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Section A :
Physique théorique.

Odd anharmonic oscillators and shape resonances

by

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ABSTRACT. — By adapting a stability argument of Vock and Hunziker it is proved that the Borel sum of the Rayleigh-Schrödinger perturbation expansion of any odd anharmonic oscillator $p^2+x^2+gx^{2k+1}$, $k=1, 2, \ldots$, is the limit of a sequence of resonances in the standard sense of dilation analyticity. The same method yields, for a suitable class of potentials, an existence proof of shape resonances.

RÉSUMÉ. — En adaptant un argument de stabilité dû à Vock et Hunziker, on montre que la somme de Borel de la série de perturbations de Rayleigh Schrödinger pour un oscillateur anharmonique impair $p^2+x^2+gx^{2k+1}$, $k=1, 2, \ldots$, est la limite d'une suite de résonances au sens usuel de l'analycité par dilatation. La même méthode fournit, pour une classe convenable de potentiels, une preuve d'existence des « résonances de forme ».

1. INTRODUCTION

The purpose of this paper is to complete an earlier investigation [*2*] on the spectral and perturbation theory of any odd anharmonic oscillator $p^2+x^2+gx^{2k+1}$, $k=1, 2, \ldots$. The aim is to justify the standard physical picture of these problems (see e. g. Davydov [*4*] and, for a more recent discussion of this order of ideas, Coleman [*3*]) given in terms of unstable states generated by the shape of the potential, which should be represented by the Rayleigh-Schrödinger perturbation expansion, in the framework of dilation analyticity and Borel summability.

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Now the differential expression $H = -d^2/dx^2 + x^2 + gx^{2k+1}$, $k \in N$, $g > 0$ is, when defined on $C_0^\infty(R)$, a symmetric operator in $L^2(R)$ admitting infinitely many distinct self-adjoint extensions, each of them having discrete spectrum. Therefore the differential expression H was first realized as an operator in $L^2(R)$ for g complex, $0 < \arg(g) < \pi$, where it represents a holomorphic family of type A with compact resolvents. It turns out that H converges in the generalized sense to $p^2 + x^2$ as $|g| \to 0$, $0 < \arg(g) < \pi$, so that $\sigma_d(H) \neq \emptyset$ for $|g|$ suitably small, and that the (divergent) Rayleigh-Schrödinger perturbation expansion near any simple eigenvalue of $p^2 + x^2$ is Borel summable to the nearby eigenvalue of H. When analytically continued to $g \in R$ the eigenvalues of H can be interpreted as resonances of the problem: in particular they are shown to be second sheet poles of a unique generalized resolvent (see e. g. [*1*] for a definition) of the symmetric operator H.

From this fact one can ask whether it is possible to approximate $x^2 + gx^{2k+1}$ by dilation analytic potentials so that such generalized resonances become the limit of resonances of self-adjoint operators in the standard sense of dilation analyticity.

This is just what we prove in Sect. 2 of this paper, by means of the following family of approximating potentials

$$V_\alpha(x) = (x^2 + gx^{2k+1})(\alpha^2 x^{4k+2} + 1)^{-1/2}, \qquad \text{as } \alpha \to 0.$$

Taking into account the result in [*2*] on the discreteness of the spectrum of $H(0, \theta) = e^{-\theta} p^2 + e^\theta x^2 + g e^{(2k+1)\theta/2} x^{2k+1}$ for $\operatorname{Im} \theta > 0$ and $g > 0$, the problem reduces to a stability result for the eigenvalues of $H(0, \theta)$ with respect to the family $H(\alpha, \theta) = e^{-\theta} p^2 + V_\alpha(e^{\theta/2} x)$ as $\alpha \to 0$. In this connection we apply the stability theory recentely developed by Hunziker and Vock [*6*]: to this end we need to extend the operator class explicitely provided for applications ([*6*], Example 5) and prove the uniform boundedness of the resolvents which is not given by a control of the numerical ranges.

Since $V_\alpha(x)$ exhibits the typical shape of a barrier, this result examplifies at the same time how in some cases the so-called shape resonances [*3*] can be proved to exist in the standard sense of dilation analyticity. This last result, obtained by adapting to $L^2(R_+)$ our arguments valid on $L^2(R)$, is briefly described after proving the existence of resonances for $p^2 + V_\alpha(x)$ by the above mentioned stability argument.

2. STABILITY OF EIGENVALUES AND APPROXIMATING RESONANCES

The following theorem allows to enlarge the operator class given in [*6*], Example 5, in which our model is not included.

THEOREM 2.1. — Let $G \subset L^2_{loc}(R^\nu)$ and $\{A(V)\}_{V \in G}$ be a class of closable operators in $L^2(R^\nu)$ of the form $A(V) = p^2 + V$, with $C_0^\infty(R^\nu)$ as a core and such that for any $V \in G$ there exists a function $\gamma_V : R^\nu \to R$ satisfying the following conditions:

(1) $\mathrm{Re}\langle u\cos\gamma_V, p^2u\rangle + \mathrm{Im}\langle u\sin\gamma_V, p^2u\rangle$
$\leq \xi\,[\mathrm{Re}\langle u\cos\gamma_V, A(V)u\rangle + \mathrm{Im}\langle u\sin\gamma_V, A(V)u\rangle + \eta\langle u, u\rangle],\ \forall u \in C_0^\infty(R^\nu)$

(2) $$\inf_{x \in R^\nu} \cos\gamma_V(x) \geq C_0 > 0$$

(3) $\gamma_V \in H^1_{loc}(R^\nu)$ and $\|\nabla\cos\gamma_V\|_\infty \leq C_1$, $\|\nabla\sin\gamma_V\|_\infty \leq C_1$,

where ξ, η, C_0 and C_1 are positive constants independent of V. Here $H^1_{loc}(R^\nu)$ denotes, as usual, the space of locally square integrable functions which admit first order generalized derivatives belonging to $L^2_{loc}(R^\nu)$;

$$p^2 = -\sum_{k=1}^{\nu} \partial^2/\partial x_k^2 .$$

Then there is $b > 0$ independent of $V \in G$ such that $\forall u \in C_0^\infty(R^\nu)$

$$\|(1 + p^2)^{1/2}u\| \leq b(\|u\| + \|A(V)u\|) \tag{2.1}$$

and there exists a sequence of multiplication operators $\{M_n\}_{n \in N}$ in $L^2(R^\nu)$ satisfying (*i*) and (*ii*) of Hypothesis 3 in [*6*]. Such Hypothesis is completely fulfilled if, moreover, $\Delta \neq \emptyset$ (Δ as defined in [*6*]).

Proof. — It suffices to prove the following inequality

$$\langle u, p^2u\rangle \leq \xi_1[\mathrm{Re}\langle u\cos\gamma_V, A(V)u\rangle + \mathrm{Im}\langle u\sin\gamma_V, A(V)u\rangle + \eta_1\langle u, u\rangle] \tag{2.2}$$

$\forall u \in C_0^\infty(R^\nu)$, with ξ_1, η_1 independent of V, $\xi_1 > 0$, $\eta_1 > 0$.

In fact (2.1) easily follows from (2.2) by Schwarz' inequality. Without loss of generality we can assume $\|u\| = 1$. Then

$$\begin{aligned}
&\mathrm{Re}\langle u\cos\gamma_V, p^2u\rangle + \mathrm{Im}\langle u\sin\gamma_V, p^2u\rangle \\
&= \sum_{k=1}^{\nu}\langle(\cos\gamma_V)p_ku, p_ku\rangle + \sum_{k=1}^{\nu}\mathrm{Re}\langle up_k\cos\gamma_V, p_ku\rangle + \sum_{k=1}^{\nu}\mathrm{Im}\langle up_k\sin\gamma_V, p_ku\rangle \\
&\geq C_0\langle u, p^2u\rangle - \sum_{k=1}^{\nu}|\langle up_k\cos\gamma_V, p_ku\rangle| - \sum_{k=1}^{\nu}|\langle up_k\sin\gamma_V, p_ku\rangle| \\
&\geq C_0\langle u, p^2u\rangle - 2C_1\sum_{k=1}^{\nu}\langle u, p_k^2u\rangle^{1/2} \\
&\geq C_0\langle u, p^2u\rangle - C_1\lambda^{-1}\sum_{k=1}^{\nu}2\lambda\langle u, p_k^2u\rangle^{1/2},
\end{aligned}$$

for any $\lambda > 0$,

$$\geq C_0 \langle u, p^2 u \rangle - C_1 \lambda^{-1} \sum_{k=1}^{\nu} [\lambda^2 + \langle u, p_k^2 u \rangle] = (C_0 - C_1 \lambda^{-1}) \langle u, p^2 u \rangle - C_1 \lambda \nu$$

where the second inequality follows from Schwarz' inequality and assumption (3). Here, as usual, $p_k = -i\partial/\partial x_k$, $k = 1, 2, \ldots, \nu$. Now we choose $\lambda > 0$ such that $C_0 - C_1 \lambda^{-1} > 0$ and (2.2) follows from assumption (1).

In order to show that Hypothesis 3 in [6] is satisfied we proceed as in Lemma 3.2 of [6], choosing $M_n = 1 - \chi_n$, where $\chi_n(x) = \chi(x/n)$, $\chi \in C_0^\infty(R^\nu)$, $\chi(x) = 1$ for $|x| < 1$, and the theorem is thus proved.

Consider in $L^2(R)$ the formal differential expression

$$H = p^2 + x^2 + g x^{2k+1}, \qquad k = 1, 2, \ldots; \qquad g \in R \backslash \{0\}$$

where, as usual, $p^2 = -d^2/dx^2$. Since specular arguments apply for $g < 0$, we can take $g > 0$ without loss of generality.

By $\overline{H}$ we mean the closure of the symmetric operator defined by H on $C_0^\infty(R)$. Set

$$H(0, \theta) = e^{-\theta} p^2 + e^\theta x^2 + g e^{(2k+1)\theta/2} x^{2k+1}.$$

For $\theta \in R$, $H(0, \theta)$ can be written as $U(\theta) H U(\theta)^{-1}$, where $U(\theta) f(x) = e^{\theta/4} f(e^{\theta/2} x)$ defines the group of unitary dilations $U(\theta)$ in $L^2(R)$.

It is well known ([8], [9]) that $\overline{H}$ admits infinitely many distinct self-adjoint extensions. The following proposition is proved in [2].

THEOREM 2.2. — *i*) Each self-adjoint extension of $\overline{H}$ has discrete spectrum.

ii) Let θ be complex, $0 < \operatorname{Im} \theta < \min(\pi/4, 2\pi/(2k+3))$. Then $H(0, \theta)$ represents a holomorphic family of type A of compact resolvent operators in $L^2(R)$ if defined on $D(p^2) \cap D(x^{2k+1}) = H^2(R) \cap L^2_{(2k+1)/2}(R)$ where $H^2(R)$ and $L^2_{(2k+1)/2}(R)$ denote the usual Sobolev and weighted L^2 spaces respectively.

iii) For any compact Γ contained in the strip

$$0 < \operatorname{Im} \theta < \min(\pi/4, 2\pi/(2k+3)),$$

$\exists g(\Gamma) > 0$ such that, $\forall g < g(\Gamma)$, $H(0, \theta)$ admits eigenvalues (independent of θ) for $\theta \in \Gamma$.

iv) There is a dense set S of dilation analytic vectors

$$(\text{for } |\operatorname{Im} \theta| < \min(\pi/4, 2\pi/(2k+3)))$$

and a generalized resolvent $\mathscr{R}(E)$ of the closed symmetric operator $\overline{H}$ (see e. g. [1] for a definition) such that if $\psi \in S$ the function

$$f_\psi(E) = \langle \mathscr{R}(E)\psi, \psi \rangle$$

which is *a priori* defined as an analytic function in the upper half-plane $\{ E \mid \operatorname{Im} E > 0 \}$ has a meromorphic continuation to the lower half-plane $\{ E \mid \operatorname{Im} E < 0 \}$ across the real axis. The set of singularities $\{ E \mid f_\psi$ has a pole at E for some $\psi \in S \}$ coincides with

$$\bigcup_\theta \{ \sigma(H(0, \theta)) \mid 0 < \operatorname{Im} \theta < \min (\pi/4, 2\pi/(2k + 3)) \},$$

and $\mathscr{R}(E)$ is uniquely determined by the identity

$$\langle \mathscr{R}(E)\phi, \psi \rangle = \langle [H(0, \theta) - E]^{-1}\phi(\theta), \psi(\bar{\theta}) \rangle,$$
$$0 < \operatorname{Im} \theta < \min (\pi/4, 2\pi/(2k+3)) ; \ \phi, \psi \in S, \ \psi(\theta)(x) = e^{\theta/4}\psi(e^{\theta/2}x); \ \operatorname{Im} E > 0.$$

For $\alpha > 0$ and θ complex let us define

$$H(\alpha, \theta) = e^{-\theta}[p^2 + V_\alpha(x, \theta)] = e^{-\theta}\left[p^2 + \frac{e^{2\theta}x^2 + ge^{(2k+3)\theta/2}x^{2k+1}}{(\alpha^2 e^{(2k+1)\theta}x^{4k+2} + 1)^{1/2}} \right] \tag{2.3}$$

on the domain $D(H(\alpha, \theta)) = H^2(R)$, as an operator in $L^2(R)$. It is easily seen that the complex-valued potential $V_\alpha(x, \theta)$ is an analytic function of θ in the strip $| \operatorname{Im} \theta | < \pi/(4k + 2)$. Since $V_\alpha(x, \theta)$ is bounded, by standard arguments (see e. g. Kato [7]) $H(\alpha, \theta)$ is a holomorphic family of type A for $| \operatorname{Im} \theta | < \pi/(4k + 2)$.

THEOREM 2.3. — Let $| \operatorname{Im} \theta | < \pi/(4k + 2)$. Then

$$\sigma_{ess}(H(\alpha, \theta)) = \{ z = -g\alpha^{-1} + \lambda e^{-\theta} \mid \lambda \geq 0 \} \cup \{ z = g\alpha^{-1} + \lambda e^{-\theta} \mid \lambda \geq 0 \}.$$

Proof. — Set $D' = \{ u \in H^2(R) \mid$ there is a compact subset $K(u)$ of R such that $\operatorname{supp} u \subset K(u) \}$ and $H'(\alpha, \theta) = H(\alpha, \theta) \upharpoonright D'$. Now we restrict D' by imposing the additional conditions $u(0) = u'(0) = 0$; let D'' denote the manifold obtained by this restriction and $H''(\alpha, \theta) = H'(\alpha, \theta) \upharpoonright D''$. Then $H''(\alpha, \theta) = H'_1(\alpha, \theta) \oplus H'_2(\alpha, \theta)$ (for the « decomposition » method see [5] or [8]) where $H'_1(\alpha, \theta)$ and $H'_2(\alpha, \theta)$ are generated in $L^2(R_-)$ and $L^2(R_+)$ respectively by the differential expression $e^{-\theta}[p^2 + V_\alpha(x, \theta)]$ in the same way as $H'(\alpha, \theta)$ was. Then $\sigma_{ess}(\overline{H}''(\alpha, \theta)) = \sigma_{ess}(\overline{H}'_1(\alpha, \theta)) \cup \sigma_{ess}(\overline{H}'_2(\alpha, \theta))$, where $\overline{H}''(\alpha, \theta)$ denotes any finite dimensional closed extension of $H''(\alpha, \theta)$ and similarly for $\overline{H}'_1$ and $\overline{H}'_2$ (again see [5]): in particular $\overline{H}''(\alpha, \theta)$ can be $H(\alpha, \theta)$. Now, $(e^{-\theta}V_\alpha(x, \theta) - g\alpha^{-1})$ is relatively compact with respect to $(e^{-\theta}p^2 + g\alpha^{-1})$ in $L^2(R_+)$, since $(e^{-\theta}V_\alpha(x, \theta) - g\alpha^{-1}) \to 0$ as $x \to +\infty$ (see [7]), so that

$$\sigma_{ess}(\overline{H}'_2(\alpha, \theta)) = \{ z = g\alpha^{-1} + \lambda e^{-\theta} \mid \lambda \geq 0 \}.$$

An analogous argument works for $\overline{H}'_1(\alpha, \theta)$ and the theorem is thus proved.

From now on, θ will be fixed in the strip $0 < \operatorname{Im} \theta < \pi/(4k + 2)$; for any $\alpha \geq 0$ we shall use the simplified notation $H(\alpha)$ to denote the differential operator defined by $H(\alpha) = e^\theta H(\alpha, \theta) = p^2 + V_\alpha$, $D(H(0)) = H^2(R) \cap L^2_{(2k+1)/2}(R)$ and $D(H(\alpha)) = H^2(R)$ $\forall \alpha > 0$.

LEMMA 2.4. — The operator family $\{ H(\alpha) \}_{\alpha \geq 0}$ satisfies assumptions (1), (2), (3) of Theorem 2.1.

Proof. — For simplicity we assume θ of the form $\theta = i\theta_0$, with

$$0 < \theta_0 = \operatorname{Im} \theta < \pi/(4k+2).$$

Let $\alpha \geq 0$ and set $f_\alpha(x) = (\alpha^2 e^{(2k+1)i\theta_0} x^{4k+2} + 1)^{1/2}$. Then

$$H(\alpha) = p^2 + (e^{2i\theta_0} x^2 + g e^{(2k+3)i\theta_0/2} x^{2k+1}) \bar{f}_\alpha(x) \,|\, f_\alpha(x) \,|^{-2}.$$

Now we proceed to calculate the terms of the right hand side of the inequality to be proved.

$$\begin{aligned}\operatorname{Re}\langle u, \cos \gamma_\alpha H(\alpha) u \rangle &= \operatorname{Re}\langle u, \cos \gamma_\alpha p^2 u \rangle \\ &+ \langle u, \cos \gamma_\alpha (x^2 \cos 2\theta_0 + g x^{2k+1} \cos (2k+3)\theta_0/2) \,|\, f_\alpha \,|^{-2} \operatorname{Re} f_\alpha u \rangle \\ &+ \langle u, \cos \gamma_\alpha (x^2 \sin 2\theta_0 + g x^{2k+1} \sin (2k+3)\theta_0/2) \,|\, f_\alpha \,|^{-2} \operatorname{Im} f_\alpha u \rangle.\end{aligned}$$

Similarly

$$\begin{aligned}\operatorname{Im}\langle u, \sin \gamma_\alpha H(\alpha) u \rangle &= \operatorname{Im}\langle u, \sin \gamma_\alpha p^2 u \rangle \\ &+ \langle u, \sin \gamma_\alpha (x^2 \cos 2\theta_0 + g x^{2k+1} \cos (2k+3)\theta_0/2) \,|\, f_\alpha \,|^{-2} \operatorname{Im} f_\alpha u \rangle \\ &- \langle u, \sin \gamma_\alpha (x^2 \sin 2\theta_0 + g x^{2k+1} \sin (2k+3)\theta_0/2) \,|\, f_\alpha \,|^{-2} \operatorname{Re} f_\alpha u \rangle.\end{aligned}$$

Then

$$\begin{aligned}&\operatorname{Re}\langle u, \cos \gamma_\alpha H(\alpha) u \rangle + \operatorname{Im}\langle u, \sin \gamma_\alpha H(\alpha) u \rangle \\ &= \operatorname{Re}\langle u, \cos \gamma_\alpha p^2 u \rangle + \operatorname{Im}\langle u, \sin \gamma_\alpha p^2 u \rangle \\ &+ \langle u, x^2 \,|\, f_\alpha \,|^{-2} [\operatorname{Re} f_\alpha \cos (2\theta_0 + \gamma_\alpha) + \operatorname{Im} f_\alpha \sin (2\theta_0 + \gamma_\alpha)] u \rangle \\ &+ \langle u, g x^{2k+1} \,|\, f_\alpha \,|^{-2} [\operatorname{Re} f_\alpha \cos (\gamma_\alpha + (2k+3)\theta_0/2) + \operatorname{Im} f_\alpha \sin (\gamma_\alpha + (2k+3)\theta_0/2)] u \rangle\end{aligned}$$

Let us construct $\gamma_\alpha(x)$ so that the second term of the right hand side of the last inequality is positive and the third one vanishes.

If $\gamma_\alpha(x) \neq -(2k+3)\theta_0/2$ then $\sin [\gamma_\alpha(x) + (2k+3)\theta_0/2] \neq 0$ and the condition $\operatorname{Re} f_\alpha [\cos \gamma_\alpha + (2k+3)\theta_0/2] + \operatorname{Im} f_\alpha \sin [\gamma_\alpha + (2k+3)\theta_0/2] = 0$ is equivalent to the following

$$-\tan \arg (f_\alpha) = \cot [\gamma_\alpha + (2k+3)\theta_0/2] = \tan [\pi/2 - \gamma_\alpha - (2k+3)\theta_0/2].$$

One solution to this equation is given by

$$\gamma_\alpha(x) = \pi/2 + \arg (f_\alpha(x)) - (2k+3)\theta_0/2 \tag{2.4}$$

Note that for any $\alpha > 0$ $\inf_x \arg (f_\alpha(x)) = 0$ and $\sup_x \arg (f_\alpha(x)) = (2k+1)\theta_0/2$. On the other hand, $f_0(x) = 1$, for all x; hence

$$\inf_x \cos \gamma_\alpha(x) \geq \cos (\pi/2 - \theta_0) = \sin \theta_0 = C_0 > 0,$$

for any $\alpha \geq 0$, and for this choice of $\gamma_\alpha(x)$ assumption (2) of Theorem 2.1 is satisfied.

Now we need to show that if $\gamma_\alpha(x)$ is given by (2.4) then

$$\operatorname{Re} f_\alpha(x) \cos(2\theta_0 + \gamma_\alpha(x)) + \operatorname{Im} f_\alpha(x) \sin(2\theta_0 + \gamma_\alpha(x)) \geq 0, \quad \forall x \tag{2.5}$$

We have $2\theta_0 + \gamma_\alpha(x) = \pi/2 + \arg(f_\alpha(x)) - (2k-1)\theta_0/2$. Thus for $\alpha = 0$, (2.5) is trivially verified. For $\alpha > 0$,

$$\sup_x (\gamma_\alpha(x) + 2\theta_0) = \theta_0 + \pi/2 \quad \text{and} \quad \inf_x (\gamma_\alpha(x) + 2\theta_0) = \pi/2 - (2k-1)\theta_0/2;$$

therefore $\sin(\gamma_\alpha(x) + 2\theta_0) > 0$ for all x and (2.5) is now equivalent to $\cot(2\theta_0 + \gamma_\alpha(x)) \geq -\tan\arg(f_\alpha(x))$, i. e.

$$\tan((2k-1)\theta_0/2 - \arg(f_\alpha(x))) \geq \tan(-\arg(f_\alpha(x)))$$

and this last inequality is certainly satisfied $\forall x$.

Now the inequality in assumption (1) of Theorem 2.1 holds with $\xi = 1$ and $\eta = 0$ for any $\alpha \geq 0$. In order to prove part (3) we need a more explicit expression for $\arg(f_\alpha(x))$, $\alpha > 0$: $\operatorname{Im} f_\alpha^2(x) = a^2 x^{4k+2} \sin(2k+1)\theta_0$ and $\operatorname{Re} f_\alpha^2(x) = \alpha^2 x^{4k+2} \cos(2k+1)\theta_0 + 1$. Thus,

$$\forall x \neq 0, \ \arg(f_\alpha(x)) = (1/2) \operatorname{arc\,cot} [\cot(2k+1)\theta_0 + (\alpha^2 x^{4k+2} \sin(2k+1)\theta_0)^{-1}].$$

An easy calculation yields

$$\frac{d}{dx}(\cos\gamma_\alpha)(x) = -(2k+1)\sin\gamma_\alpha(x)[\alpha^2 x^{4k+1}\sin(2k+1)\theta_0][h(x)]^{-1}$$

where $h(x) = (1+a^2)\alpha^4 x^{8k+4} \sin^2(2k+1)\theta_0 + 2a\alpha^2 x^{4k+2} \sin(2k+1)\theta_0 + 1$, with $a = \cot(2k+1)\theta_0$. If we assume $\theta_0 < \pi/(8k+4)$ then the rational term in the last equation is easily seen to be bounded by 1, uniformly in α. Since $\frac{d}{dx}(\cos\gamma_0)(x) = 0$, $\forall x$, we conclude that

$$\left\| \frac{d}{dx} \cos\gamma_\alpha \right\|_\infty \leq (2k+1) = C_1 < +\infty, \qquad \forall \alpha \geqslant 0.$$

Similarly one can prove $\left\| \frac{d}{dx} \sin\gamma_\alpha \right\|_\infty \leq C_1$ and this completes the proof of Lemma 2.4.

Now set $\Delta = \{ z \in \mathbb{C} \mid$ there is $\bar{\alpha}(z) > 0$ s. t. $z \notin \sigma(\mathrm{H}(\alpha))$ and $\| (\mathrm{H}(\alpha) - z)^{-1} \|$ is uniformly bounded for $0 \leq \alpha < \bar{\alpha}(z) \}$.

It is well known (see [7]) that Δ is open and, given any compact $\Gamma \subset \Delta$ there is $\bar{\alpha}(\Gamma) > 0$ such that $\| (\mathrm{H}(\alpha) - z)^{-1} \|$ is uniformly bounded for $z \in \Gamma$ and $0 \leq \alpha < \bar{\alpha}(\Gamma)$.

Unlike the situation presented in [6], Example 5, we are dealing with operators whose numerical ranges invade the whole complex plane as $\alpha \to 0$. For this reason we need the following.

LEMMA 2.5. — The operator family $\{ \mathrm{H}(\alpha) \}_{\alpha \geqslant 0}$ satisfies $\Delta \neq \emptyset$.

Proof. — By Theorem 2.2 $\sigma_{ess}(\mathrm{H}(0)) = \emptyset$ and by Theorem 2.3 for any $z \in \mathrm{C}$ there exists $\bar{\alpha}(z) > 0$ such that $z \notin \sigma_{ess}(\mathrm{H}(\alpha))$ for $0 \leq \alpha < \bar{\alpha}(z)$. Thus, Lemma 5.1 of [6] applies and if $z \notin \sigma_d(\mathrm{H}(0))$ then $z \in \Delta$, unless there exist two sequences $\{ \alpha_n \} \subset \mathrm{R}_+$ and $\{ u_n \} \subset \mathrm{L}^2(\mathrm{R})$ such that $\alpha_n \to 0, u_n \in \mathrm{D}(\mathrm{H}(\alpha_n))$, $\| u_n \| \nrightarrow 0$, $u_n \underset{\mathrm{W}}{\to} 0$ and $\| (z - \mathrm{H}(\alpha_n))u_n \| \to 0$. In order to exclude the second alternative we notice that u_n can be chosen in $\mathrm{C}_0^\infty(\mathrm{R})$, since it is a core for $\mathrm{H}(\alpha_n)$. Now fix $\chi \in \mathrm{C}_0^\infty(\mathrm{R})$ and let $\mathrm{M}_n = 1 - \chi_n$ be the sequence of multiplication operators specified in the proof of Theorem 2.1. If we define

$$\mathrm{M}_n^+(x) = \mathrm{M}^+(x/n) = \begin{cases} 0 & , \text{ if } x < 0 \\ 1 - \chi(x/n), & \text{if } x \geq 0 \end{cases}$$

and

$$\mathrm{M}_n^-(x) = \mathrm{M}^-(x/n) = \begin{cases} 1 - \chi(x/n), & \text{if } x \leq 0 \\ 0 & , \text{ if } x > 0, \end{cases}$$

then $\mathrm{M}_n = \mathrm{M}_n^+ + \mathrm{M}_n^-$ and by (*ii*) of Hypothesis 3 in [6], which follows from Theorem 2.1 and Lemma 2.4, there exists $a > 0$ such that, for each n,

$$\limsup_{m \to +\infty} \| \mathrm{M}_n u_m \|^2 = \limsup_{m \to +\infty} (\| \mathrm{M}_n^+ u_m \|^2 + \| \mathrm{M}_n^- u_m \|^2) \geq a^2 .$$

Thus, for each n

either $$\limsup_m \| \mathrm{M}_n^+ u_m \| \geq a/2 \tag{2.6a}$$

or $$\limsup_m \| \mathrm{M}_n^- u_m \| \geq a/2 \tag{2.6b}$$

Then the second alternative can be specified either with a sequence $v_n^+ = \mathrm{M}_n^+ u_{m(n)}$, such that $v_n^+(x) = 0$ for $x < n$, or with a sequence $v_n^- = \mathrm{M}_n^- u_{m(n)}$, such that $v_n^-(x) = 0$ for $x > -n$, by suitably choosing $m = m(n)$.

In fact, let (2.6*a*) hold. We have

$$\| (z - \mathrm{H}(\alpha_m))\mathrm{M}_n^+ u_m \| \leq \| (z - \mathrm{H}(\alpha_m))u_m \| + \| [\mathrm{M}_n^+, \mathrm{H}(\alpha_m)]u_m \| \tag{2.7}$$

and, since

$$[\mathrm{M}_n^+, \mathrm{H}(\alpha)] = [\mathrm{M}_n^+, p^2] = 2in^{-1} \frac{d\mathrm{M}^+}{dx}(x/n) - n^{-2} \frac{d^2\mathrm{M}^+}{dx^2}(x/n),$$

it follows from (2.1) that

$$\| [\mathrm{M}_n^+, \mathrm{H}(\alpha)]u \| \leq cn^{-1}(\| \mathrm{H}(\alpha)u \| + \| u \|) \tag{2.8}$$

where $c > 0$ is a constant independent of α and n. Now, combining (2.6*a*), (2.7) and (2.8), a suitable choice of $m = m(n)$ and normalization yield

$$\lim_n \| (z - \mathrm{H}(\alpha_{m(n)})v_n^+ \| = 0, \qquad v_n^+ \underset{\mathrm{W}}{\to} 0 \quad \text{and} \quad \| v_n^+ \| = 1 .$$

The analogous properties can be specified for the sequence $\{ v_n^- \}$ if (2.6*b*) holds. Now the contradiction follows from

$$\| [\mathrm{H}(\alpha_{m(n)}) - z]v_n^+ \| \geq \operatorname{dist}(z, \mathrm{E}_n^+(\alpha_{m(n)}))$$

where $E_n^+(\alpha) = \{ \langle H(\alpha)v, v \rangle \mid v \in C_0^\infty(R), \|v\|=1, v(x)=0 \text{ for } x < n \}$. In fact we have

$$\mathrm{Re} \langle (p^2+V_\alpha)v, v \rangle \geq \langle (\mathrm{Re}\, V_\alpha)v, v \rangle \quad \text{and} \quad \lim_{\substack{x\to+\infty \\ \alpha\to 0}} \mathrm{Re}\, V_\alpha(x) = +\infty$$

i. e. there are not two sequences $\{ x_k \} \to +\infty$ and $\{ \alpha_k \} \to 0$ such that $\mathrm{Re}\, V_{\alpha_k}(x_k)$ is bounded from above. Thus, since $\alpha_{m(n)} \to 0$, for any $\delta > 0$ there exists $n_0 \in N$ such that $\| [H(\alpha_{m(n)}) - z]v_n^+ \| \geq \delta > 0$, $\forall n \geq n_0$. Similarly we obtain a contradiction if (2.6*b*) holds, since $\lim_{\substack{x\to-\infty \\ \alpha\to 0}} \mathrm{Im}\, V_\alpha(x) = -\infty$, and this completes the proof of Lemma 2.5.

COROLLARY 2.6. — Every eigenvalue of H(0) is stable (in the sense of Kato [7]) with respect to the family $\{ H(\alpha) \}_{\alpha \geq 0}$.

Proof. — By Lemma 2.4 and 2.5 the family $\{ H(\alpha) \}_{\alpha \geq 0}$ satisfies Hypothesis 3 of [6] with the operators M_n specified in the proof of Theorem 2.1. Moreover, by Theorems 2.2 and 2.3, for any fixed $\lambda \in C$ and $\delta > 0$ there exists $\alpha_0 > 0$ such that $\mathrm{dist}(\lambda, \sigma_{ess}(H(\alpha))) \geq \delta$ for $0 \leq \alpha \leq \alpha_0$. Finally we can decompose $M_n = M_n^+ + M_n^-$, as specified in the proof of Lemma 2.5, where M_n^+ and M_n^- also satisfy (*iii*) of Hypothesis 3 in [6]. Since $\lim_{\substack{x\to+\infty \\ \alpha\to 0}} \mathrm{Re}\, V_\alpha(x) = +\infty$, proceeding as in the last part of the preceding lemma, we find $\alpha_1 > 0$ and $n_0 \in N$ such that

$$d_n^+(\lambda, \alpha) = \inf_{u \in D(H(\alpha)), \|M_n^+ u\| = 1} \| (\lambda - H(\alpha))M_n^+ u \| \geq \delta > 0$$

for $\alpha < \alpha_1$ and $n > n_0$. Similarly, since $\lim_{\substack{x\to-\infty \\ \alpha\to 0}} \mathrm{Im}\, V_\alpha(x) = -\infty$, $d_n^-(\lambda, \alpha) \geq \delta$ for $\alpha < \alpha_1$ and $n > n_0$. Therefore we can apply Theorem 5.8 of [6] and the stability of eigenvalues is proved.

As a consequence, we can now prove the result announced in the introduction. For convenience, we return to the more explicit notation of Theorem 2.2, by replacing $H(\alpha)$ with $e^\theta H(\alpha, \theta)$.

THEOREM 2.7. — Let $H(\alpha, \theta)$ be the holomorphic family of type A in $L^2(R)$ defined for $|\mathrm{Im}\, \theta| < \pi/(4k+2)$ and $\alpha > 0$ by (2.3), self-adjoint for $\theta \in R$. Then there exists $g_0 > 0$ such that for $0 < g < g_0$ there is $\alpha_g > 0$ with the property that $\forall \alpha \in (0, \alpha_g)$, $H(\alpha, 0)$ admits second sheet poles of the resolvent in the following sense: if $S \subset L^2(R)$ is the dense set of all dilation analytic vectors for $|\mathrm{Im}\, \theta| < \pi/(4k+2)$ and $\psi \in S$, the scalar product

$$g_\psi(E) = \langle [H(\alpha, 0) - E]^{-1}\psi, \psi \rangle,$$

which is *a priori* analytic in $C \backslash \sigma(H(\alpha, 0))$, admits a meromorphic continuation onto the second sheet across the cut $[-g\alpha^{-1}, +\infty)=\sigma_{ess}(\mathrm{H}(\alpha, 0))$ (from the upper half-plane of the first sheet) to the region

$$\{\,\mathrm{E} \mid -\tan(\operatorname{Im}\theta)(\operatorname{Re}\mathrm{E} + g\alpha^{-1}) \leq \operatorname{Im}\mathrm{E} \leq 0\,\}\backslash\sigma_{ess}(\mathrm{H}(\alpha, \theta)),$$

for any θ, $0 < \operatorname{Im}\theta < \pi/(4k+2)$. For $i \in \mathrm{N}$, let $\alpha(i) > 0$ be such that for $\alpha < \alpha(i)$ the eigenvalue $\mathrm{E}_i(\alpha)$ of $\mathrm{H}(\alpha, \theta)$ exists by stability with respect to the i-th eigenvalue of $\mathrm{H}(0, \theta)$. Then for fixed $\bar{i} \in \mathrm{N}$ and $\alpha \leq \min(\alpha(1), \ldots, \alpha(\bar{i}))$ the set $\{\,\mathrm{E}(\alpha) \mid \mathrm{E}(\alpha)$ is a pole of $g_\psi(\mathrm{E})$ for some $\psi \in \mathrm{S}\,\}$ contains at least the finite set of eigenvalues $\{\,\mathrm{E}_i(\alpha) \mid i=1, 2, \ldots, \bar{i}\,\}$. For each $i \in \mathrm{N}$, $\mathrm{E}_i(\alpha)$ tends to the corresponding second sheet pole of the generalized resolvent $\mathscr{R}(\mathrm{E})$ (specified in Theorem 2.2) of the closure of the symmetric operator $p^2+x^2+gx^{2k+1}$.

Proof. — It is a consequence of Corollary 2.6 since, by the usual analyticity arguments (see e. g. [*10*])

$$g_\psi(\mathrm{E}) = \langle\, [\mathrm{H}(\alpha, \theta) - \mathrm{E}]^{-1}\psi(\theta), \psi(\bar{\theta})\,\rangle$$

identically, for $0 < \operatorname{Im}\theta < \pi/(4k+2)$, $\operatorname{Im}\mathrm{E} > 0$, where $\psi(\theta)(x)=e^{\theta/4}\psi(e^{\theta/2}x)$ for $\psi \in \mathrm{S}$.

An analogous stability argument can be used to obtain an existence proof of shape resonances for a suitable class of Schrödinger operators in $\mathrm{L}^2(\mathrm{R}_+)$. In this case the results of Vock and Hunziker [*6*] can be directly applied, having care to set the necessary Dirichlet condition at the origin. Explicitly, if we set

$$\mathrm{V}_0(z) = z^2 - gz^{2k+1}, \quad k \in \mathrm{N}, \quad g > 0, \quad z \in \mathrm{C}, \tag{2.9}$$

then for the operator family defined in $\mathrm{L}^2(\mathrm{R}_+)$ by the differential expression $\mathrm{A}(0, \theta) = e^{-\theta}p^2 + \mathrm{V}_0(e^{\theta/2}x) = e^{-\theta}p^2 + e^{2\theta}x^2 - ge^{(2k+3)\theta/2}x^{2k+1}$, $(0 < \operatorname{Im}\theta < \min(\pi/4, 2\pi/(2k+3)))$ we can prove results analogous to Theorem 2.2. In particular $\mathrm{A}(0, \theta)$ is a holomorphic family of type A in $\mathrm{L}^2(\mathrm{R}_+)$ for $0 < \operatorname{Im}\theta < \min(\pi/4, 2\pi/(2k+3))$ when defined on

$$\mathrm{D} = \mathrm{D}(\mathrm{A}(0, \theta)) = \{\, u \mid u \in \mathrm{H}^2(\mathrm{R}_+) \cap \mathrm{L}^2_{(2k+1)/2}(\mathrm{R}_+),\ u(0)=0\,\} \tag{2.10}$$

and we can show the existence of eigenvalues $\mathrm{E}_i = \mathrm{E}_i(g)$ (for $0<g<g_0$, for some $g_0 > 0$) with $\operatorname{Im}\mathrm{E}_i < 0$, $i \in \mathrm{N}$, which tend to the odd eigenvalues of the harmonic oscillator as $g \to 0$.

THEOREM 2.8. — Let $\mathrm{V}_\alpha(x)$, $\alpha > 0$, $x \in \mathrm{R}_+$ be a family of real-valued functions enjoying the following properties:

i) there is $\theta_0 \in (0, \min(\pi/4, 2\pi/(2k+2))$ such that, for any $\alpha > 0$, $\mathrm{V}_\alpha(x)$ is the restriction to $z \in \mathrm{R}_+$ of a function $\mathrm{V}_\alpha(z)$ holomorphic at least in the sector $\{\, z \in \mathrm{C} \mid |\arg z| < \theta_0\,\}$, and bounded near $z = 0$;

ii) for $\alpha > 0$, $|\operatorname{Im}\theta| < \theta_0$, $\mathrm{V}_\alpha(e^{\theta/2}x) \to c(\alpha)$ as $x \to +\infty$, for some $c(\alpha) \in \mathrm{R}$, and $\lim_{\alpha\to 0} c(\alpha) = -\infty$;

iii) for any fixed θ with $0 < \operatorname{Im}\theta < \theta_0$, $V_\alpha(e^{\theta/2}x) \to V_0(e^{\theta/2}x)$ as $\alpha \to 0$ uniformly on the compact subsets of $R_+(V_0(z)$ as defined in (2.9));

iv) the family $\{ e^\theta V_\alpha(e^{\theta/2}x) \}_{\alpha\geq 0}$ satisfies the hypotheses of Theorem 6.1 in [*6*], with $C_0^\infty(R)$ replaced by D, as defined in (2.10). Then:

(1) for any $\alpha > 0$ the differetial expression $e^{-\theta}p^2 + V_\alpha(e^{\theta/2}x)$ defines a holomorphic family of type A in $L^2(R_+)$, denoted by $A(\alpha, \theta)$, for $|\operatorname{Im}\theta| < \theta_0$, if defined on $D(A(\alpha, \theta)) = \{ u \mid u \in H^2(R_+),\ u(0) = 0 \}$, with D as a core. Furthermore $\sigma_{ess}(A(\alpha, \theta)) = \{ z = c(\alpha) + \lambda e^{-\theta} \mid \lambda \geq 0 \}$.

Let E_0 be an eigenvalue of $A(0, \theta)$ (whence $\operatorname{Im} E_0 < 0$); suppose that for some $\varepsilon > 0$, $n_0 \in N$ and $\alpha_0 > 0$, $\operatorname{dist}(E_0, F_n(\alpha)) \geq \varepsilon > 0, \forall n \geq n_0, 0 < \alpha < \alpha_0$, where $F_n(\alpha) = \{ \langle A(\alpha, \theta)u, u \rangle \mid \| u \| = 1,\ u \in D(A(\alpha, \theta)),\ \operatorname{supp} u \subset [n, +\infty) \}$. Then:

(2) E_0 is a stable eigenvalue with respect to the family $\{ A(\alpha, \theta) \}_{\alpha\geq 0}$; in particular there is $\bar{\alpha} > 0$ such that for all $\alpha \in (0, \bar{\alpha})$ there exists $E(\alpha) \in \sigma_d(A(\alpha, \theta))$ with $\operatorname{Im} E(\alpha) < 0$ and $E(\alpha) \to E_0$ as $\alpha \to 0$;

(3) if S is the dense set of dilation analytic vectors for $|\operatorname{Im}\theta| < \theta_0$, then for all $\alpha \in (0, \bar{\alpha})$ and $\psi \in S$ the function $h_\psi(E) = \langle [A(\alpha, 0) - E]^{-1}\psi, \psi \rangle$, which is *a priori* analytic in $C \backslash \sigma(A(\alpha, 0))$, admits a meromorphic continuation onto the second sheet to an open connected region of the lower half plane (from the upper half plane of the first sheet) across the cut

$$\sigma_{ess}(A(\alpha, 0)) = [c(\alpha), +\infty).$$

Moreover there is $\psi \in S$ such that the continuation of h_ψ has a simple pole at $E(\alpha)$.

Natural examples are self-adjoint operators of the form

$$A(\alpha, 0) = -d^2/dx^2 + (x^2 + \alpha P^{(2k)}(x) - gx^{2k+1})(\alpha x^{\nu\rho} + 1)^{1/\nu}.$$

Here k, ν and ρ are fixed ($k \in N$, $\nu > 0$ and $\rho = 2k + 1$) $\alpha > 0$, $P^{(2k)}(x)$ is an arbitrary polynomial of order not larger than $2k$,

$$D(A(\alpha, 0)) = \{ u \mid u \in H^2(R_+),\ u(0) = 0 \}.$$

Since in this case $\lim_{\substack{\alpha\to 0\\ x\to+\infty}} \operatorname{Im}[e^\theta V_\alpha(e^{\theta/2}x)] = -\infty$, every eigenvalue E_i of $A(0, \theta)$ satisfies the required inequality $\operatorname{dist}(E_i, F_n(\alpha)) \geq \varepsilon_i > 0$ for $n \geq n_i$, $0 < \alpha < \alpha_i$. So, by the previous theorem, $A(\alpha, 0)$ provides an example of shape resonances in the standard sense of dilation analyticity.

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