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A singular lagrangian model for N-body relativistic interactions

by

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ABSTRACT. — We reformulate a multitemporal model for N-particles by means of a constrained system described by a singular lagrangian. We deduce the canonical transformation adapted to second-class constraints and discuss the quantization of the model.

RÉSUMÉ. — On reformule un modèle à plusieurs temps pour N particules sous forme d'un système avec contraintes décrit par un Lagrangien singulier. On obtient la transformation canonique adaptée aux contraintes de deuxième espèce et on discute la quantification du modèle.

I. INTRODUCTION

During the last years the problem of the instantaneous action-at-a distance Relativistic Dynamics has received increasing attention [*1*]. This has essentially developed along the following approaches: *a*) Predictive Relativistic Mechanics (P.R.M.) [*2*] [*3*]. *b*) Constrained Systems (C.S.) [*4*]. The construction of explicit models depends on the choice of the set of constraints, and this can be made in several ways. One can use the Todorov-Komar approach with N first class constraints—mass shell constraints—(N = number of particles) [*5*] [*6*] [*7*], but this approach, as has been recognized by several authors [*8*] [*9*] [*10*], is gauge dependent, i. e. dynamically incomplete. Another possibility consists in giving one first class constraint—the mass shell condition of the system—and $2(N-1)$ second class constraints [11] [12] [13] [14] [15] or, as recently suggested, one can give 2N second class constraints [16] [17], one of them depending explicitly on the evolution parameter. *c*) Canonical realizations of Poincaré Group [18]. Historically this was the first. The first explicit realizations proposed [19] [20] were not physically relevant because one could not construct world lines. Recently by means of constrained systems one has given physical sense to this realizations, relating the canonical coordinates of the realization and the physical positions of the world lines [21] [22].

It has also been made clear that there are deep connections between different approaches; in refs. [23] [24] [25] one can see the relation between approaches *a*) and *b*), while the relation between approaches *b*) and *c*) is given in refs. [21] [22] [25] [26].

In all cases the cluster decomposition property [20] is the most difficult problem. Some particular solutions in closed form at quantum level [27] are known and only recently a perturbative solution [28] [29] using the Todorov-Komar approach has been developed. Another possible way to attack this problem was suggested recently by different authors [30] [17] by means of an appropriate set of 2N second class constraints.

In this work, we put aside the problem of separability and we give a very simple model of a non-separable interaction, whose principal virtue is the simplicity, which enables us to further relate the different formulations. It is also very easy to obtain the quantization of the model and can be useful for phenomenological applications for the bound state problem.

The model presented in this paper is a reformulation of the multitemporal model of references [31] [32] for energy independent interactions. It is based on a singular Lagrangian and $2N-1$ constraints. It is interesting to emphasize that this model does not give the usual Hamiltonians or mass shell constraints.

The organization of the paper is as follows. In section 2 we give the Lagrangian and the constraints. In section 3 we work out explicitly the Shanmugadhasan transformation adapted to the second class constraints which enables us to discuss in section 4 the weak quantization of the model and make some comments about other possible types of quantization. We add two appendices at the end which complete the text.

II. THE LAGRANGIAN FUNCTION

In a previous paper [14] written by some of us we carefully analysed the structure of the two-body models existing in the literature and came to the conclusion that, in spite of the variety of Lagrangians giving rise to the same set of constraints—this variety owing to the different possible sets of primary constraints—, all of them reduce to the same functional form when the variables are made to satisfy the constraint equations. Also, this form turned out to be a very simple one:

$$\mathscr{L} = - \sqrt{- \mathrm{U}(r^2)\frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2)}\,. \tag{2.1}$$

where U is a scalar function depending on the relative separation. The constraints for these model were

$$\langle \dot{x}, r \rangle = 0 \qquad \langle \dot{x}, \dot{r} \rangle = 0 \tag{2.2}$$

with $x^\mu = \frac{1}{2}(x_1^\mu + x_2^\mu)$ the C. M. coordinate, and $r^\mu = x_2^\mu - x_1^\mu$.

In all cases the masses of the particles were forced to be equal. In particular, the DGL model, which explicitly allows for different masses of the particles, has to be considered in the region $m_1 = m_2$ in order to fulfill (2.1) and (2.2).

This formulation of the two-body problem immediately suggests a generalization for more particles. The Lagrangian would become

$$\mathscr{L} = - \sqrt{- \mathrm{U}(r_k^\mu)\frac{l}{\mathrm{N}}\sum_{i=1}^{\mathrm{N}} \dot{x}_i^2} \tag{2.3}$$

where U is a scalar function of the relative variables $r_k^\mu = x_k^\mu - x_1^\mu$ $(k=2,\ldots,\mathrm{N})$, and where all variables are subjected to the $2\mathrm{N}-2$ constraints.

$$\langle \dot{x}, r_k \rangle = 0 \qquad \langle \dot{x}, \dot{r}_k \rangle = 0 \qquad k = 2 \ldots \mathrm{N} \tag{2.4}$$

x^μ being, like before, the CM coordinate

$$x^\mu = \frac{l}{\mathrm{N}}(x_1^\mu + \ldots + x_\mathrm{N}^\mu)$$

This proposal based on (2.3) and (2.4) is not subjected to contradiction and constitutes a novel point of view of the N-particle problem. Indeed, by use of the constraints (2.4) one reduces to $6\mathrm{N} + 2$ the number of independent degrees of freedom, one more being eliminated—at the Lagrangian level—by a gauge fixing constraint (see equation (3.7) below) whose existence is guaranteed by the homogeneous (of degree one) character of (2.3). This further reduces to $6\mathrm{N} + 1$ the number of independent degrees of freedom, which can be identified with positions and velocities of the particles with respect to a certain inertial observer plus a parameter, t_0, giving for example the CM initial time. It can be easily seen, moreover, that this t_0 is non-essential parameter in the sense that, upon elimination of the evolution parameter λ, the positions and velocities of the particles depend only on increments $t-t_0$. This is ultimately due to the existence of a CM variable x^μ. Thus, finally, our model remains with the right number of independent degrees of freedom, i. e. 6N.

One could try to slightly generalize the Lagrangian (2.3) by inserting constant weights $\alpha_i \left(\sum_{i=1}^{N} \alpha_i = 1 \right)$:

$$\mathscr{L}' = - \sqrt{- \mathrm{U}(r_k^\mu) \sum_{i=1}^{N} \alpha_i \dot{x}_i^2} \tag{2.5}$$

This, however, may lead to trouble as we see in appendix A, where it is shown that $\alpha_1 \neq \alpha_2$, for $\mathrm{N} = 2$, allows for the existence of tachyonic states of motion in a two-particle harmonic oscillator.

On the other hand, the introduction of such weights does not even allow to include particles with different masses in the model. In reference [*31*] it is discussed the way in which one should identify the masses of the particles within the multitemporal version of this model, even letting α_i be constants of motion $\alpha_i = \frac{\langle \mathrm{P} p_i \rangle}{\mathrm{P}^2}$ instead of *a priori* constants. There is, therefore, no point in considering (2.5) and we, in the sequel, will only work with the Lagrangian (2.3) and the set of consistent constraints (2.4).

The question naturally arises of whether it is possible or not to have one single Lagrangian function giving rise to the whole set of constraints (2.4). In appendix B we analyse this point and formally show the existence of such Lagrangian with auxiliary variables. The elimination of these variables depends on the solution of an algebraic equation of degree N in general and this, of course, is of little practical interest. Nevertheless explicit knowledge of such Lagrangian would not provide us with additional physical information. Therefore we can carry on our analysis with the simpler point of view expressed by (2.3) plus (2.4).

III. EQUATIONS OF MOTION AND HAMILTONIAN FORMALISM

The Euler-Lagrange equations

$$\frac{d}{d\lambda}\frac{\partial \mathscr{L}}{\partial \dot{x}_i^\mu} - \frac{\partial \mathscr{L}}{\partial x_i^\mu} = 0 \qquad (i = 1 \ldots \mathrm{N}) \tag{3.1}$$

can be written

$$\sum_{j=1}^{N} \mathscr{H}_{ij\mu\nu} \ddot{x}_j^\nu = \frac{\partial \mathscr{L}}{\partial x_i^\mu} - \sum_{j=1}^{N} \frac{\partial^2 \mathscr{L}}{\partial \dot{x}_i^\mu \partial x_j^\nu} \dot{x}_j^\nu \equiv \mathrm{A}_{i\mu} \qquad (i = 1 \ldots \mathrm{N}) \tag{3.2}$$

where $\mathscr{H}_{ij\mu\nu}$ is the Hessian of $\mathscr{L}$ with respect to the velocities

$$\mathscr{H}_{ij\mu\nu} \equiv \frac{\partial^2 \mathscr{L}}{\partial \dot{x}_i^\mu \partial \dot{x}_j^\nu} = -\frac{1}{N}\frac{U}{\mathscr{L}}\left(\delta_{ij} g_{\mu\nu} + \frac{1}{N} \quad \frac{U}{\mathscr{L}^2} \dot{x}_i^\mu \dot{x}_j^\nu\right) \tag{3.3}$$

Let us recall that we have assumed that U is independent of the velocities and we will also do so in the sequel; we thus find

$$A_{i\mu} = -\frac{N}{2}\frac{\mathscr{L}^2}{U^2} \sum_{j=1}^{N} \mathscr{H}_{ij\mu\nu} \frac{\partial U}{\partial x_{j\nu}} \tag{3.4}$$

The hessian matrix (3.3) has only one null vector

$$n^\mu = \begin{pmatrix} \dot{x}^\mu \\ \vdots \\ \vdots \\ \dot{x}_N^\mu \end{pmatrix} \tag{3.5}$$

but there are no constraints since $\langle A, n \rangle = \sum_{i=1}^{N} A_{i\mu} \dot{x}_i^\mu \equiv 0$ according to (3.4). This is a rather general result for Lagrangians of the type (2.3). Now we easily isolate the accelerations from (3.2)

$$\ddot{x}_i^\mu = -\frac{N}{2}\frac{\mathscr{L}^2}{U^2}\frac{\partial U}{\partial x_{i\mu}} + \mu(\lambda)\dot{x}_i^\mu \qquad (i = 1 \ldots N) \tag{3.6}$$

$\mu(\lambda)$ being an undetermined arbitrary function. It fulfils

$$\mu(\lambda) = \frac{d}{d\lambda} \ln \frac{\mathscr{L}}{U} \tag{3.7}$$

as can be verified by a straightforward derivation.

Eqs. (3.6) must be completed with the constraints

$$\langle \dot{x}, r_k \rangle = 0 \qquad \langle \dot{x}, \dot{r}_k \rangle = 0 \qquad (k = 2 \ldots N) \tag{3.8}$$

which are stable under (3.6). Let us observe that these equations of motion are those of ref. [*31*], the latter being deduced within the framework of a multitemporal model. That model is therefore the predictive extension of the one proposed in the present paper.

Let us now define canonical momenta

$$p_{i\mu} = -\frac{\partial \mathscr{L}}{\partial \dot{x}_i^\mu} = \frac{1}{N}\frac{U}{\mathscr{L}} \dot{x}_{i\mu} \qquad (i = 1 \ldots N) \tag{3.9}$$

They are not independent but satisfy the identity

$$\phi_0 \equiv \mathrm{N} \sum_{i=1}^{\mathrm{N}} p_i^2 + \mathrm{U} = 0 \tag{3.10}$$

which, according to Dirac, must be considered as a primary Hamiltonian constraint. There are no more primary constraints so that the evolution Hamiltonian is simply

$$\mathrm{H} = \mathrm{U}_0 \phi_0 \tag{3.11}$$

because the canonical Hamiltonian is zero ($\mathscr{L}$ is homogeneous of 1st degree in $\dot{x}_i$); U_0 is an undetermined function. The equations of motion are then

$$\begin{aligned} \dot{x}_i^\mu &= \{ x_i^\mu, \mathrm{H} \} = -2\mathrm{N}\mathrm{U}_0 p_i^\mu \\ \dot{p}_i^\mu &= \{ p_i^\mu, \mathrm{H} \} = \mathrm{U}_0 \frac{\partial \mathrm{U}}{\partial x_{i\mu}} \qquad (i = 1 \ldots \mathrm{N}) \end{aligned} \tag{3.12}$$

with the fundamental brackets $\{ x_{i\mu}, p_{j\nu} \} = -\delta_{ij} g_{\mu\nu}$. Through the identification

$$\frac{\mathscr{L}}{\mathrm{U}} = -2\mathrm{U}_0 \tag{3.13}$$

we recover the set (3.6). Now we must add to (3.12) the constraints (3.8) in order to keep the correct number of independent initial conditions. We have, of course, to write them in terms of canonical variables:

$$\phi_{1k} \equiv \langle \mathrm{P}, r_k \rangle = 0 \qquad \phi_{2k} \equiv \langle \mathrm{P}, p_k - p_1 \rangle = 0 \qquad (k=2 \ldots \mathrm{N}) \tag{3.14}$$

where P_μ is the total momentum of the system

$$\mathrm{P}_\mu = \sum_{i=1}^{\mathrm{N}} p_{i\mu} \tag{3.15}$$

Equations (3.14) can be replaced by the more convenient set

$$\psi_k \equiv \langle \mathrm{P}, \mathrm{S}_k \rangle = 0 \qquad \phi_k \equiv \langle \mathrm{P}, y_k \rangle = 0 \qquad (k = 2 \ldots \mathrm{N}) \tag{3.16a}$$

where

$$\mathrm{S}_k^\mu = (x_k^\mu - x^\mu) \qquad y_k^\mu = (p_k^\mu - p_1^\mu) \qquad (k = 2 \ldots \mathrm{N})$$

are canonical variables. We can also rewrite ϕ_0 :

$$\phi_0 = \mathrm{P}^2 + \mathrm{N}\left[\sum_{k=2}^{\mathrm{N}} y_k^2 - \frac{1}{\mathrm{N}} \left(\sum_{k=2}^{\mathrm{N}} y_k \right)^2 \right] + \mathrm{U}(\langle \mathrm{S}_k, \mathrm{S}_l \rangle) \tag{3.16b}$$

The new constraints do not affect the equations of motion because all of them are second class, ϕ_0 remaining the only first class constraint (if $P^2 \neq 0$, what will always be the case here):

$$\{ \phi_0, \phi_k \} = -2 \sum_{l=2}^{N} \frac{\partial U}{\partial \langle S_k S_l \rangle} \psi_l$$

$$\{ \phi_0, \psi_k \} = 2N \sum_{l=2}^{N} \left(\delta_{kl} - \frac{1}{N} \right) \phi_l \tag{3.18a}$$

$$\{ \phi_k, \psi_l \} = \delta_{kl} P^2 \qquad (k, l = 2 \ldots N) \tag{3.18b}$$

Thus we see that the Hamiltonian formalism can also be consistently constructed.

IV. SHANMUGADHASAN'S TRANSFORMATION

In order to complete the Hamiltonian formalism one can define Dirac brackets and isolate the independent canonical variables with respect to such brackets. Or else one can follow Shanmugadhasan's method [*33*] [*34*], what amounts to writing down the set of independent variables as a subset of a set of canonical variables related to the initial ones through a local canonical transformation. The existence of such transformation is guaranteed by some theorems on involutory systems [*35*] [*36*] and transformation groups [*37*]. We will rather use the latter method in this section.

The Shanmugadhasan's transformation adapted to the second class constraints can be carried out in our case by straightforward algebra. It does not depend on the interaction because the second class constraints (3.16*a*) do not depend on the potential, and it is therefore very general [*38*].

If one wants to complete the transformation so as to have also a canonical variable adapted to the first class constraint (3.16*b*), one sees that there is no closed solution for a general r_k^μ-dependent potential. This problem is actually equivalent to solving the Hamilton-Jacobi equations [*34*]. We do not touch here this question.

Let us then study the transformation adapted only to the second class constraints. Since $P^2 \neq 0$ we can define

$$\overline{R}_k \equiv \frac{\langle PS_k \rangle}{\sqrt{P^2}} \qquad \overline{K}_k \equiv -\frac{\langle Py_k \rangle}{\sqrt{P^2}} \qquad (k = 2 \ldots N) \tag{4.1}$$

These are pairs of canonical variables, for

$$\{ \overline{R}_k, \overline{K}_l \} = \delta_{kl} \qquad (k, l = 2 \dots N) \tag{4.2}$$

the rest of the brackets being zero, and they are obviously adapted to the second class constraints (3.16). Now we must complete the set of canonical variables by finding solutions of the system of partial differential equations

$$\{ \overline{R}_k, f \} = 0, \qquad \{ \overline{K}_k, f \} = 0 \qquad (k = 2 \dots N) \tag{4.3}$$

such that they are canonical amongst themselves.

Due to relations (4.2), the general theory [35], [37] guarantees the existence of $8N-2(N-1)=6N+2$ independent solutions of (4.3), just sufficient to complete the set (4.1).

There are four obvious solutions of (4.3) namely the four components of P^μ; we call X^μ their corresponding canonically conjugate solutions. Let the $6(N-1)$ remaining solutions be noted $R_{k\lambda}$, $K_{k\lambda}$ ($\lambda=1, 2, 3$; $k=2,\dots,N$), also grouped into canonical pairs.

Let us assume that the latter solutions are independent of X^μ; if that is the case (4.3) imply

$$P^\mu \frac{\partial R_{k\lambda}}{\partial S_l^\mu} = P^\mu \frac{\partial R_{k\lambda}}{\partial y_l^\mu} = P^\mu \frac{\partial K_{k\lambda}}{\partial S_l^\mu} = P^\mu \frac{\partial K_{k\lambda}}{\partial y_l^\mu} = 0 \qquad (k, l = 2 \dots N) \tag{4.4}$$

whose easiest solution is

$$R_{k\lambda} = \varepsilon_\lambda^\mu(P) S_{k\mu} \qquad K_{k\lambda} = \varepsilon_\lambda^\mu(P) y_{k\mu} \qquad (k = 2 \dots N) \tag{4.5}$$

where $\varepsilon_\lambda^\mu(P)$ are polarization vectors [39] satisfying

$$P_\mu \varepsilon_\lambda^\mu(P) = 0 \qquad \varepsilon_\lambda^\mu(P)\varepsilon_{\lambda'\mu}(P) = -\delta_{\lambda\lambda'} \tag{4.6a}$$

$$\frac{P^\mu P^\nu}{P^2} - \sum_{\lambda=1}^{3} \varepsilon_\lambda^\mu(P)\varepsilon_\lambda^\nu(P) = g^{\mu\nu} \tag{4.6b}$$

$$\varepsilon_\lambda^0 = \frac{P^\lambda}{\sqrt{P^2}} \qquad \varepsilon_\lambda^j = \delta_\lambda^j + \frac{P^j P^\lambda}{\sqrt{P^2}(\sqrt{P^2}+P^0)} \qquad (\lambda, j = 1, 2, 3) \tag{4.6c}$$

The quantities (4.5) have the nice property that they are canonical amongst themselves

$$\{ R_{k\lambda}, K_{l\lambda'} \} = \delta_{kl}\delta_{\lambda\lambda'} \qquad \{ R_{k\lambda}, R_{l\lambda'} \} = \{ K_{k\lambda}, K_{l\lambda'} \} = 0 \tag{4.7}$$

and also that they commute with P^μ; whence we have only to determine the four functions X^μ.

According to the relations $\{ P^\mu, X^\nu \} = g^{\mu\nu}$ we have

$$X^\mu = x^\mu + f^\mu(P, S, y) \tag{4.8}$$

and X^μ must commute with all the remaining variables. We find (*)

$$\{X^\mu, \overline{R}_k\} = 0 \Rightarrow P^\rho \frac{\partial f^\mu}{\partial y_k^\rho} = \underset{\sim}{S}{}_k^\mu$$

$$\{X^\mu, \overline{K}_k\} = 0 \Rightarrow P^\rho \frac{\partial f^\mu}{\partial S_k^\rho} = -\underset{\sim}{y}{}_k^\mu \tag{4.9b}$$

$$\{X^\mu, R_{k\lambda}\} = 0 \Rightarrow \varepsilon_\lambda^\rho(P) \frac{\partial f^\mu}{\partial y_k^\rho} = \frac{\partial \varepsilon_\lambda^\rho(P)}{\partial P_\mu} S_{k\rho} \tag{4.9c}$$

$$\{X^\mu, K_{k\lambda}\} = 0 \Rightarrow \varepsilon_\lambda^\rho(P) \frac{\partial f^\mu}{\partial S_k^\rho} = -\frac{\partial \varepsilon_\lambda^\rho(P)}{\partial P_\mu} y_{k\rho} \tag{4.9d}$$

Making now use of the identities

$$\frac{\partial}{\partial S_{k\nu}} = \frac{P^\nu P^\rho}{P^2} \frac{\partial}{\partial S_k^\rho} - \sum_{\lambda=1}^{3} \varepsilon_\lambda^\nu(P)\varepsilon_\lambda^\rho(P) \frac{\partial}{\partial S_k^\rho}$$

$$\frac{\partial}{\partial y_{k\nu}} = \frac{P^\nu P^\rho}{P^2} \frac{\partial}{\partial y_k^\rho} - \sum_{\lambda=1}^{3} \varepsilon_\lambda^\nu(P)\varepsilon_\lambda^\rho(P) \frac{\partial}{\partial y_k^\rho}$$

we immediately see that

$$\frac{\partial f^\mu}{\partial S_{k\nu}} = -\frac{1}{P^2} \underset{\sim}{y}{}_k^\mu P^\nu + \sum_{\lambda=1}^{3} \varepsilon_\lambda^\nu(P) \frac{\partial \varepsilon_\lambda^\rho(P)}{\partial P_\mu} y_{k\rho} \tag{4.10a}$$

$$\frac{\partial f^\mu}{\partial y_{k\nu}} = +\frac{1}{P^2} \underset{\sim}{S}{}_k^\mu P^\nu - \sum_{\lambda=1}^{3} \varepsilon_\lambda^\nu(P) \frac{\partial \varepsilon_\lambda^\rho(P)}{\partial P_\mu} S_{k\rho} \tag{4.10b}$$

Equation (4.10 *b*) can be readily integrated:

$$f^\mu(P, S, y) = \sum_{k=2}^{N} \left(\frac{\langle P, y_k \rangle}{P^2} \underset{\sim}{S}{}_k^\mu - \sum_{\lambda=1}^{3} \varepsilon_\lambda^\nu \frac{\partial \varepsilon_\lambda^\rho}{\partial P_\mu} S_{k\rho} y_{k\nu} \right) + \psi^\mu(S, P) \tag{4.11}$$

where $\psi^\mu(s, P)$ is an unknown function. Putting (4.11) into (4.10 *a*) we see that

$$\frac{\partial \psi^\mu}{\partial S_k^\nu} = 0 \Rightarrow \psi^\mu(S, P) = \varphi^\mu(P) \tag{4.12}$$

and hence that we must still find four functions of P^μ. To this end we require the final conditions

$$\{X^\mu, X^\nu\} = 0 \tag{4.13}$$

(*) For all vectors,

$$\underset{\sim}{A}{}^\mu \equiv A^\mu - \frac{\langle P, A \rangle}{P^2} P^\mu$$

One can show by direct calculation that

$$\{ X^\mu - \varphi^\mu(P), X^\nu - \varphi^\nu(P) \} = 0$$

so that an acceptable solution is $\varphi^\mu(P) = 0$.

We have then finally (making use of rels. (4.6c))

$$X^\mu = x^\mu + \frac{1}{P^2} \sum_{k=2}^{N} \left[\langle P, y_k \rangle \underset{\sim}{S}{}_k^\mu - \langle P, S_k \rangle \underset{\sim}{y}{}_k^\mu - \frac{P^2}{P^0 + \sqrt{P^2}} (\underset{\sim}{S}{}_k^0 \underset{\sim}{y}{}_k^\mu - \underset{\sim}{S}{}_k^\mu \underset{\sim}{y}{}_k^0) \right] \tag{4.14}$$

The Shanmugadhasan transformation adapted to the second class constraints is thus completed. We also write down its inverse:

$$x^\mu = X^\mu - \frac{1}{P^2} \sum_{k=2}^{N} \sum_{\lambda=1}^{3} \Big[\sqrt{P^2}(\overline{K}_k R_{k\lambda} + \overline{R}_k K_{k\lambda}) \varepsilon_\lambda^\mu - \frac{P^2}{\sqrt{P^2} + P^0} \sum_{\lambda'=1}^{3} R_{k\lambda} K_{k\lambda'} (\varepsilon_\lambda^0 \varepsilon_{\lambda'}^\mu - \varepsilon_\lambda^\mu \varepsilon_{\lambda'}^0) \Big] \tag{4.15a}$$

$$P^\mu = P^\mu \tag{4.15b}$$

$$S_k^\mu = \frac{1}{\sqrt{P^2}} \overline{R}_k P^\mu - \sum_{\lambda=1}^{3} R_{k\lambda} \varepsilon_\lambda^\mu \tag{4.15c}$$

$$y_k^\mu = -\frac{1}{\sqrt{P^2}} \overline{K}_k P^\mu - \sum_{\lambda=1}^{3} K_{k\lambda} \varepsilon_\lambda^\mu \tag{4.15d}$$

The general theorems which have allowed us to write down the canonical transformation adapted to the second class constraints, also guarantee that it is possible to find (locally) an equivalent set of first class constraints such that their Poisson brackets amongst themselves and with the second class constraints vanish identically. This enables us to substitute the first class constraint (3.16b) by a new one $\tilde{\phi}_0$ with the previous properties.

Let us rewrite (3.16b) in terms of the new variables (4.15):

$$\phi_0 = P^2 + N \left[\sum_{k=2}^{N} \overline{K}_k^2 - \frac{1}{N} \left(\sum_{k=2}^{N} \overline{K}_k \right)^2 \right] - N \sum_{\lambda=1}^{3} \left[\sum_{k=2}^{3} K_{k\lambda}^2 - \frac{1}{N} \left(\sum_{k=2}^{N} K_{k\lambda} \right)^2 \right] + U \tag{4.16}$$

As first suggested in ref. [*40*] we expect that the new $\tilde{\phi}_0$ will not depend on the variables $\overline{R}_k$, $\overline{K}_k$. Since the latter are canonical pairs this means

$$\{ \tilde{\phi}_0, \overline{R}_k \} = -\frac{\partial\tilde{\phi}_0}{\partial\overline{K}_k} = 0 \tag{4.17a}$$

$$\{ \tilde{\phi}_0, \overline{K}_k \} = \frac{\partial\tilde{\phi}_0}{\partial\overline{R}_k} = 0 \tag{4.17b}$$

thus verifying the initial requirement. A possible choice of $\tilde{\phi}_0$ is then

$$\tilde{\phi}_0 = P^2 - N \sum_{\lambda=1}^{3} \left[\sum_{k=2}^{N} K_{k\lambda}^2 - \frac{1}{N}\left(\sum_{k=2}^{N} K_{k\lambda} \right)^2 \right] + U \tag{4.18}$$

where we replace $\langle S_k, S_l \rangle$ by $\langle \underset{\sim}{S}_k, \underset{\sim}{S}_l \rangle$ wherever it appears in the expression of U.

We can write

$$\tilde{\phi}_0 = P^2 - N \sum_{\lambda=1}^{3} \sum_{k,k'=2}^{N} \Omega_{kk'} K_{k\lambda} K_{k'\lambda} + U(\langle R_{k\lambda}, R_{k'\lambda} \rangle) \tag{4.19}$$

where $\Omega_{kk'}$ is the symmetric tensor

$$\Omega_{kk'} = \delta_{kk'} - \frac{1}{N} \qquad (k, k' = 2 \dots N) \tag{4.20}$$

which can be diagonalised by means of a matrix Λ :

$$\Lambda^T \Omega \Lambda = \mathbb{1} \tag{4.21}$$

The coefficients of Λ are

$$\Lambda_{k2} = \left(\frac{N}{N-1} \right)^{1/2}$$

$$\Lambda_{kk'} = \begin{cases} (k'-2)\,[(k'-1)(k'-2)]^{-1/2} & k' > k \\ (k'-4)\,[(k'-1)(k'-2)]^{-1/2} & k' = k \\ (k'-3)\,[(k'-1)(k'-2)]^{-1/2} & k' < k \end{cases} \quad \begin{cases} k = 2 \dots N \\ k' = 3 \dots N \end{cases} \tag{4.22}$$

Defining new variables

$$\hat{K}_{k\lambda} = \sum_{k'=2}^{N} \Lambda_{kk'} K_{k'\lambda} \tag{4.23}$$

and their canonically conjugated (i. e., $\{ \hat{R}_{k\lambda}, \hat{K}_{k'\lambda'} \} = \delta_{kk'}\delta_{\lambda\lambda'}$)

$$\hat{R}_{k\lambda} = \sum_{k'=2}^{N} \Lambda_{kk'}^{-1} R_{k'\lambda} \tag{4.24}$$

$\tilde{\phi}_0$ reduces to the simple form

$$\tilde{\phi}_0 = P^2 - N \sum_{\lambda=1}^{3} \sum_{k=2}^{N} \hat{K}_{k\lambda}^2 + \tilde{U}(\langle \hat{R}_{k\lambda}, \hat{R}_{k'\lambda} \rangle)$$

where $\tilde{U}$ is the potential function written in terms of the new canonical variables.

V. QUANTIZATION

To perform the quantization of this model we follow the prescriptions of [*38*] and [*40*], according to which the prescription

$$\{\ ,\ \} \to -i[\ ,\ ] \tag{5.1}$$

is taken up in the complete phase space. The physical states $|\psi\rangle$ are required to satisfy the conditions

$$\tilde{\phi}_0 |\psi\rangle = 0 \tag{5.2}$$

$$a_k |\psi\rangle = 0 \Leftrightarrow \langle\psi| a_k^+ = 0 \qquad (k = 2 \dots N) \tag{5.3}$$

where

$$a_k = \frac{1}{\sqrt{2\beta_k}}(\overline{R}_k + i\beta_k\overline{K}_k) \qquad (k = 2 \dots N) \tag{5.4}$$

the β_k being real constants. The following commutation rules are verified

$$[a_k,\ a_l] = [a_k^+, a_l^+] = 0 \tag{5.5a}$$

$$[a_k, a_l^+] = \delta_{kl} \qquad (k, l = 2 \dots N) \tag{5.5b}$$

Now from equation (4.19) we have

$$[\tilde{\phi}_0, a_k] = [\tilde{\phi}_0, a_k^+] = 0 \qquad (k = 2 \dots N) \tag{5.6}$$

and therefore no inconsistencies arise.

The solutions of (5.3) have the form

$$\langle X^\mu, \overline{R}_k, \hat{R}_{k\lambda} | \psi \rangle = \left(\prod_{k=2}^{N} e^{-\frac{1}{2\beta_k}\overline{R}_k^2} \right) f(X^\mu, \hat{R}_{k\lambda}) \tag{5.7}$$

The dependence of the wave functions on the variables X^μ, $\hat{R}_{k\lambda}$ is determined by (5.2) and, of course, depend on the potential.

For a harmonic oscillator potential,

$$U = -\alpha \sum_{i<j} (x_i - x_j)^2 = -\alpha N \left[\sum_{\lambda=1}^{3} \sum_{k,k'=2}^{N} \Omega'_{kk'} R_{k\lambda} R_{k'\lambda} \right] \tag{5.8}$$

where $\Omega'_{kk'}$ is the quadratic form

$$\Omega'_{kk'} = \delta_{kk'} + 1 = \Omega^{-1}_{kk'} \tag{5.9}$$

which is inverse of the one in (4.20). It is diagonalised by Λ^{-1} and the harmonic potential adopts the simple form

$$U = -\alpha N \sum_{\lambda=1}^{3} \sum_{k=2}^{N} \hat{R}^2_{k\lambda} \tag{5.10}$$

Introducing the following operators

$$\begin{aligned} a_{k\lambda} &= \frac{1}{\sqrt{2\beta}} (\hat{R}_{k\lambda} + i\beta\hat{K}_{k\lambda}) \\ a^+_{k\lambda} &= \frac{1}{\sqrt{2\beta}} (\hat{R}_{k\lambda} - i\beta\hat{K}_{k\lambda}) \qquad (k = 2 \dots N) \end{aligned} \tag{5.11}$$

such that

$$[a_{k\lambda}, a^+_{k'\lambda'}] = \delta_{\lambda\lambda'}\delta_{kk'} \tag{5.12}$$

eq. (5.2) can be written as

$$P^2 - N\sqrt{\alpha}\left\{ (N-1) + 2 \sum_{\lambda=1}^{3} \sum_{k=2}^{N} a^+_{k\lambda} a_{k\lambda} \right\} | \psi_{n_2 \dots n_N} \rangle = 0 \tag{5.13}$$

with $\beta = \sqrt{\alpha}$.

Therefore the mass spectrum in this case is

$$M^2 = N\sqrt{\alpha}\left((N - 1) + 2 \sum_{k=2}^{N} n_k \right) \tag{5.14}$$

It is interesting to point out that for the case $N = 2$, the corresponding wave functions (5.7) are those of Kim and Noz [*41*].

Let us now make some comments about the weak quantization performed here with one first class constraint—the mass shell constraint of the system—and $N-1$ non hermitian combination of second class constraints, and those proposed by other authors [*42*] [*43*] in which they quantize N first class constraints—the mass shell constraints of the individual particles. The Leutwyler-Stern model [*44*] belongs to this class as shown in ref. [*45*]. Due to general theorems on function algebra, it is possible to extract from a set of $2N-1$ constraints one of them first class and the others second class $-a$ (maximal) subset of N constraints commuting among themselves. So, the alternative quantization cited above can also be performed in our model.

To this end we extract from this model the corresponding N first class constraints

$$H_i = \sum_{j=1}^{N} P_j^2 - P^2 + N \langle P, p_i \rangle + \frac{1}{N} U(\underset{\sim}{r}_k^2, \langle \underset{\sim}{r}_k, \underset{\sim}{r}_l \rangle) \qquad i=1 \dots N \quad (5.15)$$

$$\left(\underset{\sim}{r}_k^\mu = r^\mu - \frac{\langle P, r \rangle}{P^2} P^\mu \right)$$

(5.15) are equivalent to

$$\begin{aligned} \sum_{i=1}^{N} H_i &= N \sum_{j=1}^{N} p_j^2 + U \\ H_k - H_{k'} &= N \langle P, p_k - p_{k'} \rangle \qquad k, k' = 2 \dots N \end{aligned} \qquad (5.16)$$

If we use the canonical transformation (4.15) and (4.23), (4.24), these constraints can be written as

$$\tilde{\phi}_0 = P^2 - N \sum_{\lambda=1}^{3} \sum_{k=2}^{N} \hat{K}_{k\lambda}^2 + \tilde{U}(\langle \hat{R}_{k\lambda}, \hat{R}_{k'\lambda} \rangle)$$

$$\overline{K}_k = 0 \qquad k = 2 \dots N \qquad (5.17)$$

and therefore if we perform the quantization we obtain the following N wave equations

$$\tilde{\phi}_0 \mid \psi \rangle = 0 \qquad (5.18\,a)$$

$$\overline{K}_k \mid \psi \rangle = 0 \qquad k = 2 \dots N \qquad (5.18\,b)$$

We point out that the quantization performed with these variables is equivalent to the old one due to the fact the canonical transformation is a point transformation in the momenta.

If we use wave functions in momentum space

$$\psi(P, \overline{K}_k, \hat{K}_{k\lambda}) = \langle P, \overline{K}_k, \hat{K}_{k\lambda} \mid \psi \rangle \qquad (5.19)$$

the solutions of eq. (5.18) are

$$\delta(\overline{K}_k) f(P, \hat{K}_{k\lambda}) \qquad (5.20)$$

where $f(P, \hat{K}_{k\lambda})$ is a solution of (5.18 a), while the solutions of (5.2), (5.3) in momentum space are:

$$e^{-\frac{1}{2\beta_k} \overline{K}_k^2} f(P, \hat{K}_{k\lambda}) \qquad (5.21)$$

where $f(P, \hat{K}_{k\lambda})$ satisfies (5.2) and also (5.18 a).

As shown in ref. [*46*] it is possible to introduce a scalar product for the wave functions (5.20) and (5.21) in a such a way that the corresponding Hilbert spaces are isometric, therefore we must conclude that at quantum level the two procedures are equivalent.

APPENDIX A

We study in this appendix the equations of motion of a two-particle harmonic oscillator as described by the Lagrangian function

$$\mathscr{L} = -\sqrt{(-kr^2)(\alpha_1\dot{x}_1^2 + \alpha_2\dot{x}_2^2)} \tag{A.1}$$

in which, for the moment, we do not assume $\alpha_1 = \alpha_2$; we only assume $\alpha_1 + \alpha_2 = 1$. Defining $x^\mu = \alpha_1 x_1^\mu + \alpha_2 x_2^\mu$ and $r^\mu = x_2^\mu - x_1^\mu$, the equations of motion are easily obtained

$$\ddot{x}^\mu = \mu(\lambda)\dot{x}^\mu \qquad \ddot{r}^\mu = -\frac{1}{\alpha_1\alpha_2}\frac{\mathscr{L}^2}{(kr^2)}\mathrm{K}r^\mu + \mu(\lambda)\dot{r}^\mu \tag{A.2}$$

where $\mu(\lambda) = \frac{d}{d\lambda}\ln\frac{\mathscr{L}}{kr^2}$ is the gauge function. We can choose it equal to zero and then take $\frac{\mathscr{L}}{kr^2} = 1$. This imposes the gauge constraint

$$\dot{x}^2 + \alpha_1\alpha_2\dot{r}^2 + kr^2 = 0 \tag{A.3}$$

to be added to the previous ones

$$\langle \dot{x}, r \rangle = 0 \qquad \langle \dot{x}, \dot{r} \rangle = 0 \tag{A.4}$$

It simplifies the equations of motion to

$$\ddot{x}^\mu = 0 \qquad \ddot{r}^\mu + \frac{k}{\alpha_1\alpha_2}r^\mu = 0 \tag{A.5}$$

whose solutions are

$$x^\mu = a^\mu\lambda + b^\mu \tag{A.6 a}$$

$$r^\mu = \mathrm{B}^\mu\cos\omega'\lambda + \mathrm{C}^\mu\sin\omega'\lambda \qquad \omega' \equiv \sqrt{\frac{k}{\alpha_1\alpha_2}} \tag{A.6 b}$$

If we impose the constraints (A.3) and (A.4), the constants in (A.6) are seen to verify

$$\langle a, \mathrm{B} \rangle = \langle a, \mathrm{C} \rangle = 0 \qquad a^2 + k(\mathrm{B}^2 + \mathrm{C}^2) = 0 \tag{A.7}$$

Let us study the situation in the CM rest frame. In this frame $\vec{a} = 0$, $b^\mu = 0$, so that the equations of the trajectories and the constraints read

$$x^0 = a^0\lambda \qquad \vec{x} = 0 \tag{A.8 a}$$

$$r^0 = 0 \qquad \vec{r} = \vec{\mathrm{B}}\cos\omega'\lambda + \vec{\mathrm{C}}\sin\omega'\lambda \tag{A.8 b}$$

$$\mathrm{B}^0 = \mathrm{C}^0 = 0 \qquad a^{02} = k(\vec{\mathrm{B}}^2 + \vec{\mathrm{C}}^2) \tag{A.9}$$

We can now eliminate the parameter λ in favour of the time for this reference frame, i. e., $x^0 \equiv t$. Thus we have

$$\vec{r} = \vec{\mathrm{B}}\cos\omega t + \vec{\mathrm{C}}\sin\omega t \qquad \omega = \frac{\omega'}{a^0} = \frac{1}{\sqrt{\alpha_1\alpha_2}\sqrt{(\vec{\mathrm{B}}^2 + \vec{\mathrm{C}}^2)}} \tag{A.10}$$

whence

$$\vec{x}_1 = -\alpha_2\vec{r} \qquad \vec{x}_2 = \alpha_1\vec{r} \tag{A.11}$$

and

$$\vec{v}_1 = \frac{d\vec{x}_1}{dt} = -\sqrt{\frac{\alpha_2}{\alpha_1}}\frac{\vec{\mathrm{B}}\cos\omega t + \vec{\mathrm{C}}\sin\omega t}{\sqrt{\vec{\mathrm{B}}^2 + \vec{\mathrm{C}}^2}} \tag{A.12 a}$$

$$\vec{v}_2 = \frac{d\vec{x}_2}{dt} = \sqrt{\frac{\alpha_1}{\alpha_1}}\frac{\vec{\mathrm{B}}\cos\omega t + \vec{\mathrm{C}}\sin\omega t}{\sqrt{\vec{\mathrm{B}}^2 + \vec{\mathrm{C}}^2}} \tag{A.12 b}$$

It is apparent in these expressions the possibility of having tachyonic states of motion unless $\alpha_1 = \alpha_2 = \frac{1}{2}$ as pointed out in section II.

APPENDIX B

In this appendix we attempt to solve the problem of finding a Lagrangian function such that it automatically yields the whole set of constraints (3.8) and (3.10). To this end let us introduce N + 1 auxiliary variables v_0, v_i ($i = 1, \ldots, N$) such that

$$\sum_{i=1}^{N} v_i = 0 \tag{B.1}$$

We now construct a Hamiltonian function

$$H_c = -\frac{1}{2} v_0 \phi_0 - \frac{1}{2} \sum_{k=2}^{N} v_k \phi_k \tag{B.2}$$

ϕ_0, ϕ_k being the same constraints as before, and its associated equations of motion.

$$\dot{x}^\mu = \{ x^\mu, H_c \} = v_0 P^\mu + \frac{1}{2} \sum_{k=2}^{N} v_k y_k^\mu \tag{B.3 a}$$

$$\dot{S}_k^\mu = \{ S_k^\mu, H_c \} = v_0 N\left(y_k^\mu - \frac{1}{N} \sum_{l=2}^{N} y_l^\mu \right) + \frac{1}{2} v_k P^\mu \qquad (k = 2 \ldots N) \tag{B.3 b}$$

This system of equations enables us to write down expressions for P^μ, y_k^μ in therms of $\dot{x}$, $\dot{S}_k^\mu$ and the auxiliary variables:

$$P^\mu = \left(v_0 - \frac{B}{4v_0} \right)^{-1} A^\mu \tag{B.4 a}$$

$$y_k^\mu = \frac{1}{v_0 N} \dot{\xi}_k^\mu - \frac{2 \hat{v}_k}{N(4v_0^2 - B)} A^\mu \qquad (k = 2 \ldots N) \tag{B.4 b}$$

with

$$\xi_k^\mu \equiv (S_k^\mu - S_1^\mu) \qquad \hat{v}_k \equiv (v_k - v_1) \qquad (k = 2 \ldots N)$$

and

$$A^\mu \equiv \dot{x}^\mu - \frac{1}{2v_0 N} \sum_{k=2}^{N} v_k \dot{\xi}_k^\mu \qquad B \equiv \frac{1}{N} \sum_{i=1}^{N} v_i^2$$

and $S_i^\mu = x_1^\mu - x^\mu$. Hence

$$p_i^\mu = \frac{1}{v_0 N} (\dot{x}_i^\mu - \dot{x}^\mu) + \frac{1}{N}\left(1 - \frac{v_i}{2v_0} \right) P^\mu \qquad (i = 1 \ldots N) \tag{B.5}$$

Now the Lagrangian is

$$\mathscr{L} = \sum_{i=1}^{N} \dot{x}_i^\mu \frac{J\mathscr{L}}{J\dot{x}_i^\mu} - H_c = - \sum_{i=1}^{N} \langle \dot{x}_i p_i \rangle - H_c .$$

where H_c is defined by (B.2). Upon substitution we find

$$\mathscr{L} = \frac{1}{2v_0}\left(\dot{x}^2 - \frac{1}{N} \sum_{i=1}^{N} \dot{x}_i^2 \right) - \frac{2v_0}{4v_0^2 - B} A^2 + \frac{1}{2} v_0 U \tag{B.6}$$

In order to eliminate the auxiliary variables we make use of their corresponding equations of motion which, since the velocities $\dot{v}_0$, $\dot{v}_i$ are cyclic in (B.6) are

$$\frac{\partial \mathscr{L}}{\partial v_0} = 0 \qquad \frac{\partial \mathscr{L}}{\partial v_i} = 0 \qquad (i = 1 \ldots \mathrm{N}) \tag{B.7}$$

This is equivalent to

$$0 = \dot{x}^2 - \frac{1}{\mathrm{N}}\sum_{i=1}^{\mathrm{N}} \dot{x}_i^2 - v_0(\langle \mathrm{P}, \mathrm{A} \rangle - 2\langle \mathrm{P}, \dot{x} \rangle) - 2v_0^2\left(\mathrm{P}^2 + \frac{\mathrm{U}}{2}\right) \tag{B.8 a}$$

$$0 = \langle \mathrm{P}, \dot{x}_i + \frac{v_i}{2}\mathrm{P} \rangle \qquad (i = 1 \ldots \mathrm{N}) \tag{B.8 b}$$

where P^μ is given by (B.4 *a*). This is an unsolvable system in practice because we need to solve in general an algebraic equation of N th degree. But from formal point of view the Lagrangian (B.6) is sufficient to study the dynamical properties of the system. Furthermore it is sufficient to work with (2.3) plus (2.4).

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