cdot (Df \circ t)(y) \} du$$
$$= \varphi_{ab}(x) \cdot f(t(x)) + \nabla_{(a}(h^k_{b)}\sigma^l)\varphi_{kl}(z(s)) \cdot f(s)$$
$$- \int_0^1 \frac{(1-u)}{u^2} \{ \nabla_{(a}(h^q_{b)}\sigma^p\sigma^r \nabla_{\{p}\varphi_{qr\}}) \cdot (f \circ t)(y)$$
$$+ h^p_{(b}\nabla_{a)} t(y) \cdot \sigma^p \sigma^r \nabla_{\{p}\varphi_{qr\}} \cdot (Df \circ t)(y)$$
$$+ \nabla_{(a}(h^q_{b)}\sigma^p\sigma^r \nabla_{\{p} t(y) \varphi_{qr\}}) \cdot (Df \circ t)(y)$$
$$+ h^q_{(b}\nabla_{a)} t(y) \sigma^p \sigma^r \nabla_{\{p} t(y) \varphi_{qr\}} \cdot (D^2 f \circ t)(y) \} du . \tag{7.15}$$

Define

$$\mathfrak{n}_{ab}^{(0,0)}[\varphi](s, x) := \varphi_{ab}(x) + \nabla_{(a}(h_{b)}^{k}\sigma^{l})\varphi_{kl} - \int_0^1 \frac{(1-u)}{u^2} \nabla_{(a}(h_{b)}^{q}\sigma^{p}\sigma^{r}\nabla_{\{p}\varphi_{qr\}})du, \quad (7.16)$$

$$\mathfrak{n}_{ab}^{(0,1)}[\varphi](s, x) := - \int_0^1 \frac{(1-u)}{u^2} \{ h_{(b}^{q}\nabla_{a)}t(y)\sigma^{p}\sigma^{r}\nabla_{\{p}\varphi_{qr\}} + \nabla_{(a}(h_{b)}^{q}\sigma^{p}\sigma^{r}\nabla_{\{p}t(y)\varphi_{qr\}}) \} du, \quad (7.17)$$

$$\mathfrak{n}_{ab}^{(0,2)}[\varphi](s, x) := - \int_0^1 \frac{(1-u)}{u^2} h_{(b}^{q}\nabla_{a)}t(y)\sigma^{p}\sigma^{r}\nabla_{\{p}t(y) \cdot \varphi_{qr\}}du. \quad (7.18)$$

Obviously (cf. Prop. 4.17) $\mathfrak{n}^{(0,p)}[\cdot]$, $p = 0, 1, 2$, are continuous and linear maps $\mathscr{E}_{[2]}^{0}(\mathrm{M}) \rightarrow \mathscr{E}_{0,[2]}^{0,0}(\mathrm{U})$. For $x \in \Sigma(s)$ the integration in (7.15) occurs only in the hypersurface $\Sigma(s)$; therefore, if $x \in \Sigma(s)$, we have

$$(\mathfrak{c}[\varphi \cdot (f \circ t)])_{ab}(s, x) = \sum_{p=0}^{2} (\mathrm{D}^p f)(s)\mathfrak{n}_{ab}^{(0,p)}[\varphi](s, x). \quad (7.19)$$

Similar representations hold for the derivatives of $\mathfrak{c}[\varphi \cdot (f \circ t)]$ if $x \in \Sigma(s)$. This implies the existence of continuous and linear maps $\mathfrak{N}^{(q,p)}$: $\mathscr{E}_{[2]}^{0}(\mathrm{M}) \rightarrow (\mathrm{T}_{[2]}^{0}, \mathrm{T})l$, $p = 0, \ldots, q+2$, such that for $\mathrm{X} \in \mathrm{H}_{z(s)}$:

$$\mathrm{V}^{i_1} \ldots \mathrm{V}^{i_q}\nabla_{*i_1 \ldots i_q}(\exp_{z(s)}^{*} \mathfrak{c}[\varphi \cdot (f \circ t)])(s, \mathrm{X}) = \sum_{p=0}^{q+2} (\mathrm{D}^p f)(s) \cdot \mathfrak{N}^{(q,p)}[\varphi](s, \mathrm{X}). \quad (7.20)$$

Now, by Prop. 6.16.

$$\begin{aligned}
\langle \hat{\mathrm{T}}_{ex,z(s)}, \exp_{z(s)}^{*} \mathfrak{c}[\varphi \cdot (f \circ t)] \rangle
&= \sum_{q=0}^{2} (-1)^q \langle \overline{\mathrm{Q}}_{z(s)}^{(q)}, \mathrm{V}^{i_1} \ldots \mathrm{V}^{i_q}\nabla_{*i_1 \ldots i_q} \exp_{z(s)}^{*} \mathfrak{c}[\varphi \cdot (f \circ t)] \rangle \\
&= \sum_{q=0}^{2} (-1)^q \langle \mathrm{Q}_{z(s)}^{(q)}, (\mathrm{V}^{i_1} \ldots \mathrm{V}^{i_q}\nabla_{*i_1 \ldots i_q} \exp_{z(s)}^{*} \mathfrak{c}[\varphi \cdot (f \circ t)]) \circ \gamma \rangle \\
&= \sum_{q=0}^{2} \sum_{p=0}^{q+2} (-1)^q (\mathrm{D}^p f)(s) \langle \mathrm{Q}_{z(s)}^{(q)}, \mathfrak{N}^{(q,p)}[\varphi] \circ \gamma \rangle \\
&= \sum_{q=0}^{2} \sum_{p=0}^{q+2} (-1)^q (\mathrm{D}^p f)(s) \langle \overline{\mathrm{Q}}_{z(s)}^{(q)}, \mathfrak{N}^{(q,p)}[\varphi] \rangle \qquad (7.21)
\end{aligned}$$

The functions $s \mapsto \langle \overline{\mathrm{Q}}_{z(s)}^{(q)}, \mathfrak{N}^{(q,p)}[\varphi] \rangle$ are smooth and have compact support if $\varphi \in \mathscr{F}$ (Clearly $\mathfrak{N}^{(q,p)}[\varphi]|_{\mathrm{H}_{z(s)}} = 0$ if $\varphi = 0$ in U_s).

Hence, using (7.12), (7.13) and partial integration, we find

$$\langle \mathrm{T}, \varphi\cdot(f\circ t)\rangle = \int ds\, f(s)\Big\{ (\mathrm{P}^k \dot z^l - \mathrm{D}_s(u_m \mathrm{S}^{mk}\dot z^l))\varphi_{kl} + (\delta^k_n - \dot z^k u_n)\mathrm{S}^{nl}\dot z^m \nabla_k \varphi_{lm} + \sum_{q=0}^{2}\sum_{p=0}^{q+2}(-1)^{p+q}\left(\frac{d}{ds}\right)^p \langle \overline{\mathrm{Q}}^{(q)}_{z(s)}, \mathfrak{N}^{(q,p)}[\varphi]\rangle \Big\}. \quad (7.22)$$

The bracket $\{\ \ldots\ \}$ defines a C^∞-function $\int_{\Sigma(s)} \mathrm{T}\varphi$. Due to the properties of $\mathfrak{N}^{(q,p)}$ and of $\overline{\mathrm{Q}}^{(q)}_{z(s)}$, $\varphi_n \xrightarrow{\mathscr{F}} 0$ implies $\int_{\Sigma(s)} \mathrm{T}\varphi_n \xrightarrow{\mathscr{E}} 0$. Hence T is restrictable.

APPENDIX 1

Test field spaces and distributions

For a manifold M let $\mathscr{K}$ denote the family of compact subsets, $\mathscr{C}$ the family of closed subsets, and, for a given closed $W \subseteq M$, $\mathscr{J}$ the family

$$\mathscr{J} := \{ F \subseteq M \mid F \text{ is closed and } F \cap W \text{ is compact} \} \tag{A1.1}$$

Let $\mathfrak{T} \in \{ \mathscr{K}, \mathscr{C}, \mathscr{J} \}$. Define a family

$$\mathfrak{T}^{\perp} := \{ G \subseteq M \mid G \text{ is closed}, \forall T \in \mathfrak{T} : T \cap G \text{ is compact} \} ; \tag{A1.2}$$

clearly $\mathscr{C}^{\perp} = \mathscr{K}$, $\mathscr{K}^{\perp} = \mathscr{C}$.

For $T \in \mathfrak{T}$ $\mathfrak{T}^r_s(T)$ is the locally convex vectorspace of all C^{∞}-tensor fields of type $\binom{r}{s}$ with support in T together with the family of seminorms $(P_{D,m})$, $D \in \mathfrak{T}^{\perp}$, $m \in \mathbb{N}_0$, where, for $t \in \mathfrak{T}^r_s(T)$:

$$P_{D,m}(t) := \begin{cases} 0 & \text{if } D \cap T = \emptyset \\ \sup\limits_{x \in D, \alpha \leqslant m} |\nabla^{\alpha} t(x)|_e & \text{if } D \cap T \neq \emptyset \end{cases} \tag{A1.3}$$

(e is a Riemannian metric on M, $|t^{b_1 \dots b_r}_{a_1 \dots a_s}|_e := |e^{a_1 a'_1} \dots e^{a_s a'_s} e_{b_1 b'_1} \dots e_{b_r b'_r} t^{b_1 \dots b_r}_{a_1 \dots a_s} t^{b'_1 \dots b'_r}_{a'_1 \dots a'_s}|^{1/2}$). Obviously, $\mathscr{C}^r_s(\cdot)$, $\mathscr{K}^r_s(\cdot)$ are well-known spaces $\mathscr{E}^r_s(\cdot)$ resp. $\mathscr{D}^r_s(\cdot)$, instead of $\mathscr{J}^r_s(\cdot)$ we write $\mathscr{F}^r_s(\cdot)$. All these spaces are Fréchet-spaces [20].

Let $\mathfrak{T}_i \in \{ \mathscr{K}, \mathscr{C}, \mathscr{J} \}$, $T_i \in \mathfrak{T}_i$, $i = 1, 2$; a linear map $\Lambda : \mathfrak{T}_1(T_1) \to \mathfrak{T}_2(T_2)$ is continuous iff for $D_2 \in \mathfrak{T}_2^{\perp}$, $m_2 \in \mathbb{N}_0$ there exist a $D_1 \in \mathfrak{T}'_1$, a $m_1 \in \mathbb{N}_0$ and a $c > 0$ such that for all

$$t_1 \in \mathfrak{T}_1(T_1) : P_{D_2,m_2}(\Lambda(t_1)) \leqslant C \cdot P_{D_1,m_1}(t_1) . \tag{A1.4}$$

$\mathfrak{T}^r_s := \bigcup\limits_{T \in \mathfrak{T}} \mathfrak{T}^r_s(T)$ is a vector space; we endow $\mathfrak{T}^r_s$ with the inductive limit topology [2].

If V is a locally convex space, a linear map $\Lambda : \mathfrak{T}^r_s \to V$ is continuous iff its restriction to each $\mathfrak{T}^r_s(T)$ is continuous.

One has continuous inclusions $\mathscr{D}^r_s \hookrightarrow \mathscr{F}^r_s \hookrightarrow \mathscr{E}^r_s$.

A sequence (t_n) in $\mathfrak{T}^r_s$ is said to converge in the sense of $\mathfrak{T}$ to 0 ($t_n \overset{\mathfrak{T}}{\to} 0$) iff

i) there is a fixed $T \in \mathfrak{T}$ such that $t_n \in \mathfrak{T}^r_s(T)$ for all n;

ii) on each $D \in \mathfrak{T}^{\perp}$, $\nabla^{\alpha} t_n$ converges uniformly to zero for all $\alpha \in \mathbb{N}_0$.

$(\mathfrak{T}^r_s)'$ denotes the topological dual of $\mathfrak{T}^r_s$, i. e. the space of all (real) continuous linear functions distributions) on $\mathfrak{T}^r_s$.

For a linear functional on $\mathfrak{T}^r_s$ the following three statements are equivalent:

i) L is continuous (A1.5)

ii) $$t_n \overset{\mathfrak{T}}{\to} 0 \Rightarrow \lim_{n \to \infty} \langle L, t_n \rangle = 0 \tag{A1.6}$$

iii) $$\forall T \in \mathfrak{T}\ \exists D \in \mathfrak{T}^{\perp}\ \exists c > 0\ \exists N \in \mathbb{N}_0\ \forall t \in \mathfrak{T}^r_s(T) : |\langle L, t \rangle| \leqslant c \cdot p_{D,N}(t) \tag{A1.7}$$

One has inclusions $(\mathscr{E}^r_s)' \hookrightarrow (\mathscr{F}^r_s)' \hookrightarrow (\mathscr{D}^r_s)'$ (continuous w. r. t. the weak topologies).

It is well-known [20] that

$$(\mathscr{E}^r_s)' = \{ L \in (\mathscr{D}^r_s)' \mid L \text{ has compact support, i. e. } \operatorname{supp}(L) \in \mathscr{C}^{\perp} = \mathscr{K} \} \tag{A1.8}$$

in a similar way one can characterize the space $(\mathscr{F}^r_s)'$:

$$(\mathscr{F}^r_s)' = \{ L \in (\mathscr{D}^r_s)' \mid \operatorname{supp}(L) \in \mathscr{J}^{\perp} \} . \tag{A1.9}$$

This allows us to apply the whole well-known theory of distributions also to elements of $(\mathscr{F}^r_s)'$. On various spaces of two-point-tensor fields and tensor field over π we have topological structures of the $\mathscr{E}$-type. As those spaces play only an auxiliary role and their properties are quite straight-forward, we are not going to formalize them.

(We will use symbols like $\mathscr{E}^{p,q}_{r,s}$, etc. without further explanation).

APPENDIX 2

Some special bitensor fields

Let (M, ∇) be an affine manifold. In a neighbourhood of the diagonal set of M $\times$ M one can define the world vector fields σ^k, σ^a (and their derivatives), the Jacobi propagators $K^a{}_k$, $H^a{}_k$ and the Jacobi co-propagators $k_a{}^k$, $h_a{}^k$ (see [17] for details). One has

$$\sigma^k(z, x) = -\exp_z^{-1}(x), \tag{A2.1}$$

$H^a{}_k$ is the differential of $\exp_z$,

$$H^a{}_k \cdot \sigma^k{}_b = -\delta^a_b. \tag{A2.2}$$

In a normal neighbourhood of z, the inhomogeneous adjoint Jacobi equation (cf. [17])

$$D_{\dot{x}}^2 l_a + l_d R^d{}_{cba}\dot{x}^b\dot{x}^c = e_a \tag{A2.3}$$

has the solution

$$l_a(x) = k_a{}^k(z, x)l_k(z) - h_a{}^k(z, x)\sigma^l(z, x)\nabla_l l_k(z) + \int_0^1 (1-u)h_a{}^q(y_x(u), x)e_q(y_x(u))du, \tag{A2.4}$$

where the index q refers to the point

$$y_x(u) := \exp_z(-u\sigma^k(z, x)). \tag{A2.5}$$

Furthermore we introduce a bitensor G^m_{kl} (cf. [8]) by

$$G^m_{kl} := H^a{}_k H^b{}_l \sigma^m_{ab}. \tag{A2.6}$$

In the coincidence limit,

$$\langle G^m_{kl} \rangle = 0. \tag{A2.7}$$

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