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Feynman rules for Wightman functions

by

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ABSTRACT. — We consider the vacuum expectation values of partially time ordered products of field operators. For these we give graphical computation rules analogous to the Feynman rules. As a special case this includes the Wightman functions. It is shown that these functions satisfy the Wightman conditions as formulated in Wightman's reconstruction theorem, except positivity.

RÉSUMÉ. — On considère les valeurs moyennes dans le vide de produits partiellement ordonnés en temps d'opérateurs de champ. On donne pour celles-ci des règles de calcul graphiques analogues aux règles de Feynman. Les fonctions de Wightman sont incluses comme cas particulier. On montre que ces fonctions satisfont les conditions de Wightman telles qu'elles sont formulées dans le théorème de reconstruction de Wightman, à l'exception de la positivité.

1. INTRODUCTION

In studying the properties of local quantum field theories the Wightman functions are important tools, because the basic assumptions (« axioms ») of the theory are easily translated into properties of them [*1*] [*2*]. Since exact non-trivial models are hard to come by, we are largely dependent on approximation schemes, one of the more useful of which still is perturbation theory. It is therefore desirable to find explicit rules for the calculation of perturbative Wightman functions which are comparable in simplicity to the Feynman rules for the time-ordered functions. In the present paper such Feynman-like diagrammatic rules for Wightman functions will be derived for the example of the φ^4-theory. It is convenient to solve a somewhat more general problem, namely that of finding diagrammatic rules for the « sector τ-functions », defined by

$$\begin{aligned} \langle 0 \,|\, T\{ \varphi(x_1^{(1)}) \dots \varphi(x_{n_1}^{(1)}) \} T\{ \varphi(x_1^{(2)}) \dots \varphi(x_{n_2}^{(2)}) \} \dots \\ \dots T\{ \varphi(x_1^{(N)}) \dots \varphi(x_{n_N}^{(N)}) \} \,|\, 0 \rangle \quad (1.1) \\ =: \tau(x_1^{(1)}, \dots, x_{n_1}^{(1)} \,|\, x_1^{(2)}, \dots, x_{n_2}^{(2)} \,|\, \dots \,|\, x_1^{(N)}, \dots, x_{n_N}^{(N)}) \\ \text{boundary} \qquad \text{sector} \\ =: \tau(X_1 \,|\, \dots \,|\, X_N) \qquad X_i := \{ x_1^{(i)}, \dots, x_{n_i}^{(i)} \} \\ T : \text{time order operator} \end{aligned}$$

They include the usual τ-functions and the Wightman functions as special cases. Their existence in perturbation theory as distributions has been proved in [*3*], without giving an explicit representation of the type presented here.

Let

$$W(x_1, \dots, x_n) = \langle 0 \,|\, \varphi(x_1) \dots \varphi(x_n) \,|\, 0 \rangle$$

and W_σ be the term of order σ in the perturbative expansion of W and similarly for the sector τ-functions. Our construction is based on the following conditions:

a) W_σ satisfies the linear Wightman conditions [*2*]

b) In $\tau_\sigma(X_1 \,|\, \dots \,|\, X_N)$ the sector boundaries can be dropped between adjacent sectors, if their arguments are in the proper time ordering:

$$\tau_\sigma(X_1 \,|\, \dots \,|\, X_k \,|\, X_{k+1} \,|\, \dots \,|\, X_N) = \tau_\sigma(X_1 \,|\, \dots \,|\, X_k, X_{k+1} \,|\, \dots \,|\, X_N)$$

if $\quad x_j^{0,(k)} > x_{j'}^{0,(k+1)} \quad$ for all $\quad j = 1, \dots, n_k$, $\quad j' = 1, \dots, n_{k+1}$

c) For $\tau_\sigma(x_1, \dots, x_n)$ the usual Feynman rules are valid.

Alternatively we could replace condition *c*) by the requirement that the $\tau_\sigma(X_1 \,|\, \dots \,|\, X_N)$ satisfy appropriate equations of motion.

The plan of the paper is as follows. In sections 2 and 3 we derive unrenormalized diagrammatic rules satisfying conditions *b*) and *c*). In section 4 we show that the ultraviolett divergences can be removed by standard renormalization procedures. In section 5 we demonstrate that the proposed rules indeed satisfy the linear Wightman conditions. Positivity is, of course, not expected to hold in finite-order perturbation theory.

The rules given here can be easily generalized to other local field theories. However, in theories with massless particles we are then confronted with infrared divergent subgraphs. Examples in QED show that these divergences will probably cancel between different graphs, but we have no general proof of this assertion.

2. RECURSIVE DEFINITION OF SECTOR τ-FUNCTIONS

To simplify the discussion and to facilitate understanding let us first handle the unrenormalized case. The essential ideas will become evident in this part. Later on we shall treat renormalization with the BPHZ formalism. We make the following ansatz for the sector τ-functions in perturbation theory:

$$\begin{aligned}
\tau_\sigma(x, X_1 \mid X_2 \mid \dots \mid X_N) :=& \\
= - i \int du\, i\Delta_F(x-u)\tau_{\sigma-1}(u,u,u,X_1 \mid \dots \mid X_N) & \\
- \Delta_+(x-u) \sum_{i=2}^{N} \tau_{\sigma-1}(X_1 \mid \dots \mid X_{i-1} \mid u,u,u \mid X_i \mid \dots \mid X_N) & \\
- \tau_{\sigma-1}(X_1 \mid \dots \mid X_{i-1} \mid u,u,u,X_i \mid \dots \mid X_N) & \qquad (2.1)\\
+ \sum_{j=1}^{n_1} i\Delta_F(x - x_j^{(1)})\tau_\sigma(X_1 \backslash \{ x_j^{(1)} \} \mid \dots \mid X_N) & \\
+ \sum_{i=2}^{N}\sum_{j=1}^{n_i} \Delta_+(x - x_j^{(i)})\tau_\sigma(X_1 \mid \dots \mid X_i \backslash \{ x_j^{(i)} \} \mid \dots \mid X_N) &
\end{aligned}$$

We start the recursion with the free field, i. e.

$$\tau_0(X_1 \mid \dots \mid X_N) = \sum_{\substack{\text{all}\\ \text{pairgs.}}} \prod_{\substack{\text{all}\\ \text{pairs}}} \Delta_\#(x_{j'}^{(i')} - x_j^{(i)}) \qquad (2.2)$$

where $\Delta_\# = i\Delta_F$ for $i' = i$, $\Delta_\# = \Delta_+$ for $i' < i$.

In addition we set $\tau_\sigma(\emptyset) = \delta_{\sigma 0}$, $\tau_\sigma(x) = 0$.

(2.1) comprises the wellknown recurrence relation for τ-functions as a special case, since τ-functions are sector τ-functions without sector boundaries.

In the following we shall give a proof of the time ordering property mentioned before under *b*) by induction.

Zeroth order:

We know that $i\Delta_F(x-y)=\theta(x^0-y^0)\Delta_+(x-y)+\theta(y^0-x^0)\Delta_+(y-x)$. (2.3)
This implies

$$\Delta_+(x_{j'}^{(k)}-x_j^{(k+1)})=i\Delta_F(x_{j'}^{(k)}-x_j^{(k+1)}) \quad \text{if} \quad x_{j'}^{0,(k)}>x_j^{0,(k+1)}$$

Therefore boundaries between time ordered sectors can be discarded in zeroth order.

σ-th order

Let us call $I_k(u)$ the integrand of $\tau_\sigma(x, X_1 | \dots | X_k | X_{k+1} | \dots | X_N)$ on the right hand side of equation (2.1) and $I_k'(u)$ the integrand of

$$\tau_\sigma(x, X_1 | \dots | X_k, X_{k+1} | \dots | X_N).$$

We have to handle the cases $k=1$ and $k>1$ separately.

$\underline{k=1}$

The integrands $I_1(u)$ and $I_1'u)$ read according to (2.1)

$$\begin{aligned} I_1(u) = {} & i\Delta_F(x-u)\tau_{\sigma-1}(u, u, u, X_1 | \dots | X_N) \\ & -\Delta_+(x-u)\sum_{i=2}^{N}[\tau_{\sigma-1}(X_1 | \dots | X_{i-1} | u, u, u | X_i | \dots | X_N) \\ & \qquad - \tau_{\sigma-1}(X_1 | \dots | X_{i-1} | u, u, u, X_i | \dots | X_N)] \end{aligned}$$

$$\begin{aligned} I_1'(u) = {} & i\Delta_F(x-u)\tau_{\sigma-1}(u, u, u, X_1, X_2 | \dots | X_N) \\ & -\Delta_+(x-u)\sum_{i=3}^{N}[\tau_{\sigma-1}(X_1, X_2 | \dots | X_{i-1} | u, u, u | X_i | \dots | X_N) \\ & \qquad - \tau_{\sigma-1}(X_1, X_2 | \dots | X_{i-1} | u, u, u, X_i | \dots | X_N)]. \end{aligned}$$

From the induction assumption follows that

$$\tau_{\sigma-1}(X_1, X_2 | \dots | X_N) = \tau_{\sigma-1}(X_1 | X_2 | \dots | X_N)$$

under the condition that $x^{0,(1)} > x^{0,(2)}$. We obtain

$$\begin{aligned} & I_1(u)-I_1'(u)=i\Delta_F(x-u)\tau_{\sigma-1}(u, u, u, X_1 | X_2 | \dots | X_N) \\ & - \Delta_+(x-u)[\tau_{\sigma-1}(X_1 | u, u, u | X_2 | \dots | X_N)-\tau_{\sigma-1}(X_1 | u, u, u, X_2 | \dots | X_N)] \\ & - i\Delta_F(x-u)\tau_{\sigma-1}(u, u, u, X_1, X_2 | \dots | X_N). \end{aligned}$$

Since $x^0, x_j^{0,(1)} > x_j^{0,(2)}$ there exists a y^0 with $x^0, x_j^{0,(1)} > y^0 > x_j^{0,(2)}$.

We distinguish two cases:

$\underline{u^0 > y^0}$

We assume that the sector τ-functions are invariant under permutations of the arguments within a sector. This will be shown later on. With the induction assumption we obtain

$$\begin{aligned}
&I_1(u)-I'_1(u)=i\Delta_F(x-u)\tau_{\sigma-1}(u,u,u,X_1\,|\,X_2\,|\,\ldots\,|\,X_N)\\
&-\Delta_+(x-u)\,[\tau_{\sigma-1}(X_1\,|\,u,u,u,|\,X_2\,|\,\ldots\,|\,X_N)-\tau_{\sigma-1}(X_1\,|\,u,u,u\,|\,X_2\,|\,\ldots\,|\,X_N)]\\
&-i\Delta_F(x-u)\tau_{\sigma-1}(u,u,u,X_1\,|\,X_2\,|\,\ldots\,|\,X_N)\\
&=0\,.
\end{aligned}$$

$\underline{u^0 < y^0}$

Since $x^0 > y^0 > u^0$ we have $i\Delta_F(x-u)=\Delta_+(x-u)$.
We obtain

$$\begin{aligned}
I_1(u)-I'_1(u)=\Delta_+(x-u)\,[&\tau_{\sigma-1}(X_1\,|\,u,u,u\,|\,X_2\,|\,\ldots\,|\,X_N)\\
&-\tau_{\sigma-1}(X_1\,|\,u,u,u\,|\,X_2\,|\,\ldots\,|\,X_N)+\tau_{\sigma-1}(X_1\,|\,u,u,u,X_2\,|\,\ldots\,|\,X_N)\\
&-\tau_{\sigma-1}(X_1\,|\,u,u,u,X_2\,|\,\ldots\,|\,X_N)]\\
&=0\,.
\end{aligned}$$

We see that for $k = 1$ the integrands are identical for the given time order.

$\underline{k > 1}$

With the same technique as described above it is easy to show that all contributions cancel, so $I_k(u) - I'_k(u) = 0$ for any k. It follows that for the given time order both sector τ-functions differ at most in the unintegrated terms of the recursion equation (2.1). For these we proceed in the same way as above until sector τ-functions with no or one argument remain. These terms are equal to zero.

Since equation (2.1) reduces for ordinary τ-functions to the usual recursion equation we see that the perturbative sector τ-functions have in fact the time ordering properties one would expect for the VEVs of partially time ordered products of field operators (condition b)).

3. RULES FOR EVALUATING SECTOR τ-FUNCTION GRAPHS

In this section we give an explicit solution of (2.1) in terms of Feynman like graphs.

First we need some definitions.

A graph corresponding to $\tau_\sigma(x_1, \ldots, x_n)$ in φ^4 theory consists of n external points and σ four line vertices.

A graph corresponding to $\tau_\sigma(X_1\,|\,\ldots\,|\,X_n)$ also consists of $n = \sum_{i=1}^{N} n_i$

external points and σ four line vertices. Furthermore every graph gets an additional structure which we call sector partition. We will define sector partitions in the following.

To every external point and to every vertex we relate a real number called sector number. The sector numbers for internal vertices are for the present arbitrary. Every external point gets the number of its sector, i. e. $x_j^{(i)} \in X_i$ gets sector number i. For graphs we extend the definition of a sector as the set of all external points and vertices which have the same sector number. The sectors must have the following properties:

External sectors are sectors which contain external points. Each of its vertices is connected to at least one external point within this sector.

Internal sectors are sectors which contain no external point. They are connected.

We call two sectors adjoining if there exists a vertex in one sector which is connected to a vertex in the other sector by a line. Every partition of the graph into sectors is called sector partition. We call a sector partition proper if every internal sector has a number between the highest and the lowest sector number of the adjoining sectors. This intermediate value rule implies that all sectors have a number between 1 and N.

Sector partitions define an ordering of the vertices because we can order the vertices after the numerical value of their sector numbers. Two sector partitions are called equivalent when the ordering of all pairs of adjoining vertices are the same.

An example of different sector partitions of a given graph is shown in fig. 1. With these definitions we can formulate the rules for computing sector τ-functions with graphs; we compute $\tau_\sigma(X_1 \mid \ldots \mid X_N)$ as follows:

1. Draw all distinct diagrams with n external points $x_1^{(1)}, \ldots, x_{n_N}^{(N)}$ and vertices $u_1, \ldots, u_\sigma$. $\left[\sum_{i=1}^{N} n_i = n\right]$.

To every diagram draw all versions with non-equivalent proper sector partitions.

2. Now set for each vertex a factor $-i$, for each line connecting the vertices y_i and y_j with sector numbers a_i and a_j a factor $i\Delta_F(y_i - y_j)$ if $a_i = a_j$, a factor $\Delta_+(y_i - y_j)$ if $a_i < a_j$, and a factor $\Delta_+(y_j - y_i) = \Delta_-(y_i - y_j)$ if $a_i > a_j$.

3. Every diagram G has to be divided by a combinatorial factor $\gamma(G) = g2^\alpha(3\,!)^{\beta-\sigma}2^\gamma$, where α is the number of double lines (pairs of vertices connected by two lines), β the number of triple lines (pairs of vertices connected by three lines), γ the number of lines connecting a vertex with itself and g the number of automorphism of the diagram (permutation of

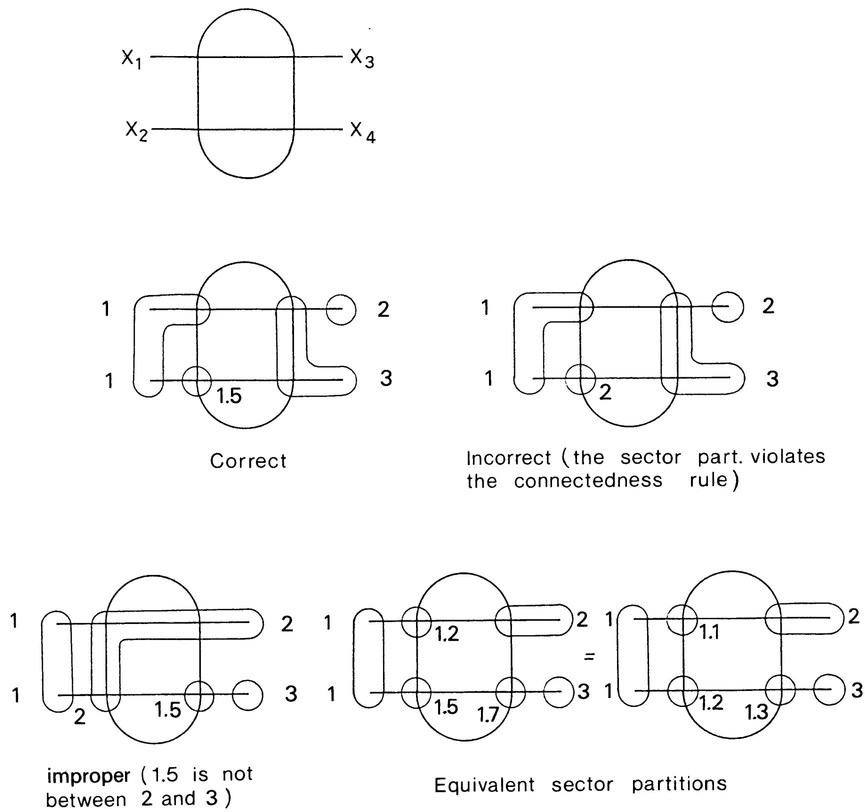

FIG. 1

vertices which leave the diagram unchanged) [*4*]. In addition every diagram gets a factor $(-1)^\eta$, where η is the number of internal sectors.

4. Sum up for each diagram all sector partitions, integrate over all u_i, and sum up all diagrams.

Let us consider an example of a second order graph $G_2(x_1 \mid x_2, x_3 \mid x_4)$, a contribution to $\tau_2(x_1 \mid x_2, x_3 \mid x_4)$.

We have to sum over all possible sector numbers for u_1 and u_2 which lead to different proper sector partitions. The sector partitions in fig. 2 are all proper sector partitions for $G_2(x_1 \mid x_2, x_3 \mid x_4)$.

c reads for example

$$\frac{6^2}{2}(-i)^2 \int i\Delta_F(x_1-u_1)\Delta_-(x_4-u_1)\Delta_+(u_1-u_2)\Delta_+(u_1-u_2)i\Delta_F(u_2-x_2)\, i\Delta_F(u_2-x_3)du_1du_2$$

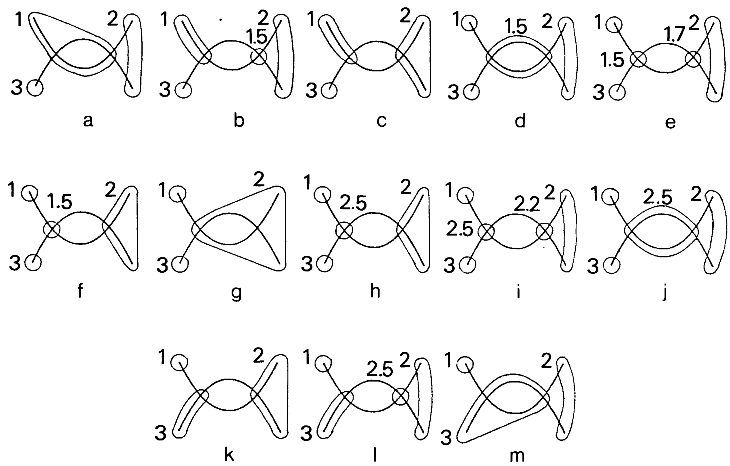

FIG. 2

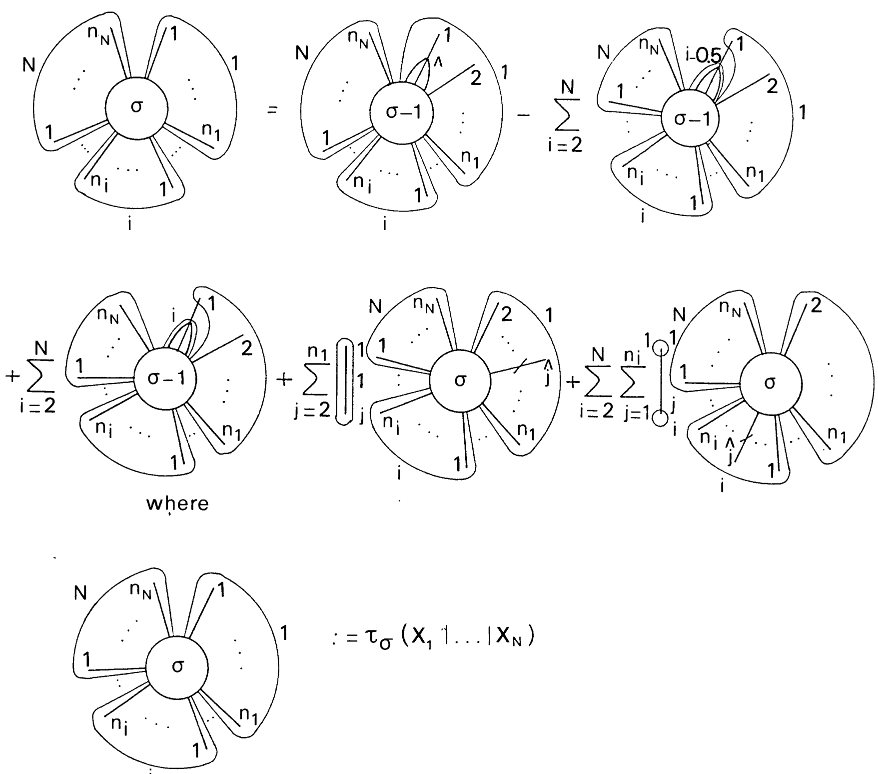

FIG. 3

Proof of the rules:

We can write the recursion equation in the graphical form shown in fig. 3.

We will prove the rules by induction. In zeroth order the rules lead to the correct expressions because for every graph exists only one sector partition since there are no inner vertices, and these sector partitions obviously lead to the correct propagators.

Now suppose the rules are true in all orders up to $\sigma - 1$. Let us first consider the integral term of the right hand side of eq. (2.1). Obviously equivalent sector partitions lead to the same analytic expressions for a graph because it is only the order of the sector numbers of adjoining vertices which determines the kind of propagator between them. In the graphical form we have chosen $S(\Lambda)=1, 1.5, \ldots, N-0.5, N$. This special selection can lead to improper sector partitions and to a repetition of the same graph for different sector numbers $S(\Lambda)$. For example the sector partitions shown in fig. 4 are of the form shown in fig. 5.

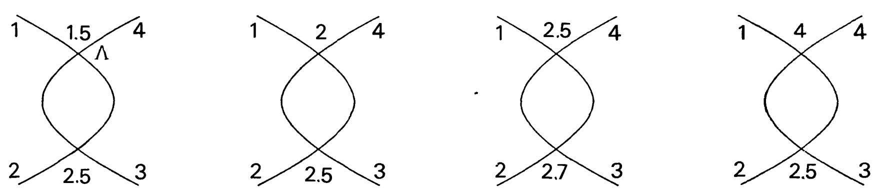

FIG. 4

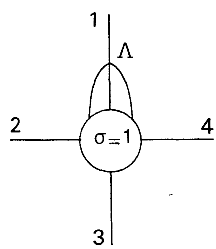

FIG. 5

The graphs with $S(\Lambda) = 1.5, 2, 2.5$ have equivalent sector partitions, the graphs with $S(\Lambda) = 2, 4$ have improper sector partitions.

We will prove that graphs with improper sector partitions cancel and that equivalent sector partitions sum up to yield one graph with proper sign.

For this purpose we have to answer the following question: How often do we get equivalent sector partitions for a given graph, when $S(\Lambda)$ runs through the values $S(\Lambda) = 1, 1.5, \ldots, N$?

LEMMA. — Let G be a graph with sector partitions S_G, γ an internal sector of G with sector number $S(\gamma)$. There exists an intervall $(k_\gamma, l_\gamma) \subset \mathbb{R}$ such that we can find a sector partition equivalent to S_G with $S'(\gamma)$ instead of $S(\gamma)$ iff $S'(\gamma) \in (k_\gamma, l_\gamma)$, where k_γ and l_γ are sector numbers of external points.

Proof. — The internal sectors γ_i of S_G are restricted by a number of inequalities like $k < S(\gamma_{i_1}) < S(\gamma_{i_2}) < S(\gamma_{i_3}) < l$, where due to the intermediate value rule k and l are sector numbers of external points. For a given sector γ there exists a highest k such that $k < S(\gamma)$ and a lowest l such that $S(\gamma) < l$. We call these bounds k_i and l_i for a sector γ_i, so that $k_i < S(\gamma_i) < l_i$. Only if $S'(\gamma)$ is within these bounds we can find an equivalent sector partition with $S'(\gamma)$ instead of $S(\gamma)$. It remains to be shown that for every $S'(\gamma) \in (k_\gamma, l_\gamma)$ there exists a sector partition equivalent to S_G. Suppose that $S'(\gamma) < S(\gamma)$. Then the sector partition with $S'(\gamma_i) = (1-\lambda)k_i + \lambda S(\gamma_i)$ is equivalent to S_G if $\lambda \in (0, 1]$, and λ can be adjusted such that $S'(\gamma) = (1 - \lambda)k + \lambda S(\gamma)$. If $S'(\gamma) > S(\gamma)$ choose $S'(\gamma_i) = (1-\lambda)l_i + \lambda S(\gamma_i)$.

With this lemma we can complete the proof of the Feynman rules for the sector τ-functions.

For a given sector partition Λ is part of a sector $\gamma(\Lambda)$ with sector number $S(\Lambda)$ (see fig. 6).

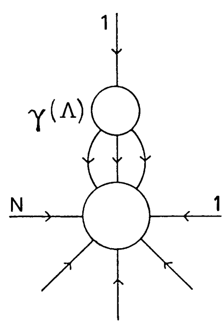

FIG. 6

If $\gamma(\Lambda)$ is external the sector partitions for other values of $S(\Lambda)$ will be different. We see from the induction assumption that the resulting sector partition has the right properties.

If $\gamma(\Lambda)$ is internal and $S(\Lambda)$ satisfies the intermediate value rule we know from the lemma that for $k_\gamma < S(\Lambda) < l_\gamma$ we obtain equivalent sector partitions, where k_γ and l_γ are the sector numbers of external points which are all integer. Since $S(\Lambda)$ runs through integer and halfinteger values this implies that there are an odd number of graphs with equivalent sector partitions. Due to the alternating sign the sum of them results in one graph with negative sign. If $\gamma(\Lambda)$ consists of Λ alone, $\gamma(\Lambda)$ is a new internal sector, and the negative sign is appropriate. The sector numbers inside of $\tau_{\sigma-1}$ can be chosen such that they are unequal to $S(\Lambda)$, so that all internal sectors are connected. If $\gamma(\Lambda)$ consists of more vertices than Λ this implies that $\gamma(\Lambda)$ is an external sector with respect to $\tau_{\sigma-1}$. It follows that $\gamma(\Lambda)$ is a new internal sector of τ_σ and the negative sign is again appropriate. In addition it is connected, following from the connectedness properties of external sectors.

If $\gamma(\Lambda)$ is internal and does not satisfy the intermediate value rule we obtain equivalent sector partitions for $k < S(\gamma) \leq N$. The lemma applies for the part of the graph belonging to $\tau_{\sigma-\sigma(\gamma)}$. Therefore k is integer, so that the number of equivalent sector partitions is even. Due to the alternating sign their sum vanishes.

For the unintegrated terms in (2.1) we repeat this procedure until τ_σ with no or one argument remain. These are equal to zero.

We see from our rules that the sequence of arguments of $\tau_\sigma(X_1 \mid \dots \mid X_N)$ within a sector has no influence on the graphs contributing to $\tau_\sigma(X_1 \mid \dots \mid X_N)$. Therefore the arguments can be permuted within a sector without changing the sector τ-function.

4. RENORMALIZATION OF SECTOR τ-FUNCTION GRAPHS

Until now everything has been done without renormalization. To get well defined results a renormalization is necessary and the renormalization scheme we shall use is that of BPHZ [4] [5].

We are mainly interested in the subtraction of infinities and shall comment on the renormalization conditions and finite renormalization later on.

The BPHZ renormalization scheme is formulated in p-space. We therefore have to transform the Feynman rules into this space. We use the unsymmetric Fourier transform

$$\varphi(p) = \int d^4x \varphi(x) e^{ipx}, \qquad \varphi(x) = \frac{1}{(2\pi)^4} \int d^4p \varphi(p) e^{-ipx} \tag{4.1}$$

A connected diagram in p-space is evaluated in the following way: We have to set an overall factor $\delta(p_1 + \dots + p_n)$, p_i external variables; for every line a factor $\Delta_{\#}(p)$, depending on the sector partition; for every vertex a factor $-i$. In every vertex momentum is conserved and to every independent closed loop we associate an integration variable k_i and an integration measure $\frac{d^4k}{(2\pi)^4}$. We have to sum over all sector partitions analogous to the rules in x-space.

The renormalization of a proper diagram G (i. e. a connected diagram which remains connected when an arbitrary internal line is cut) is done in the usual theory in the following way:

The integrand of every superficial divergent diagram is subtracted by a number of counterterms. We do not give a general definition of superficial divergence here, we only need that in φ^4 theory every subdiagram with four external lines has superficial divergence of degree 0, i. e. is logarithmically divergent, and every diagram with two external lines has superficial divergence of degree 2, i. e. is quadratically divergent ([*4*], for example). Here we use the definition of a subdiagram of a graph G as a subset of vertices of G and of all lines joining them in G. A subdiagram must have at least 2 vertices.

We replace the integrand I_G of a Feynman graph by

$$R = \sum_U \prod_{\gamma \in U} (-T_\gamma) I_G \qquad T_\gamma I_G := I_{G/\gamma} T_\gamma I_\gamma \tag{4.2}$$

U is the forest of renormalization parts, G/γ is the graph we obtain, when we replace γ in G by a point, and $T_\gamma I_G = I_{G/\gamma} T_\gamma I_\gamma$ is defined as the integrand of the Graph G, where $T_\gamma I_\gamma$ is inserted in place of I_γ.

The first step is to translate this renormalization scheme to sector τ-functions. We shall see that we can apply exactly the same procedure to them. This is based on the following two properties:

1. every integral over a loop not completely within one sector is finite
2. every renormalization part γ not completely within one sector satisfies the equation $T_\gamma I_\gamma = 0$.

So only those subgraphs of a sector function graph need a renormalization which consist of purely $i\Delta_F$ propagators. Seagull terms, i. e. subdiagrams connected to the diagram only by a single vertex, vanish in the course of renormalization, because they do not depend on external momenta. Therefore, we shall omit them in the following.

Let us first prove property 1. Consider the sector with lowest sector number within the loop, so that all lines coming out of the sectors correspond to Δ_+ propagators. If we call k the integration variable in the loop momentum conservation implies that there exist two lines with propagators $\Delta_+(k)$ and $\Delta_+(p-k)$, respectively. But the product $\Delta_+(k)\Delta_+(p-k)$ has its support in a finite region with respect to k. It follows that no UV divergences appear.

To prove the second property we similarly consider the sector with the lowest sector number in the renormalization part. Let us first assume that the Taylor operator has degree zero which means that the external momenta of the subgraph γ are set to zero. This implies that the sum of all momenta of the outgoing lines of the sector is equal to zero. Since all these momenta belong to Δ_+ propagators $T_\gamma I_\gamma = 0$. This remains true if the external momenta vary in a small neighbourhood of zero. Hence every derivation in the external momenta of I_γ at the origin is zero and $T_\gamma I_\gamma = 0$ is satisfied.

At last we have to examine wether G/γ leads to a finite integral in all cases. Due to point 2 above this is only important if γ consists of only one sector. If γ has four external lines G/γ has a new four line vertex with the sector number of Λ instead of the subgraph γ and we see that this is again an ordinary sector τ-function graph.

Things are different when γ has two external lines. We obtain a new type of vertex with two lines for G/γ. This kind of vertex occurs also in the course of mass renormalization.

This two line vertex leads to an undefined product of propagators such as $\Delta_+(p)i\Delta_F(p)$ or $\Delta_+(p)\Delta_+(p)$. We can hope that the summation of these undefined products leads to a well defined expression. Let us therefore consider a reduced graph with $m-1$ two line vertices between two four line vertices.

We assume the sector number of the four line vertex on the left hand to be lower than that on the right hand and take into account all proper sector partitions of the two line vertices, holding the rest of the sector numbers fixed. Due to the intermediate value rule we obtain only Δ_+ and $i\Delta_F$ propagators. There are m positions for one Δ_+ propagator, $\binom{m}{2}$ for two propagators and so on. Using the sign rule we obtain:

$$\begin{aligned} I(p) &= \sum_{j=1}^{m} \binom{m}{j} (-1)^j \Delta_+^j(p)(i\Delta_F(p))^{m-j} \\ &= \theta(p^0) \sum_{j=1}^{m} \binom{m}{j} (-1)^j (2\pi\delta(p^2-m^2))^j (i\Delta_F(p))^{m-j} \\ &= \theta(p^0)[-2\pi\delta(p^2-m^2) + i\Delta_F(p))^m - (i\Delta_F(p))^m] \\ &= \theta(p^0)[(-\Delta_+(p)-\Delta_-(p)+i\Delta_F(p))^m - (i\Delta_F(p)^m] \end{aligned}$$

where we used $\theta^j(p^0) = \theta(p^0)$.

With the formula

$$i\Delta_F(p + \overline{i\Delta_F(p)} = \Delta_+(p) + \Delta_-(p) \tag{4.3}$$

we obtain

$$I(p) = \theta(p^0)[(\overline{-i\Delta_F}(p))^m - (i\Delta_F(p))^m]\,. \tag{4.4}$$

$i\Delta_F(p)$ as well as $\overline{i\Delta_F}(p)$ can be potentiated as boundary values of holomorphic functions. The multiplication with $\theta(p^0)$ does not cause any difficulties.

Apart from this obvious difficulty with selfenergy insertions we have not considered the problem of the local existence in p-space of our integrals, i. e. the question corresponding to the existence of the $\varepsilon -> 0$ limit in ordinary Feynman graphs [6]. We feel that this is a difficult, but not a serious problem. In any case the sum over all graphs of a given order must exist by the results of [3], hence a possible local divergence of individual graphs can be easily cured by a suitable regularization.

Thus we have the result that $R = \sum_{U} \prod_{\gamma} (-T_\gamma) I_G$ is a well defined expression both for τ-functions and for sector τ-functions. Only for the τ-function subgraphs counterterms are necessary. Zimmermann proved that they lead to finite expressions when the integrations are performed. Integration over loops with vertices in different sectors are always finite as we showed above. So the whole integration can be performed and is finite.

The time ordering property mentioned in part 1 under b) is true also for the renormalized τ-functions. This can be easily seen with the multiplicative renormalization scheme. We regularize all propagators by setting $\Delta_{\#}(x\,;\,m) \to \Delta_{\#}(x\,;\,m) - \Delta_{\#}(x\,;\,M)$(Pauli-Villars). The argumentation given in part 2 can be applied then, because the reparametrization of the theory by Z, $\delta\lambda$, and δm does not influence the time ordering property.

In the BPHZ renormalization scheme renormalization conditions can be imposed as usual with the help of finite counter vertices.

5. DEMONSTRATION OF WIGHTMAN PROPERTIES FOR THE PERTURBATIVE WIGHTMAN FUNCTIONS

In this section we show that our expression for W satisfies the linear Wightman properties.

a) Relativistic Transformation Law

$$W_\sigma(Lx_1 + a, Lx_2 + a, \ldots, Lx_n + a) = W_\sigma(x_1, x_2, \ldots, x_n) \qquad a \in R^4,\ L \in L_+^\uparrow \tag{5.1}$$

Translation invariance: In the equations only differences between two arguments occur, differences between two exterior variables $(x-y)$ and differences between external variables and integration variables $\Delta_{\#}(x-u)$ or $\Delta_{\#}(u-u')$. An addition of a constant vector vanishes or can be absorbed in a translation of the integration variables.

Lorentz invariance: Δ_F, Δ_+ and Δ_- are invariant to Lorentz transformations. The integration measure $dx^0dx^1dx^2dx^3$ is invariant, just as the integration region which is the whole $\mathbb{R}^4$.

b) Spectral Condition

$$W_\sigma(p_1, \ldots, p_n) = 0$$

$$\text{if} \quad \sum_{j=1}^{k} p_j \notin \overline{V}_+ \quad \text{for any} \quad k < n, \qquad \overline{V}_+ = \{ p \mid p^0 \geqslant 0, p^2 \geqslant 0 \} \tag{5.2}$$

We shall prove a slightly stronger statement, namely

$$\tau_\sigma(P_1 \mid \ldots \mid P_N) = 0$$

$$\text{if} \quad \sum_{i=1}^{k} \mid P_i \mid \notin \overline{V}_+ \qquad \mid P_i \mid := \sum_{j=1}^{n_i} p^{(i)} \qquad k < N \tag{5.3}$$

Consider an arbitrary graph $G_\sigma(P_1 \mid \ldots \mid P_N)$ of order σ together with a given sector partition. Let us cut every line for which one vertex has sector number less or equal k and the other greater than k. Then the graph disintegrates into two pieces, which are not necessarily connected. Due to the intermediate value rule one piece has sector numbers less than k and the other bigger than k. If we call the momenta of the lines between the two pieces $q_1, \ldots, q_s$, then to every line corresponds a propagator $\Delta_+(q_i)$. This implies that G_σ vanishes if any $q_i \notin \overline{V}_+$ and particulary if $\sum_{i=1}^{s} q_i \notin V_+$. Due to momentum conservation $\sum_{i=1}^{s} q_i = \sum_{i=1}^{k} \mid P_i \mid$, which proves the Spectral Condition.

c) Hermiticity Condition

$$\overline{W_\sigma(x_1, \ldots, x_n)} = W_\sigma(x_n, \ldots, x_1) \tag{5.4}$$

At first sight this seems to be easy to prove, but it turns out to be quite involved. The reason for this is that $i\Delta_F(x)$ has a relatively complicated algebraic relation to $\overline{i\Delta_F}(x)$ (eq. 4.5). We shall use the relation

$$W_\sigma(x_1, \ldots, x_n) = \tau_0(x_1 \mid \ldots \mid x_n)$$

and proceed again by induction.

In zeroth order the hermiticity property is obvious because

$$\overline{\Delta}_+(x_i - x_j) = \Delta_+(x_j - x_i)$$

In higher orders we first consider the unrenormalized but regularized graphs. $G_\sigma(x_1 \mid \ldots \mid x_n)$ denotes a graph together with all sector partitions and we define $D_\sigma^G = G_\sigma(x_1 \mid \ldots \mid x_n) - \overline{G_\sigma(x_n \mid \ldots \mid x_1)}$.

In the following we will expand $G_\sigma(x_n \mid \ldots \mid x_1)$ from behind, i. e. with respect to x_1. This can easily be done with the help of the graphical rules for the sector τ-functions. We obtain:

$$\begin{aligned}
D_\sigma^G &= G_\sigma(x_1 \mid \dots \mid x_n) - \overline{G_\sigma(x_1 \mid \dots \mid x_n)} \\
&= (-i)\int \{ i\Delta_F(x_1 - u)G_{\sigma-1}(u, u, u \mid x_2 \mid \dots \mid x_n) \\
&- \Delta_+(x_1 - u)[G_{\sigma-1}(u, u, u \mid x_2 \mid \dots \mid x_n) - G_{\sigma-1}(u, u, u, x_2 \mid \dots \mid x_n) \\
&- \sum_{i=3}^{n} [G_{\sigma-1}(x_2 \mid \dots \mid x_{i-1} \mid u, u, u \mid x_i \mid \dots \mid x_n) \\
&- G_{\sigma-1}(x_2 \mid \dots \mid x_{i-1} \mid u, u, u, x_i \mid \dots \mid x_n)]] \} \\
&- \Big\{ (-i)\int \{ i\Delta_F(x_1 - u)G_{\sigma-1}(x_n \mid \dots \mid x_2 \mid u, u, u) \\
&- \Delta_-(x_1 - u)[G_{\sigma-1}(x_n \mid \dots \mid x_2 \mid u, u, u) - G_{\sigma-1}(x_n \mid \dots \mid x_2 \mid u, u, u) \\
&+ \sum_{i=3}^{n} [G_{\sigma-1}(x_n \mid \dots \mid x_i \mid u, u, u \mid x_{i-1} \mid \dots \mid x_2) \\
&- G_{\sigma-1}(x_n \mid \dots \mid x_i, u, u, u \mid x_{i-1} \mid \dots \mid x_2)]] \Big\}^-
\end{aligned}$$

The second term of D_σ^G yields, using the induction assumption and $\overline{\Delta_-} = \Delta_+$:

$$\begin{aligned}
G_\sigma(x_n \mid \dots \mid x_1) &= (-i)\int \{ i\Delta_F(x_1 - u)G_{\sigma-1}(u, u, u \mid x_2 \mid \dots \mid x_n) - \\
&- \Delta_+(x_1 - u)[G_{\sigma-1}(u, u, u \mid x_2 \mid \dots \mid x_n) - \overline{G_{\sigma-1}(x_n \mid \dots \mid x_2, u, u, u)} \\
&+ \sum_{i=3}^{n} [G_{\sigma-1}(x_2 \mid \dots \mid x_{i-1} \mid u, u, u \mid x_i \mid \dots \mid x_n) - \\
&- \overline{G_{\sigma-1}(x_n \mid \dots \mid x_i \mid u, u, u \mid x_{i-1} \mid \dots \mid x_2)}]] \} \, du
\end{aligned}$$

From equation (4.3) we know that $i\Delta_F + \overline{i\Delta_F} = \Delta_+ + \Delta_-$. A similar equation holds for $G_\sigma + \overline{G_\sigma}$, namely

$$\begin{aligned}
&G_{\sigma-1}(x_2 \mid \dots \mid x_{i-1} \mid u, u, u, x_i \mid \dots \mid x_n) + \overline{G_{\sigma-1}(x_n \mid \dots \mid x_i, u, u, u \mid x_{i-1} \mid \dots \mid x_2)} \\
&= G_{\sigma-1}(x_2 \mid \dots \mid x_{i-1} \mid u, u, u \mid x_i \mid \dots \mid x_n) + G_{\sigma-1}(x_2 \mid \dots \mid x_{i-1} \mid x_i \mid u, u, u \mid \dots \mid x_n),
\end{aligned} \tag{5.5}$$

This can easily be seen by distinguishing the cases $u^0 > x_i^0$ and $u^0 < x_i^0$. Using equation (5.5) several terms cancel and we obtain after some simple calculation

$$D_\sigma^G = -i\int \{ \Delta_+(x_1 - u)G_{\sigma-1}(x_2 \mid \dots \mid x_n \mid u, u, u) + \Delta_-(x_1 - u)G_{\sigma-1}(u, u, u \mid x_2 \mid \dots \mid x_n) \} \, du$$

Transformation into p-space yields

$$- i \int \{ \Delta_+(p_1)G_{\sigma-1}(p_2 \,|\, \dots \,|\, p_n \,|\, p_1 - k_1 - k_2, k_1, k_2) \\ + \Delta_-(p_1)G_{\sigma-1}(p_1 - k_1 - k_2, k_1, k_2 \,|\, p_2 \,|\, \dots \,|\, p_n) \} \, dk_1 dk_2 . \quad (5.6)$$

We know that $G_{\sigma-1}(p_2 \,|\, \dots \,|\, p_n \,|\, p_1 - k_1 - k_2, k_1, k_2)$ has support in $p_1 \in V_-$, whereas $\Delta_+(p_1)$ has support in $p_1 \in V_+$. Hence the product vanishes and the same is true for the second term in (5.6). It follows that $D_\sigma^G = 0$. In the same way as in part 4 we can conclude that the hermiticity condition is satisfied also for the non-regularized and renormalized sector τ-functions.

d) Local Commutativity

$$W_\sigma(x_1, \dots, x_j, x_{j+1}, \dots, x) = W_\sigma(x_1, \dots, x_{j+1}, x_j, \dots, x_n)$$
$$\text{if} \qquad (x_j - x_{j+1})^2 < 0 \qquad \text{for} \qquad j = 1, \dots, n-1 . \qquad (5.7)$$

From (a) we know the relativistic invariance of the sector τ-functions. Hence

$$W_\sigma(x_1, \dots, x_j, x_{j+1}, \dots, x_n) = \tau_\sigma(x_1 \,|\, \dots \,|\, x_j \,|\, x_{j+1} \,|\, \dots \,|\, x_n) \\ = \tau_\sigma(Lx_1 \,|\, \dots \,|\, Lx_j \,|\, Lx_{j+1} \,|\, \dots \,|\, Lx_n), \qquad \text{for} \qquad L \in L_+^\uparrow .$$

We choose L such that $(Lx_j - Lx^j{}_{+1})^0 > 0$. This is always possible if $(x_j - x_{j+1})^2 < 0$. Then

$$W_\sigma(\dots) = \tau_\sigma(Lx_1 \,|\, \dots \,|\, Lx_j, Lx_{j+1} \,|\, \dots \,|\, Lx_n) \\ = \tau_\sigma(Lx_1 \,|\, \dots \,|\, Lx_{j+1}, Lx_j \,|\, \dots \,|\, Lx_n)$$

where we used the fact that we can permute arguments within a sector. Now we choose $L' \in L_+^\uparrow$ such that $(L'x_j - L'x_{j+1})^0 < 0$. Then

$$W_\sigma(\dots) = \tau_\sigma(L'x_1 \,|\, \dots \,|\, L'x_{j+1}, L'x_j \,|\, \dots \,|\, L'x_n) \\ = \tau_\sigma(L'x_1 \,|\, \dots \,|\, L'x_{j+1} \,|\, L'x_j \,|\, \dots \,|\, L'x_n) \\ = \tau_\sigma(x_1 \,|\, \dots \,|\, x_{j+1} \,|\, x_j \,|\, \dots \,|\, x_n) \\ = W_\sigma(x_1, \dots, x_{j+1}, x_j, \dots, x_n) .$$

The main point in the proof is that every sector τ-function is Lorentz invariant for every selection of boundaries.

At last we have to prove the Cluster Decomposition Property. We can completely mimic the axiomatic proof. The essential assumptions are locality, spectrum condition and the fact that the Wightman functions are tempered distribution.

By the results of this section $W_\sigma(x_1, \dots, x_n)$ is the boundary value of an analytic function in a forward tube [*1*]. W_σ is thus uniquely determined by its values in the open real set $\{ x_1^0 > x_2^0 > > \dots > x_n^0 \}$, where it must coincide with $\tau_\sigma(x_1, \dots, x_n)$. Hence our solution of the problem posed in the introduction is unique.

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REFERENCES

[1] A. S. WIGHTMAN, Quantum Field Theory in Terms of Vacuum Expectation Values, *Phys. Rev.*, t. **101**, 1956, p. 860.
[2] R. STREATER and A. WIGHTMAN, *PCT, Spin and Statistics, and All That*, New York, 1964.
[3] H. EPSTEIN and V. GLASER, The role of locality in perturbation Theory, *Ann. Inst.Henri Poincaré*, t. **19**, 1973, p. 3.
[4] W. ZIMMERMANN, Local Operator Products and Renormalization in Quantum Field Theory, *in Lectures on Elementary Particle Physics and Quantum Field Theory*, Brandeis University, Summer Institute in Theoretical Physics; MIT Press, Cambridge, 1970.
[5] W. ZIMMERMANN, Convergence of Bogoliubov's Method of Renormalization in Momentum Space, *Comm. Math. Phys.*, t. **15**, 1969, p. 208.
[6] K. HEPP, Proof of the Bogoliubov-Parasiuk Theorem on Renormalization, *Comm. Math. Phys.*, t. **2**, 1966, p. 301.

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