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On the structure of relative identification operators for quantum fields and their connection with the Haag-Ruelle scattering theory

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ABSTRACT. — The paper recalls the notion of a relative identification operator K of a Wightman field $A(\cdot)$ with respect to a corresponding free field $A^0(\cdot)$, useful for the definition of wave operators with respect to the field $A(\cdot)$. The corresponding pre-wave operator $e^{itH}Ke^{-itH_0}$ can be linked directly with the Haag-Ruelle approximants of the field $A(\cdot)$. Thus the Haag-Ruelle scattering theory can be embedded formally into the framework of the abstract scattering theory. Some structural properties of K are presented. It is pointed out that K is uniquely determined by a single field operator $A(h(p)\gamma_0(\mathrm{p}))$, where $h(p)$ is a smooth function with support in a sufficiently small neighbourhood of the discrete mass hyperboloid characterized by the mass $m_0 > 0$ belonging to $A^0(\cdot)$ (as is usually introduced within the Haag-Ruelle framework) and where $\gamma_0 \in \mathscr{S}(\mathbb{R}^3)$ is multiplicative-generating.

RÉSUMÉ. — On rappelle la notion d'opérateur d'identification relatif K d'un champ de Wightman $A(\cdot)$ par rapport au champ libre correspondant $A^0(\cdot)$, utile pour la définition des opérateurs d'onde du champ $A(\cdot)$. Le pré-opérateur d'onde $e^{itH}Ke^{-itH_0}$ correspondant peut être relié directement aux approximants de Haag-Ruelle pour le champ $A(\cdot)$. Ainsi la théorie de la diffusion de Haag-Ruelle peut être formellement incorporée dans le cadre de la théorie abstraite de la diffusion. On donne quelques propriétés structurelles de K. On remarque que K est déterminé de façon unique par un seul opérateur de champ $A(h(p)\gamma_0(\mathrm{p})$, où $h(p)$ est une fonction lisse à support contenu dans un voisinage assez petit de l'hyperboloïde de

masse caractérisé par la masse $m_0 > 0$ du champ $A^0(\cdot)$ (comme on l'introduit de façon usuelle dans la théorie de Haag-Ruelle), et où $\gamma_0 \in \mathcal{S}(\mathbb{R}^3)$ est génératrice par multiplication (voir Définition 3).

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§ 1. INTRODUCTION

Let $A(\cdot)$ be a Wightman field on a (separable) Hilbert space $\mathcal{H}$. For convenience we collect the properties of such a field. The tensor algebra over the Schwartz space $\mathcal{S}(\mathbb{R}^4)$ is denoted by $\mathcal{F}$. Its elements are finite sequences $f := \{ f_0, f_1, \ldots, f_N, 0, \ldots \}$, where N depends on f and where $f_n \in \mathcal{S}(\mathbb{R}^{4n})$. $\mathcal{F}$ is equipped with the usual topology τ (locally convex direct sum of the Schwartz space topologies of the $\mathcal{S}(\mathbb{R}^{4n})$). The Wightman functional $W(\cdot) : \mathcal{F} \mapsto \mathbb{C}$ of the field $A(\cdot)$ is assumed to be linear, normed, positive, continuous and Poincaré invariant. There is a unique vacuum $\omega \in \mathcal{H}$ and the field is assumed to be spectral and local. The continuous linear functionals on $\mathcal{F}$ have the form $W = \{ W_0, W_1, W_2, \ldots \}$, where W_n is a continuous linear functional on $\mathcal{S}(\mathbb{R}^{4n})$, a so-called n-point functional, and $W(f) = \Sigma W_n(f_n)$. Recall the special form of the functionals W_0, W_1, W_2 :

$$W_0(f_0) = f_0 , \tag{1}$$

$$W_1(f_1) = \gamma f_1(0) , \qquad \gamma \in \mathbb{R} , \tag{2}$$

$$W_2(f_2) = \gamma f_2(0, 0) + \int_0^\infty \int_{H_m} f_2(-p, p)\mu_m(dp)\rho(dm) . \tag{3}$$

Note that we prefer to work with momentum coordinates, that is with functions $f_n(p_1, p_2, \ldots, p_n) \in \mathcal{S}(\mathbb{R}^{4n})$ which are Fourier transforms

$$f_n(p_1, \ldots, p_n) = (2\pi)^{-2n} \int e^{-i \sum_{j=1}^{n} (p_j, x_j)} \check{f}_n(x_1, \ldots, x_n) dx_1 \ldots dx_n$$

of functions $\check{f}_n \in \mathcal{S}(\mathbb{R}^{4n})$ depending on position coordinates. $(\cdot, \cdot)$ denotes the Cartesian scalar product in $\mathbb{R}^4$. H_m denotes the mass hyperboloid $H_m := \{ p : p_0^2 - |\mathfrak{p}|^2 = m^2, p_0 > 0 \}$, $\mu_m(\cdot)$ denotes the Lorentz invariant measure on H_m, given by $\mu_m(dp) = d\mathfrak{p}/(m^2 + |\mathfrak{p}|^2)^{1/2}$ and $\rho(\cdot)$ is a characteristic polynomially bounded Borel measure on $m \geq 0$. Formula (3) is called the Källen-Lehmann representation of the 2-point functional.

The set of all $f \in \mathcal{F}$ satisfying $W(f^*f) = 0$ is denoted by lker W (left kernel). Note that f^* denotes the usual conjugation in $\mathcal{F}$, $f_n(p_1, \ldots, p_n) = \overline{f_n(-p_n, -p_{n-1}, \ldots, -p_1)}$.

The field $A(\cdot)$ is defined on $\mathcal{F}$, i. e. $A(f)$, $f \in \mathcal{F}$, is a generalized field operator, the usual field operator is given by $A(\tilde{f}_1)$ where $\tilde{f}_1 = (0, f_1, 0, \ldots) \in \mathscr{F}$. For brevity we write also $A(f_1)$ in this case.

Furthermore, an upper and lower mass gap is assumed, the discrete mass is denoted by m_0, the corresponding one-particle subspace of $\mathscr{H}$ is denoted by $\mathscr{H}_1$, it is assumed to be irreducible with respect to the Poincaré group, the corresponding representation is labeled by m_0 and $s = 0$. Recall the representation (SNAG-theorem)

$$U_a = \int_{\mathbb{R}^4} e^{-i(a,p)} E(dp) \tag{4}$$

for the unitary representation U_a of the translation group $a \in \mathbb{R}^4$ associated with the field. In terms of $E(\cdot)$ the mass gap is expressed by

$$\operatorname{supp}^m E = \{0\} \cup \{m_0\} \cup \Lambda, \ \Lambda \subseteq [m_0 + \varepsilon, \infty), \ \varepsilon > 0, \ m_0 > 0, \tag{5}$$

where $\operatorname{supp}^m E$ denotes the mass spectrum (note that $\operatorname{supp} E \subseteq \operatorname{clo} V^+$, V^+ the forward cone, and that $\operatorname{supp} E$ is Lorentz invariant, i. e. it contains only full mass hyperboloids H_m, then the mass spectrum is the closure of all m such that $H_m \subset \operatorname{supp} E$).

Finally, the condition of « coupling of the vacuum to the one-particle states » is assumed to be satisfied (see for example M. Reed and B. Simon [*1*, p. 319]. Note that this condition is satisfied if and only if

$$m_0 \in \operatorname{supp} \rho \tag{6}$$

is valid. That is, in this case one obtains

$$\{m_0\} \subseteq \operatorname{supp} \rho \subseteq \operatorname{supp}^m E, \tag{7}$$

(the latter inclusion is obvious).

The free (scalar) field, corresponding to $m_0 > 0$ and $s = 0$, is denoted by $A^0(\cdot)$, acting on the Hilbert space $\mathscr{H}^0$. Its measure « ρ » (mass distribution of the Källen-Lehmann representation) is given by the Dirac measure $\rho(m) = \delta(m - m_0)$.

In H. Baumgärtel *et al.* [*2*] a so-called relative identification operator K is introduced, useful for the definition of wave operators with respect to the field $A(\cdot)$. For convenience, we recall the definition and simple properties of $K : \mathscr{H}^0 \mapsto \mathscr{H}$. First, by $h \in C^\infty(\mathbb{R}^4)$ we denote a fixed real-valued function, $0 \leqslant h \leqslant 1$, with the following properties:

i) $\operatorname{supp} h \subseteq \bigcup_{m \in [m_0 - \delta, m_0 + \delta]} H_m, \quad \delta > 0,$

ii) $h(\Lambda p) = h(p)$ for all $\Lambda \in \mathscr{L}_+^\uparrow$ (proper Lorentz group),

iii) $h \restriction H_{m_0} = 1$.

Second, we define a certain linear manifold $\mathscr{L} \subset \mathscr{F}$ by: $g \in \mathscr{L}$ if and only if $g_0 \in \mathbb{C}$ arbitrary, $g_n(p_1, p_2, \ldots, p_n) = h(p_1)h(p_2) \ldots h(p_n)\gamma_n(\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n)$, $\gamma_n \in \mathscr{S}(\mathbb{R}^{3n})$, γ_n symmetric with respect to $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n$.

Then, if $(W^0(.)$ denotes the Wightman functional and J^0 denotes the (absolute) identification operator of the free field, it turns out that $\mathscr{L}$ contains exactly one element from each equivalence class mod lker W^0, that is, ima $(J^0 \restriction \mathscr{L})$ = ima J^0 and : $f \in \mathscr{L}$ and $J^0 f = 0$ imply $f = 0$, in other words, $\mathscr{L} \cap$ lker $W^0 = \{ 0 \}$ and $\mathscr{L} \oplus$ lker $W^0 = \mathscr{F}$. Now an operator K: $\mathscr{H}^0 \mapsto \mathscr{H}$ can be defined by

(8) $$K \{ A^0(f)\omega^0 \} := A(f)\omega, \qquad f \in \mathscr{L},$$

or

(9) $$K \{ J^0 f \} := Jf, \qquad f \in \mathscr{L},$$

where J denotes the (absolute) identification operator of the field $A(\cdot)$. (9) means that K is a certain factorization of J, $J = KJ^0$, on $\mathscr{L}$. K is called the *relative identification operator* between $A^0(\cdot)$ and $A(\cdot)$ with respect to $\mathscr{L}$. Recall the following simple properties of K:

I) K is densely defined, dom K = ima J^0, which is dense in $\mathscr{H}^0$.

II) dom K is invariant with respect to U_g^0 (the unitary representation of the Poincaré group belonging to the free field).

III) K is continuous with respect to the Schwartz space topology τ of $\mathscr{F}$ (more precisely : dom K may be equipped with this topology by the bijection $\mathscr{L} \ni f \leftrightarrow J^0 f \in$ dom K, then K is continuous with respect to this topology).

IV) $K\omega^0 = \omega$.

V) The intertwining relation

(10) $$U_g K = K U_g^0$$

is valid if $g = \{ \Lambda_0, (0, \mathfrak{a}) \}$, where $\mathfrak{a} \in \mathbb{R}^3$ is a pure spatial translation and where Λ_0 is a pure rotation in the $\mathfrak{a}$-space (the intertwining relation (10) is not valid in general for time translations).

Using K, the standard two-space pre-wave operator is given by

(11) $$e^{itH} K e^{-itH^0} u, \qquad u \in \text{dom } K,$$

where $e^{-itH} = U_{\{t,0\}}$, $e^{-itH^0} = U^0_{\{t,0\}}$ denote the unitary representations of the time translations in $\mathscr{H}$, $\mathscr{H}^0$, respectively.

In this paper some further structural properties of K are presented. In fact, it is shown that the expression (11) is intimately connected with the Haag-Ruelle approximants with respect to the field $A(\cdot)$.

If $\mathscr{M}$ is a subset of $\mathscr{F}$, for brevity we denote by $\mathscr{M}^{(n)}$ the set of all $f \in \mathscr{M}$ with $f = \{ 0, \ldots, 0, f_n, 0, \ldots \}$, i. e. the intersection of $\mathscr{M}$ with $\mathscr{S}(\mathbb{R}^{4n})$.

Finally recall the assignment between one-particle states and field operators.

If $f \in \mathscr{L}^{(1)}$ then $A(f)\omega \in \mathscr{H}_1$. Moreover, the assignment $\mathscr{L}^{(1)} \ni f \mapsto A(f)\omega \in \mathscr{H}_1$ is an injection, the image $\{ A(f)\omega, f \in \mathscr{L}^{(1)} \}$ coincides with

$$\{ A(f)\omega, f \in \mathscr{F}^{(1)}, \text{supp } f \cap \text{supp } E \subseteq H_{m_0} \}$$

and this linear manifold is dense in $\mathscr{H}_1$.

That is, the vector $u = \mathrm{A}(f)\omega \in \mathcal{H}_1$ with $f \in \mathcal{L}^{(1)}$ is in one-to-one correspondence with f. Therefore, one has an assignment of one-particle states $u \in \mathcal{H}_1$ to certain field operators $\mathrm{B}_u = \mathrm{A}(f)$. This assignment satisfies the property $\mathrm{B}_u\omega = u$.

On the other hand, the assignment can be considered as the assignment of vectors of the free one-particle space to vectors $u \in \mathcal{H}_1$ via K. Namely, if $f \in \mathcal{L}^{(1)}$, then $\mathrm{A}^0(f)\omega^0 = \{ 0, \gamma(\mathfrak{p}), 0, \dots \}$, where $f_1(p) = h(p)\gamma(\mathfrak{p})$ and where $\gamma(\mathfrak{p})$ is to be considered as an element of $\mathrm{L}^2(\mathbb{R}^3, d\mathfrak{p}/\mu(\mathfrak{p}))$. That is, we have

$$\mathcal{H}_1^0 = \mathrm{L}^2(\mathbb{R}^3, d\mathfrak{p}/\mu(\mathfrak{p})) \ni \gamma(\mathfrak{p}) \overset{\mathrm{K}}{\mapsto} \mathrm{A}(f)\omega = u \in \mathcal{H}_1 ,$$

where $\mu(\mathfrak{p}) := (m_0^2 + |\,\mathfrak{p}\,|^2)^{1/2}$.

§ 2. CALCULATION OF K

In this paragraph we calculate K on a dense subdomain of dom K = ima J^0 = ima ($\mathrm{J}^0 \restriction \mathcal{L}$). For this purpose we use the structure of J^0, which is given in [2]. According to this paper (Corollary 1) we have

$$\mathrm{J}^0 f = \{ f_0, (\mathrm{T}f)_1 \restriction \mathrm{H}_{m_0}, \mathrm{S}_2(\mathrm{T}f)_2 \restriction \mathrm{H}_{m_0} \times \mathrm{H}_{m_0}, \dots \} , \qquad f \in \mathcal{F} ,$$

where T denotes a certain continuous linear operator, acting in $\mathcal{F}$ (see [2, Theorem 2]) and where S_n denotes the symmetrization operator

$$(\mathrm{S}_n f_n)(p_1, \dots, p_n) = (n\,!)^{-1} \sum_{\pi} f_n(p_{\pi(1)}, p_{\pi(2)}, \dots, p_{\pi(n)}) .$$

Note that $\mathrm{T}f = f$ for $f \in \mathcal{L}$ (cf. [2, Lemma 5]), that is, one obtains

$$\mathrm{J}^0 f = \{ f_0, f_1 \restriction \mathrm{H}_{m_0}, \mathrm{S}_2 f_2 \restriction \mathrm{H}_{m_0} \times \mathrm{H}_{m_0}, \dots \} , \qquad f \in \mathcal{L} .$$

Recall that $f \in \mathcal{L}$ means $f_n(p_1, \dots, p_n) = \prod_{j=1}^{n} h(p_j)\gamma_n(\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_n)$, where γ_n is symmetric. Therefore

$$\mathrm{J}^0 f = \{ f_0, \gamma_1(\mathfrak{p}_1), \gamma_2(\mathfrak{p}_1, \mathfrak{p}_2), \dots \} , \qquad f \in \mathcal{L} ,$$

where $\gamma_n(\mathfrak{p}_1, \dots, \mathfrak{p}_n)$ is to be considered as an element of

$$\mathrm{L}^2\left(\mathbb{R}^{3n}, \bigotimes_{j=1}^{n} d\mathfrak{p}_j/\mu(\mathfrak{p}_j) \right),$$

where $\mu(\mathfrak{p}) := (m_0^2 + |\,\mathfrak{p}\,|^2)^{1/2}$.

Now let K_n be the n-particle component of K, that is

$$\mathrm{K}_n := \mathrm{K} \restriction \mathrm{S}_n(\mathcal{H}_1 \otimes \mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_1) ,$$

such that

$$\mathrm{K}u = \sum_{n=0}^{\infty} \mathrm{K}_n u_n, \; u = \mathrm{J}^0 f, \; f \in \mathscr{L}, \; u_n = \mathrm{J}^0 f_n .$$

Furthermore, let

$$(12) \quad \gamma_n(\mathfrak{p}_1, \ldots, \mathfrak{p}_n) = \mathrm{S}_n(\alpha_1(\mathfrak{p}_1) \otimes \alpha_2(\mathfrak{p}_2) \otimes \ldots \otimes \alpha_n(\mathfrak{p}_n)), \; \alpha_j \in \mathscr{S}(\mathbb{R}^3).$$

According to (8) we obtain

$$(13) \quad \mathrm{K}_n \{ \mathrm{S}_n(\alpha_1 \otimes \alpha_2 \otimes \ldots \otimes \alpha_n) \} = \mathrm{S}_n \left\{ \prod_{j=1}^{n} \mathrm{A}(h(p)\alpha_j(\mathfrak{p})) \right\} \omega, \qquad n = 1, 2, \ldots,$$

where we write for brevity $\mathrm{A}(h(p)\alpha_j(\mathfrak{p}))$ instead of $\mathrm{A}(\{ 0, h(p)\alpha_j(\mathfrak{p}), 0, \ldots \})$. We denote the linear submanifold of $\mathscr{L}$ defined by (12) by $\mathscr{L}_0 \subset \mathscr{L}$. $\mathscr{L}_0$ is dense in $\mathscr{L}$ with respect to τ. Then we have

PROPOSITION 1. — *The relative identification operator* $\mathrm{K} : \mathscr{H}^0 \mapsto \mathscr{H}$ *is given on the* (*dense*) *submanifold* ima $(\mathrm{J}^0 \restriction \mathscr{L}_0)$ *of* dom K *by formula* (13).

Proof. — Obvious by the preceding arguments of this paragraph. ■

According to (13), $\mathrm{K} \restriction \mathrm{ima}\,(\mathrm{J}^0 \restriction \mathscr{L}_0)$, hence K itself, is already uniquely determined by the field operators $\mathrm{A}(h(p)\alpha(\mathfrak{p}))$, $\alpha \in \mathscr{S}(\mathbb{R}^3)$. But the Poincaré covariance property of the field $\mathrm{A}(\cdot)$ implies a strong connection between these field operators. Namely, we have

LEMMA 2. — *Let* $\alpha \in \mathscr{S}(\mathbb{R}^3)$, $f \in \mathscr{S}(\mathrm{R}^4)$. *Then*

$$(14) \qquad \mathrm{A}(\hat{\alpha}(\mathfrak{p})\hat{f}(p)) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} \alpha(\mathfrak{a}) \mathrm{U}_{\mathfrak{a}} \mathrm{A}(f) \mathrm{U}_{-\mathfrak{a}} d\mathfrak{a} ,$$

where $\hat{\alpha}$ *denotes the spatial Fourier transform of* α *and* $\hat{f}$ *denotes the* 4-*dimensional Fourier transform of* f. *The integral on the right hand side is weakly convergent for vectors* $u, v \in \mathscr{D} =$ ima J (*domain of the field operators* $\mathrm{A}(\cdot)$).

Proof. — From the covariance property of $\mathrm{A}(\cdot)$ we obtain

$$\mathrm{A}(\mathrm{V}_{\mathfrak{a}} f) = \mathrm{U}_{\mathfrak{a}} \mathrm{A}(f) \mathrm{U}_{-\mathfrak{a}} ,$$

where, as usual, $(\mathrm{V}_{\mathfrak{a}} f)(x) = f(x_0, \mathfrak{x} - \mathfrak{a})$. Further we obtain

$$\mathrm{A}\left(\int_{\mathbb{R}^3} \alpha(\mathfrak{a}) \mathrm{V}_{\mathfrak{a}} f d\mathfrak{a} \right) = \int_{\mathbb{R}^3} \alpha(\mathfrak{a}) \mathrm{U}_{\mathfrak{a}} \mathrm{A}(f) \mathrm{U}_{-\mathfrak{a}} d\mathfrak{a} .$$

But $(V_{\mathfrak{a}} f)^{\wedge}(p) = e^{-i(\mathfrak{a},p)}\hat{f}(p)$ and

$$\int_{\mathbb{R}^3} \alpha(\mathfrak{a})(V_{\mathfrak{a}}f)^{\wedge}(p)d\mathfrak{a} = \int_{\mathbb{R}^3} \alpha(\mathfrak{a})e^{-i(\mathfrak{a},p)}d\mathfrak{a}\,\hat{f}(p) = (2\pi)^{3/2}\hat{\alpha}(\mathfrak{p})\hat{f}(p).$$

This concludes the proof. ■

The relation (14) implies that the field operators $A(h(p)\alpha(\mathfrak{p}))$ are uniquely determined by a single field operator $A(h(p)\gamma_0(\mathfrak{p}))$, where γ_0 is multiplicative-generating in $\mathscr{S}(\mathbb{R}^3)$ (together with the representation $U_{\mathfrak{a}}$).

DEFINITION 3. — *The function $\gamma_0 \in \mathscr{S}(\mathbb{R}^3)$ is called multiplicative-generating with respect to $\mathscr{S}(\mathbb{R}^3)$, if $\{ \alpha\gamma_0 : \alpha \in \mathscr{S}(\mathbb{R}^3) \}$ is dense in $\mathscr{S}(\mathbb{R}^3)$.*

For example, if $\gamma_0 \in \mathscr{S}(\mathbb{R}^3)$ and $\gamma(\mathfrak{p}) \neq 0$ for all $\mathfrak{p} \in \mathbb{R}^3$, then γ_0 is multiplicative-generating. For example let $\gamma_0(\mathfrak{p}) = \exp(-|\mathfrak{p}|^2)$.

PROPOSITION 4. — *Let $\gamma_0 \in \mathscr{S}(\mathbb{R}^3)$ be multiplicative-generating. Then $\{ \hat{\alpha}\gamma_0 : \alpha \in \mathscr{S}(\mathbb{R}^3) \}$ is dense in $\mathscr{S}(\mathbb{R}^3)$ and*

$$A(h(p)\hat{\alpha}(\mathfrak{p})\gamma_0(\mathfrak{p})) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} \alpha(\mathfrak{a})U_{\mathfrak{a}}A(h(p)\gamma_0(\mathfrak{p}))U_{-\mathfrak{a}}d\mathfrak{a}. \tag{15}$$

Proof. Obvious. ■

Proposition 1 and Proposition 4 together lead to an explicit description of K, showing, that K is uniquely determined by $A(h(p)\gamma_0(\mathfrak{p}))$ for some multiplicative-generating function γ_0.

COROLLARY 5. — *Let $\gamma_0 \in \mathscr{S}(\mathbb{R}^3)$ be multiplicative-generating and put $B_0 := A(h(p)\gamma_0(\mathfrak{p}))$. Then $K \restriction \mathrm{ima}\,(J^0 \restriction \mathscr{L}_0)$ is explicitly given by*

$$\begin{aligned} (16)\quad & K_n\{ S_n(\hat{\alpha}_1\gamma_0 \otimes \hat{\alpha}_2\gamma_0 \otimes \cdots \otimes \hat{\alpha}_n\gamma_0) \} \\ &= S_n \prod_{j=1}^{n} \left\{ (2\pi)^{-3/2} \int_{\mathbb{R}^3} \alpha_j(\mathfrak{a})U_{\mathfrak{a}}B_0U_{-\mathfrak{a}}d\mathfrak{a} \right\} \omega, \qquad n = 1, 2, \ldots \end{aligned}$$

Proof. Obvious. ■

§ 3. CALCULATION OF THE PRE-WAVE OPERATOR

Let $\gamma_0 \in \mathscr{S}(\mathbb{R}^3)$ be multiplicative-generating and put $B_0 := A(h(p)\gamma_0(\mathfrak{p}))$. The next Proposition calculates the pre-wave operator (11).

PROPOSITION 6. — *Let $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathscr{S}(\mathbb{R}^3)$. Then*

$$\begin{aligned} (17)\quad & (e^{itH}Ke^{-itH^0})_n S_n(\hat{\alpha}_1\gamma_0 \otimes \cdots \otimes \hat{\alpha}_n\gamma_0) \\ &= \left(S_n \prod_{j=1}^{n} \left\{ (2\pi)^{-3/2} \int_{\mathbb{R}^3} f_j(t, \mathfrak{x})U_{\{-t,\mathfrak{x}\}}B_0U_{\{t,-\mathfrak{x}\}}d\mathfrak{x} \right\} \right)\omega, \end{aligned}$$

where

$$f_j(t, \mathfrak{x}) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{-it\mu(\mathfrak{p}) + i(\mathfrak{x},\mathfrak{p})} \hat{\alpha}_j(\mathfrak{p}) d\mathfrak{p} ,$$

i. e. $f_j(t, \mathfrak{x})$ *is the spatial inverse Fourier transform of* $e^{-it\mu(\mathfrak{p})}\hat{\alpha}_j(\mathfrak{p})$, *thus* $f_j(t, \mathfrak{x})$ *is a so-called smooth solution of the Klein-Gordon equation with « negative » frequencies.*

Proof. — According to Proposition 4 we have the assignment

$$e^{-it\mu(\mathfrak{p})}\hat{\alpha}_j(\mathfrak{p})\gamma_0(\mathfrak{p}) \mapsto (2\pi)^{-3/2} \int_{\mathbb{R}^3} f_j(t, \mathfrak{x}) U_{\mathfrak{x}} B_0 U_{-\mathfrak{x}} d_{\mathfrak{x}} .$$

Furthermore,

$$e^{-it\sum_{j=1}^{n}\mu(\mathfrak{p}_j)} \hat{\alpha}_1\gamma_0 \otimes \ldots \otimes \hat{\alpha}_n\gamma_0 = \bigotimes_{j=1}^{n} e^{-it\mu(\mathfrak{p}_j)}\hat{\alpha}_j(\mathfrak{p}_j)\gamma_0(\mathfrak{p}_j) = (e^{-itH^0})_n(\hat{\alpha}_1\gamma_0 \otimes \ldots \otimes \hat{\alpha}_n\gamma_0)$$

is valid, thus we obtain

$$(Ke^{-itH^0})_n S_n(\hat{\alpha}_1\gamma_0 \otimes \ldots \otimes \hat{\alpha}_n\gamma_0) = \left(S_n \prod_{j=1}^{n} \left\{ (2\pi)^{-3/2} \int_{\mathbb{R}^3} f_j(t, \mathfrak{x}_j) U_{\mathfrak{x}_j} B_0 U_{-\mathfrak{x}} d_{\mathfrak{x}_j} \right\} \right) \omega .$$

Therefore we finally obtain (17) for $\{ e^{itH}(Ke^{-itH^0}) \}_n S_n(\hat{\alpha}_1\gamma_0 \otimes \ldots \otimes \hat{\alpha}_n\gamma_0$ because of $e^{itH} = U_{\{-t,0\}}$ and $U_{\{t,0\}}\omega = \omega$. ■

Proposition 6 gives a link between the pre-wave operator $e^{itH}Ke^{-itH_0}$ and the expressions (Haag-Ruelle approximants) used in the Haag-Ruelle scattering theory for Wightman fields. That is, Haag-Ruelle's theory appears as a special part of the abstract two-space scattering theory (see H. Baumgärtel and M. Wollenberg [*4*]). The basic concept of the abstract scattering theory is given by the pre-wave operator mentioned above. Note that the identification operator K in the field-theoretic case is unbounded and not closable but densely defined whereas the identification operators appearing usually (e. g. in the non-relativistic scattering theory) are bounded.

It should be mentioned that the characteristic t-dependence of the special field operators $A(e^{it(p_0 - \mu(\mathfrak{p}))}f(p))$ occuring in the Haag-Ruelle approximants (see for example K. Hepp [*5*, p. 96]) suggests the introduction of a pre-wave operator, i. e. of an operator K. In fact, one obtains, in an intimate connection with Proposition 6, that

$$(e^{itH}Ke^{-itH_0})_n \{ S_n(h(p)\hat{\alpha}_1(\mathfrak{p}) \otimes \ldots \otimes h(p)\hat{\alpha}_n(\mathfrak{p})) \} = \left\{ S_n \prod_{\rho=1}^{n} A(e^{it(p_0 - \mu(\mathfrak{p}))}h(p)\hat{\alpha}_\rho(\mathfrak{p})) \right\} \omega$$

is valid (see for example [2, Lemma 6]). Proposition 6 shows explicitly that the pre-wave operator, i. e. also K, depends on a single field operator $A(h(p)\gamma_0(\mathfrak{p}))$ only.

Under the assumptions listed at the beginning of paragraph 1 the strong limits for $t \to \pm \infty$ of (11) exist and they turn out to be isometric (Haag-Ruelle). Because K is uniquely determined by B_0 the question arises, what properties of B_0 are decisive for the existence and isometry of the wave operators. Some authors (for example A. S. Schwartz [3]) have shown that the (sufficient) assumptions of Haag-Ruelle can be weakened. It would be nice to have some insights what properties of K resp. B_0 are necessary for the existence and isometry of the wave operators in order to attack the corresponding inverse problem.

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