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Upper bounds on the decay of correlations in SO(N)-symmetric spin systems with long range interactions

by

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ABSTRACT. — Upper bounds on the decay at large distances of the spin-spin correlation functions are derived for classical SO(N)-symmetric spin systems in one and two dimensions with $N \geqslant 2$ and long range interactions.

RÉSUMÉ. — Nous démontrons des bornes supérieures pour la décroissance à longues distances de la fonction de corrélation spin-spin pour des systèmes classiques de spin avec une symétrie SO(N) à une ou deux dimensions lorsque $N \geqslant 2$ et que l'interaction est de longue portée.

1. INTRODUCTION AND RESULTS

Two-dimensional SO(N)-symmetric classical spin systems for $N \geqslant 2$ do not have spontaneous magnetization at non-zero temperature, except if the interaction is very long range. Consequently the spin-spin correlation functions decay to zero at large distances.

Fisher and Jasnow [*1*] established a first upper bound on this decay, of the form $\log^{-1} |x|$, using Bogoliubov's inequality.

(*) Chercheur Contrat DRET n° 82/1029.

By a different technique, McBryan and Spencer [2] obtained a better upper bound, having the form of the power low $|x|^{-(k/\beta)}$ (where k is a constant and $\beta = 1/\mathrm{T}$ the inverse temperature).

Schlossman [3] has extended this result to classical systems with a compact connected Lie group of symmetries. His approach is based on the work by Dobrushin and Schlossman [4].

All these upper bounds concern systems with short range interactions (typically finite range interactions or with an exponential decay at large distances). To report on a number of results concerning this problem in the case of long range interactions is the purpose of the present paper.

Our approach starts with the same estimate used in [2], which following some ideas of [4], [5] and [6] we control by the introduction of small rotations (in our approach, however, these rotations are imaginary). The problem is finally reduced to the computation of numerical series. The method applies, even more directly, to one-dimensional systems, which for this reason are also included in our discussion.

In this way we obtain upper bounds on the decay of correlations for all the interactions for which the absence of spontaneous magnetization has been proved.

To describe the classical SO(N)-symmetric spin systems, we consider an infinite square lattice with sites labelled by indices $x = (x_1, \ldots, x_d) \in \mathbb{Z}^d$ where d is the space dimension. To each site $x \in \mathbb{Z}^d$, we associate a N-dimensional vector $\vec{s}_x$, with $\mathrm{N} \geqslant 2$, of unit length $\|\vec{s}_x\| = 1$. The spin-spin correlation function, at inverse temperature β, is given by

$$\langle \vec{s}_0 \vec{s}_x \rangle (\beta) = \mathrm{Z}^{-1}(\beta) \int \prod_x d\Omega_\mathrm{N}(\vec{s}_x) \vec{s}_0 \vec{s}_x \exp\left\{ \beta \sum_{x,y} \mathrm{J}(|x-y|) \vec{s}_x \vec{s}_y \right\}$$

$$\mathrm{Z}(\beta) = \int \prod_x d\Omega_\mathrm{N}(\vec{s}_x) \exp\left\{ \beta \sum_{x,y} \mathrm{J}(|x-y|) \vec{s}_x \vec{s}_y \right\}$$

where $\mathrm{J}(r)$ is a real function defined for $r \geqslant 1$, which describes the interaction and $d\Omega_\mathrm{N}$ is the invariant measure on the unit sphere of the N-dimensional Euclidean space. The formula giving $\langle \vec{s}_0 \vec{s}_x \rangle (\beta)$ is to be interpreted as the thermodynamic limit $\Lambda \to \mathbb{Z}^d$, of the corresponding finite volume quantities $\langle \vec{s}_0 \vec{s}_x \rangle_\Lambda(\beta)$, defined by the same expression but with sites restricted to a finite box Λ.

THEOREM 1. — *We assume that the interaction potential verifies*

$$|\mathrm{J}(r)| \leqslant \mathrm{A} r^{-\alpha}$$

where A *is a constant. Then, the spin-spin correlation functions are bounded according to* *a) in dimension* $d = 1$ *and if* $\alpha > 2$

$$|\langle \vec{s}_0 \vec{s}_x \rangle(\beta)| \leqslant B_0 |x|^{-(\alpha-1)}$$

b) in dimension $d = 2$ *and if* $\alpha > 4$

$$|\langle \vec{s}_0 \vec{s}_x \rangle(\beta)| \leqslant B_1 |x|^{-\lambda_1(\beta)}$$

c) in dimension $d = 2$ *and if* $\alpha = 4$

$$|\langle \vec{s}_0 \vec{s}_x \rangle(\beta)| \leqslant B_2(\log |x|)^{-\lambda_2(\beta)}$$

d) in dimension $d = 1$ *and if* $\alpha = 2$ *and* β *is small enough*

$$|\langle \vec{s}_0 \vec{s}_x \rangle(\beta)| \leqslant B_3 |x|^{-\lambda_3}$$

where $B_0, B_1, B_2, B_3, \ldots, \lambda_3$, *are strictly positive constants and* $\lambda_1(\beta)$, $\lambda_2(\beta)$ *strictly positive non increasing functions of* β *proportional to* β^{-1} *for large* β.

e) in dimensional $d = 1$ *and if* $J(r)$ *decays exponentially when* $r \to \infty$ *also* $|\langle \vec{s}_0 \vec{s}_x \rangle(\beta)|$ *decays exponentially for large* $|x|$.

Under conditions *a*), *b*) or *c*) there is no spontaneous magnetization and we see that in general the spin-spin correlation functions have then a power law decaying upper bound.

For the special case $d = 2$ and $\alpha = 4$, our method does not lead to a power law decay (see the proof below in particular formulae (16), (19)). We do not know whether the upper bound stated in *c*) could indicate a different behaviour in this case.

On the other hand, for ferromagnetic systems with the interaction $J(r) = r^{-\alpha}$ for large r, the existence of a spontaneous magnetization at low temperatures has been proved in dimension $d = 1$ if $1 < \alpha < 2$ and in dimension $d = 2$ if $2 < \alpha < 4$ [*7*], [*8*], (the conditions $\alpha > 1$ in $d = 1$, and $\alpha > 2$, in $d = 2$, are needed for the existence of the thermodynamic free energy). In the case $d = 1$ and $\alpha = 2$ the absence of a spontaneous magnetization when $N \geqslant 2$ has recently been proved by Simon [*12*].

This last case however cannot be treated even with the improvements introduced in the proof of Theorem 2, although we can derive a power law decay at high temperatures (statement *d*)). We notice that the power law decay of the correlations at high temperatures is a general property of the systems under consideration, as has been shown by Brydges, Fröhlich and Spencer [*9*] (see also [*10*]).

Statement *e*), which is well-known, has a simple proof in our context. On the other hand we remark that if the interaction is not of a short range we cannot expect an exponential decay of the correlations, at least for $N=2$ where Ginibre inequalities (by taking all the interactions equal to zero, except that between the points 0 and x which is kept constant) imply

$$\langle \vec{s}_0 \vec{s}_x \rangle(\beta) \geqslant \frac{1}{2} \beta J(|x|) + O(J^3)$$

for small $J(|x|)$.

Finally, we notice that our bounds hold for all $N \geqslant 2$ and, therefore, taking into account correlation inequalities which compare $N = 2$ with $N \geqslant 3$ (see [*11*]), they cannot be better than the upper bounds valids for $N = 2$.

Next we consider interactions which are near to the special cases $d = 1$, $\alpha = 2$, and $d = 2$, $\alpha = 4$.

THEOREM 2. — a) *In dimension $d = 1$ we assume that*

$$| J(r) | \leqslant A r^{-2} (\log^{(p)} r)^{-1}$$

where $p \geqslant 1$ is an integer and

$$\log^{(1)} r = \max \{ 1, \log r \}, \qquad \log^{(p)} r = \max \{ 1, \log \log^{(p-1)} r \}.$$

b) *In dimension $d = 2$ we assume that*

$$| J(r) | \leqslant A r^{-4} \log^{(2)} r \ldots \log^{(p)} r$$

with $p \geqslant 2$.

Then, the spin-spin correlation function is bounded by

$$| \langle \vec{s}_0 \vec{s}_x \rangle (\beta) | \leqslant B_4 (\log^{(p)} | x |)^{-\lambda_4(\beta)}$$

where B_4, is a strictly positive constant and $\lambda_4(\beta)$ a strictly positive non increasing function proportional to β^{-1} for β large.

In dimension $d = 2$, condition *b*) has been considered by Pfister [*5*] proving the absence of symmetry breaking. If $J(r) = r^{-4} \log r$ for large r a first order phase transition exists for ferromagnetic systems [*5*].

A similar approach can also be developped to study the decay of Wilson-loop expectations, in three-dimensional U(1)-gauge theories on a lattice, the results of which will be reported elsewhere. (Long-range pure gauge interactions appear in the treatment of a lattice theory of gauge and fermion fields).

After finishing this work we become aware of the papers by Bonato *et al.* [*13*] and Ito [*14*] where related results were obtained for classical and quantum models with long range interactions, including the systems considered here. Their bounds are improved in our Theorems 1 and 2. For example in the case of statement *b*) of Theorem 1 we get an inverse power law decay while in [*13*] [*14*] temperature independent bounds of the form $\log^{-1} | x |$ are proved. Our upper bounds in the case of statement *b*) of Theorem 1 are the same type as those previously established [*2*] [*3*] for the case of finite range interaction and solve the conjecture mentioned in [*14*]. We notice also the important result by Fröhlich and Spencer [*15*] proving that for $N = 2$ and nearest neighbour interactions the correlation functions $\langle \vec{s}_0 \vec{s}_x \rangle (\beta)$ have precisely an inverse power law decay at low temperatures.

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2. PROOF OF THEOREM 1

Our starting point is the following estimate due to McBryan and Spencer [2].

LEMMA 1. — *Let* R *denote the point on the* x_1 *axis with* $x_1 = \mathrm{R}$. *For any given family of real numbers* $\{ a_x \}$ *indexed by the sites* $x \in \mathbb{Z}^d$, *the following inequality holds*

$$\langle \vec{s}_0 \vec{s}_{\mathrm{R}} \rangle (\beta) \leqslant \exp \left\{ -(a_{\mathrm{R}} - a_0) + \beta \sum_{x,y} | \mathrm{J}(|x-y|) | (\cosh(a_x - a_y) - 1) \right\} \quad (1)$$

We refer the reader to [2] for the proof of the Lemma. Next we control this estimate by means of Lemmas 2 and 3.

LEMMA 2. — *We assume that the interaction potential verifies* $| \mathrm{J}(r) | \leqslant \mathrm{A} r^{-\alpha}$ *for all* $r \geqslant 1$. *Then, for any given sequence* $\{ a_n \}$ *of real numbers,*

a) *in dimension* $d = 1$

$$| \langle \vec{s}_0 \vec{s}_{\mathrm{R}} \rangle (\beta) | \leqslant \exp \left\{ -(a_{\mathrm{R}} - a_0) + \beta \mathrm{A}_1 \sum_{n=0}^{\mathrm{R}} \mathrm{Q}_{\alpha}(n) \right\} \quad (2)$$

b) *in dimension* $d = 2$

$$| \langle \vec{s}_0 \vec{s}_{\mathrm{R}} \rangle (\beta) | \leqslant \exp \left\{ -(a_{\mathrm{R}} - a_0) + \beta \mathrm{A}_2 \sum_{n=0}^{\mathrm{R}} n \mathrm{Q}_{\alpha-1}(n) \right\} \quad (3)$$

where A_1 *and* A_2 *are constants and*

$$\mathrm{Q}_{\mu}(n) = \sum_{m=1}^{\infty} m^{-\mu} (\cosh (a_{n+m} - a_n) - 1) \quad (4)$$

If the dimension $d = 1$, we take $a_x = a_{|x|}$. Then, using Lemma 1, we obtain

$$| \langle \vec{s}_0 \vec{s}_{\mathrm{R}} \rangle (\beta) | \leqslant \exp \Big\{ -(a_{\mathrm{R}} - a_0)$$
$$+ \beta \sum_{n=0}^{\mathrm{R}} \sum_{m=1}^{\infty} 2(| \mathrm{J}(m) | + | \mathrm{J}(m+2n) |) (\cosh (a_{n+m} - a_n) - 1) \Big\} \quad (5)$$

and, since $| \mathrm{J}(m) | \leqslant \mathrm{A} m^{-\alpha}$, part *a*) of Lemma 2 follows.

If the dimension $d = 2$, we introduce the notation C_n to represent the set of sites on the boundary of a square of side $2n + 1$, centered at the origin. We take $a_x = a_n$ whenever $x \in C_n$. Then, using Lemma 1, we obtain

$$|\langle \vec{s}_0 \vec{s}_R \rangle (\beta)| \leqslant \exp\left\{ -(a_R - a_0) + \beta \sum_{n=0}^{R} \sum_{m=1}^{\infty} \left(\sum_{\substack{x \in C_n \\ y \in C_{n+m}}} |J(|x-y|)| (\cosh(a_{n+m} - a_n) - 1) \right) \right\} \quad (6)$$

Assuming that $|J(r)| \leqslant Ar^{-\alpha}$ with $\alpha \geqslant 2$, we have

$$\begin{aligned} \sum_{\substack{x \in C_n \\ y \in C_{n+m}}} |J(|x-y|)| &\leqslant 4(2n+1) \sum_{y \in C_{n+m}} A|x-y|^{-\alpha} \\ &\leqslant 4(2n+1) \sum_{l=0}^{\infty} 8A(\sqrt{m^2+l^2})^{-\alpha} \\ &\leqslant 4(2n+1) \sum_{l=0}^{\infty} 16A(m+l)^{-\alpha} \\ &\leqslant 4(2n+1) 16Am^{-(\alpha-1)} \end{aligned} \quad (7)$$

From (7) part *b*) of Lemma 2 follows.

LEMMA 3. — *Let* $Q_\mu(n)$ *denote the series defined by* (4), *for* $n \geqslant 1$, *and let* b_K *and* c_ρ *be the positive numbers defined by* $\cosh(K\theta) - 1 \leqslant b_K(K\theta)^2$ *for all* $|\theta| \leqslant 1$ *and* $C_\rho = \sum_{m=1}^{\infty} m^{-\rho}$ *for* $\rho > 1$.

If we take $a_j - a_{j-1} = Kj^{-1}$ *and* $K \leqslant \mu - 1$, *then*

$$Q_\mu(n) \leqslant \frac{b_K}{3-\mu} K^2 n^{-(\mu-1)} + 2^{\mu-1} n^{-(\mu-1)} \log n \quad \text{if} \quad 2 < \mu < 3 \quad (8)$$

$$Q_3(n) \leqslant (b_K K^2 + 4) n^{-2} \log n \quad \text{if} \quad \mu = 3 \quad (9)$$

$$Q_\mu(n) \leqslant b_K c_{\mu-2} K^2 n^{-2} + 2^{\mu-1} n^{-(\mu-1)} \log n \quad \text{if} \quad \mu > 3 \quad (10)$$

If we take $a_j - a_{j-1} = K(j \log j)^{-1}$ *and* $K \leqslant 1$, *then*

$$Q_2(n) \leqslant (b_K K^2 + 1)(n \log n)^{-1} + Q_2'(n) \quad (11)$$

$$Q_3(n) \leqslant b_K K^2 n^{-2} (\log n)^{-1} + (n \log n)^{-2} + Q_3'(n) \quad (12)$$

where the rests $Q_2'(n)$ *and* $Q_3'(n)$ *decay, for large n, faster than the precedent terms in the sum.*

The proof of Lemma 2 is based on a decomposition of the series into a finite sum, where we can use the estimate $\cosh(K\theta) - 1 \leqslant b_K(K\theta)^2$, for $|\theta| \leqslant 1$, and a rest, which is an infinite series, where it is enough to use the estimate $\cosh(K\theta) \leqslant \exp(K\theta)$.

We consider first the case $a_j - a_{j-1} = Kj^{-1}$ and we write

$$Q_\mu(n) = S_\mu(n) + T_\mu(n) \tag{13}$$

where

$$S_\mu(n) = \sum_{m=1}^{n} \frac{1}{m^\mu}\left(\cosh\left(\frac{K}{n} + \ldots + \frac{K}{n+m-1}\right) - 1\right)$$

$$T_\mu(n) = \sum_{m=n+1}^{\infty} \frac{1}{m^\mu}\left(\cosh\left(\frac{K}{n} + \ldots + \frac{K}{n+m-1}\right) - 1\right)$$

We observe that

$$S_\mu(n) \leqslant \sum_{m=1}^{n} \frac{1}{m^\mu}\left(\cosh\frac{Km}{n} - 1\right)$$

$$\leqslant b_K \frac{K^2}{n^2} \sum_{m=1}^{n} \frac{1}{m^{\mu-2}}$$

$$T_\mu(n) \leqslant \sum_{m=n}^{\infty} \frac{1}{m^\mu} \exp K(\log(n+m) - \log n)$$

$$\leqslant \sum_{m=n}^{\infty} \frac{1}{m^\mu}\left(\frac{2m}{n}\right)^K$$

and therefore

$$S_2(n) \leqslant b_K K^2 n^{-1} \tag{14}$$

$$S_\mu(n) \leqslant \frac{b_K}{3-\mu} K^2 n^{-(\mu-1)} \quad \text{if} \quad 2 < \mu < 3 \tag{15}$$

$$S_3(n) \leqslant b_K K^2 n^{-2} \log n \tag{16}$$

$$S_\mu(n) \leqslant b_K c_{\mu-2} K^2 n^{-2} \quad \text{if} \quad \mu > 3 \tag{17}$$

$$T_\mu(n) \leqslant 2^{\mu-1} n^{-(\mu-1)} \log n \quad \text{if} \quad K = \mu - 1 \tag{18}$$

$$T_\mu(n) \leqslant \frac{2^{\mu-1}}{\mu-1-K} n^{-(\mu-1)} \quad \text{if} \quad K < \mu - 1 \tag{19}$$

From this, the inequalities (8), (9), and (10) of Lemma 3 follow.

We next study the case $a_j - a_{j-1} = \mathrm{K}(j \log j)^{-1}$. We write the decomposition (13) in the form

$$\mathrm{S}_\mu(n) = \sum_{m=1}^{n\log n} \frac{1}{m^\mu}\left(\cosh\left(\frac{\mathrm{K}}{n\log n} + \ldots + \frac{\mathrm{K}}{(n+m-1)\log(n+m-1)}\right) - 1\right)$$

$$\mathrm{T}_\mu(n) = \sum_{m=n\log n+1}^{\infty} \frac{1}{m^\mu}\left(\cosh\left(\frac{\mathrm{K}}{n\log n} + \ldots + \frac{\mathrm{K}}{(n+m-1)\log(n+m-1)}\right) - 1\right)$$

We observe that

$$\mathrm{S}_\mu(n) \leqslant \sum_{m=1}^{n\log n} \frac{1}{m^\mu} b_\mathrm{K}\left(\frac{\mathrm{K}m}{n\log n}\right)^2$$

$$\leqslant \frac{b_\mathrm{K}\mathrm{K}^2}{(n\log n)^2} \sum_{m=1}^{n\log n} \frac{1}{m^{\mu-2}}$$

$$\mathrm{T}_\mu(n) \leqslant \sum_{m=n\log n}^{\infty} \frac{1}{m^\mu} \exp \mathrm{K}\,(\log\log((n+m)\log(n+m)) - \log\log(n\log n)$$

$$\leqslant \sum_{m=n\log n}^{\infty} \frac{1}{m^\mu}\left(\frac{\log(2m)}{\log(2^n\log(2n))}\right)^{\mathrm{K}}$$

Therefore, restricting ourselves to the case $\mathrm{K} \leqslant 1$

$$\mathrm{S}_2(n) \leqslant b_\mathrm{K}\mathrm{K}^2(n\log n)^{-1} \tag{20}$$

$$\mathrm{S}_3(n) \leqslant b_\mathrm{K}\mathrm{K}^2(n\log n)^{-2}\log(n\log n)$$
$$\leqslant b_\mathrm{K}\mathrm{K}^2 n^{-2}(\log n)^{-1} + \mathrm{S}'_3(n) \tag{21}$$

$$\mathrm{T}_2(n) \leqslant (n\log n)^{-1} + \mathrm{T}'_2(n) \tag{22}$$

$$\mathrm{T}_3(n) \leqslant (n\log n)^{-2} + \mathrm{T}'_3(n) \tag{23}$$

where, the rests S'_3, T'_2 and T'_3 decrease, for large n, faster than the precedent terms in the sum. From this, inequalities (11) and (12) follow.

This ends the proof of Lemma 3.

Next we conclude the proof of Theorem 1.

In the case $d = 1$ and $\alpha > 2$, we take $a_j - a_{j-1} = (\alpha - 1)j^{-1}$. From Lemma 3, (inequalities (8), (9) and (10)), we see that the series

$$\sum_{n=1}^{\infty} \mathrm{Q}_\alpha(n)$$

is convergent (the term $Q_\alpha(0)$ is always finite, as can be seen by direct computation). On the other hand, $a_R - a_0 \cong (\alpha - 1) \log R$. Lemma 2 (inequality (2)) implies then the statement *a*) of Theorem 1.

In the case $d = 1$ and $d = 2$, we take $a_j - a_{j-1} = Kj^{-1}$, with $K \leqslant 1$. Then from (14) and (19)

$$\sum_{n=1}^{R} Q_2(n) \leqslant \sum_{n=1}^{R} (b_K K^2 + 1)n^{-1} \leqslant (b_K K^2 + 1) \log R$$

Using inequality (2) we have

$$| \langle \vec{s}_0 \vec{s}_R \rangle (\beta) | \leqslant \exp \{ (- K + \beta(b_K K^2 + 1) \log R \}$$

which implies a power law decay for $\langle \vec{s}_0 \vec{s}_R \rangle (\beta)$ only when

$$- K + \beta(b_K K^2 + 1) < 0$$

that is, if β is small enough. This proves the statement *d*) of Theorem 1.

In the case $d = 1$ and if we assume that the interaction potential $| J(r) |$ is bounded by $\exp - \lambda_0 r$, we take $a_j - a_{j-1} = K$ with $K < \lambda_0$. From (5) the statement *e*) of Theorem 1 is established in a similar way.

In the case $d = 2$ and $\alpha > 4$, we take $a_j - a_{j-1} = Kj^{-1}$ and $K \leqslant 1$ (for instance). Then, the inequality (8) of Lemma 3 implies

$$\sum_{n=1}^{R} nQ_{\alpha-1}(n) \leqslant \sum_{n=1}^{R} b_1 c_{\alpha-3} K^2 n^{-1} + n^{-(\alpha-3)}$$
$$\leqslant b_1 c_{\alpha-3} K^2 \log R + c_{\alpha-3}$$

From inequality (3) of Lemma 2 we get

$$| \langle \vec{s}_0 \vec{s}_R \rangle (\beta) | \leqslant \exp \{ (- K + \beta K^2 b_1 c_{\alpha-3} A_2) \log R + \beta A_2 c_{\alpha-3} \}$$

which implies a power law decay provided that $- K + \beta K^2 (b_1 c_{\alpha-3} A_2) < 0$. This inequality is satisfied for allowed values of K by taking $K < 1$ if $\beta \leqslant B_0 (b_1 c_{\alpha-3} A_2)^{-1}$ and $K < B_0 \beta^{-1}$ if $\beta > B_0$.
This proves the statement *b*) of Theorem 1.

Finally, in the case $d = 2$ and $\alpha = 4$, we take $a_j - a_{j-1} = K(j \log j)^{-1}$ for $j \geqslant 2$, $a_1 - a_0 = K$, and $K \leqslant 1$. Then the inequality (12) of Lemma 3 implies

$$\sum_{n=1}^{R} nQ_{\alpha-1}(n) \leqslant \sum_{n=1}^{R} b_1 K^2 (n \log n)^{-1} + n^{-1} (\log n)^{-2} + nQ'_3(n)$$
$$\leqslant b_1 K^2 \log \log R + c'$$

where, since the last two series converge, c' is a constant. On the other hand, in the case, $a_R - a_0 \cong \log \log R$, and from (3) we get

$$|\langle \vec{s}_0 \vec{s}_R \rangle (\beta)| \leqslant \exp \{ (-K + \beta K^2 b_1 A_2) \log \log R + c' A_2 \}$$

This proves the statement *c*) of Theorem 1.

3. PROOF OF THEOREM 2

For simplicity we shall restrict the proof of Theorem 2 to the case $p = 2$. In this case one takes

$$a_j - a_{j-1} = K(j \log^{(1)} j \log^{(2)} j)^{-1} \tag{24}$$

and therefore

$$a_R - a_0 \cong K \log^{(3)} R$$

For $p \neq 2$, the proof goes along the same lines by taking the natural generalization of (24). On the other hand, in order to simplify the notations, we make the convention that the inequalities which will be written should to be understood as inequalities among the larger terms, neglecting the contribution of terms with a faster decay at infinity.

To deal with the assumptions of Theorem 2 we have to modify appropriately Lemmas 2 and 3.

By arguying as in Lemma 2 we get

$$|\langle \vec{s}_0 \vec{s}_R \rangle (\beta)| \leqslant \exp \left\{ -(a_R - a_0) + \beta A_1 \sum_{n=0}^{R} Q_\phi(n) \right\} \tag{26}$$

if the dimension $d = 1$, and

$$|\langle \vec{s}_0 \vec{s}_R \rangle (\beta)| \leqslant \exp \left\{ -(a_R - a_0) + \beta A_2 \sum_{n=0}^{R} n Q_\phi(n) \right\} \tag{27}$$

if the dimension $d = 2$, where A_1 and A_2 are positive constants and

$$Q_\phi(n) = \sum_{m=1}^{\infty} \phi(m) (\cosh (a_{n+m} - a_n) - 1) \tag{28}$$

Under the hypothesis of Theorem 2, with $p = 2$, the function ϕ is defined by

$$\phi(m) = m^{-2} (\log^{(2)} m)^{-1} \quad \text{if} \quad d = 1 \tag{29}$$

$$\phi(m) = m^{-3} \log^{(2)} m \quad \text{if} \quad d = 2 \tag{30}$$

As in proof of Lemma 3 we write

$$Q_\phi(n) = S_\phi(n) + T_\phi(n) \tag{31}$$

and we assume that the summation index m runs, in the finite sum $S_\phi(n)$ from 1 to $q(n) = n \log^{(1)} n \log^{(2)} n$. $T_\phi(n)$ is the remaining part of $Q_\phi(n)$ in which m runs from $q(n) + 1$ to infinity.

If the function ϕ is given by (29), we have

$$S_\phi(n) \leqslant \sum_{m=1}^{q(n)} \frac{b_K K^2}{m^2 \log^{(2)} m} \left(\frac{Km}{n \log^{(1)} n \log^{(2)} n} \right)^2$$

$$\leqslant b_K K^2 (n \log^{(1)} n \log^{(2)} n)^{-1}$$

$$T_\phi(n) \leqslant \sum_{m=q(n)}^{\infty} \frac{1}{m^2 \log^{(2)} m} \left(\frac{\log^{(2)} m}{\log^{(2)} n} \right)^K$$

$$\leqslant \frac{1}{\log^{(2)} n} \sum_{m=q(n)}^{\infty} \frac{1}{m^2}$$

$$\leqslant (n \log^{(1)} n)^{-1} (\log^{(2)} n)^{-2}$$

Hence

$$\sum_{n=1}^{R} S_\phi(n) \leqslant b_K K^2 \log^{(3)} R$$

$$\sum_{n=1}^{\infty} T_\phi(n) < +\infty$$

Then, the statement *a*) of Theorem 2 follows from (25) and (26). If the function ϕ is given by (30), we have

$$S_\phi(n) \leqslant \sum_{m=1}^{q(n)} \frac{\log^{(2)} m}{m^3} b_K \left(\frac{Km}{n \log^{(1)} n \log^{(2)} n} \right)^2$$

$$\leqslant \frac{b_K K^2}{(n \log^{(1)} n)^2 \log^{(2)} n} \sum_{m=1}^{q(n)} \frac{1}{m}$$

$$\leqslant b_K K^2 n^{-2} (\log^{(1)} n \log^{(2)} n)^{-1}$$

$$T_\phi(n) \leqslant \sum_{m=q(n)}^{\infty} \frac{\log^{(2)} m}{m^3} \left(\frac{\log^{(2)} m}{\log^{(2)} n} \right)^K$$

$$\leqslant \frac{1}{\log^{(2)} n} \sum_{m=q(n)}^{\infty} \frac{(\log^{(2)} m)^2}{m^3}$$

$$\leqslant (n \log^{(1)} n \log^{(2)} n)^{-2} \log^{(2)} n$$

Hence

$$\sum_{n=1}^{R} n S_\phi(n) \leqslant b_K K^2 \log^{(3)} R$$

$$\sum_{n=0}^{\infty} n T_\phi(n) < +\infty$$

Then, the statement *b*) of Theorem 2 follows from (25) and (27).

This ends the proof of Theorem 2.

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