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A note on cluster expansions

by

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ABSTRACT. — We make, on a particular field theory model, an estimate of the weakest long range behaviour necessary to obtain easily a convergent cluster expansion.

RÉSUMÉ. — On estime, dans un modèle particulier de Théorie des Champs, le comportement à grande distance le plus faible permettant d'obtenir facilement un développement en « clusters » convergent.

1. INTRODUCTION

We prove the existence of the infinite volume limit, with a cluster expansion, of a φ_2^4 model with a covariance

$$C(x, y) = \int d^2p \exp i(x - y) \frac{|p|^{1,5+\varepsilon}}{(p^2 + 1)^2} \quad \text{for} \quad 0 < \varepsilon < 1/2$$

so that

$$|C(x, y)| \leq \frac{O(1)}{|x - y|^{3,5+\varepsilon/2} + 1}$$

Such models have been considered by Federbush [1].

In field theory the cluster expansion is a creation of Glimm Jaffe Spencer; the first version [3] used only characteristic functions, and Brydges gave of it a convenient form that we use [0]. Here the two estimates (lemma 1 and 2) are taken from Glimm and Jaffe [4], [2].

2. THE EXPANSION

For simplicity we consider a two point function:

$$\frac{S_{\Lambda}(f, g)}{Z_{\Lambda}} = Z_{\Lambda}^{-1} \int \varphi(f) \varphi(g) \exp \left(- \lambda \int_{\Lambda} \varphi^4(x) dx \right) d\mu$$

where

$$Z_{\Lambda} = \int \exp \left(- \lambda \int_{\Lambda} \varphi^4(x) dx \right) d\mu \quad \text{and} \quad \varphi(f) = \int \varphi(x) f(x) dx$$

where Λ is a space cutoff and $d\mu$ is the Gaussian measure of covariance $C(x, y)$ and mean zero.

We consider a unit lattice $\mathcal{D}$ on $\mathbb{R}^2$. The support of f (resp. g) is in $\Delta f \in \mathcal{D}$ [resp. Δg].

To a square $\Delta_1 \in \mathcal{D}^*$ we associate a variable s_1 and define

$$C(s_1) = s_1 C + (1 - s_1) C_1, \quad C_1 = \chi_{\Delta_1} C \chi_{\Delta_1} + (1 - \chi_{\Delta_1}) C (1 - \chi_{\Delta_1})$$

where χ_{Δ_1} is the characteristic function of Δ_1 .

Let $d\mu(s_1)$ be the measure of covariance $C(s_1)$ so that for some function Q of the field:

$$S(s_1) = \int Q d\mu(s_1)$$

A step of the expansion consists in the decomposition

$$\begin{aligned} S &= S(s_1) |_{s_1=1} = S(0) + \int_0^1 \left(\frac{d}{ds_1} S(s_1) \right) ds_1 \\ \frac{d}{ds_1} \int Q d\mu(s_1) &= \sum_{\Delta_2} \int P(\Delta_1, \Delta_2) Q d\mu(s_1) \end{aligned}$$

with (see [4]).

$$P(\Delta_1, \Delta_2) = \int \frac{dC(s_1)}{ds_1}(x, y) \frac{\delta}{\delta \varphi(x)} \frac{\delta}{\delta \varphi(y)} \chi_{\Delta_1}(x) \chi_{\Delta_2}(y) dx dy$$

In our case $Q = \varphi(f) \varphi(g) e^{-\int \varphi^4}$ so that, with $Q|_{\mathbb{R}}$ = the part of Q with support in $\mathbb{R}$, $C_{\Delta} = \chi_{\Delta} C \chi_{\Delta}$ and $S_{\sim \Delta} = \int Q|_{\sim \Delta} d\mu|_{\sim \Delta}$:

$$S = \int Q|_{\Delta_1} d\mu|_{\Delta_1} \cdot S_{\sim \Delta} + \sum_{\Delta_2} \int_0^1 ds_1 \int P(\Delta_1, \Delta_2) Q d\mu(s_1)$$

(*) We can take $\Delta_1 = \Delta f$ or Δg , but this is not necessary.

Then we define

$$C(s_1, s_2) = s_2 C(s_1) + (1 - s_2) C(s_1)_2$$

$$C(s_1)_2 = \chi_{\Delta_1 \cup \Delta_2} C(s_1) \chi_{\Delta_1 \cup \Delta_2} + (1 - \chi_{\Delta_1 \cup \Delta_2}) C(s_1) (1 - \chi_{\Delta_1 \cup \Delta_2})$$

and

$$\int P(\Delta_1, \Delta_2) Q d\mu(s_1) = \int P(\Delta_1, \Delta_2) Q \left|_{\Delta_1 \cup \Delta_2} d\mu(s_1, s_2) \right|_{s_2=0} \cdot S_{\sim \Delta_1 \cup \Delta_2}$$

$$+ \sum_{j=1}^2 \sum_{\Delta_3} \int_0^1 ds_2 P(\Delta_j, \Delta_3) P(\Delta_1, \Delta_2) Q d\mu(s_1, s_2)$$

and so forth; we obtain

$$S = \int \varphi(f) e^{-\int_{\Delta_1} \varphi^4 d\mu} \left|_{\Delta_1} \cdot S_{\sim \Delta_1} \right.$$

$$+ \sum_{\Delta_2} \int ds_1 \int P(\Delta_1, \Delta_2) Q \left|_{\Delta_1 \cup \Delta_2} d\mu(s_1, s_2) \right|_{s_2=0} \times S_{\sim \Delta_1 \cup \Delta_2}$$

$$+ \sum_{i=3}^{\infty} \sum_{j_1=1}^1 \sum_{j_2=1}^2 \cdots \sum_{j_{i-1}=1}^{i-1} \sum_{\substack{\{\Delta_j\} \\ i \geq j \geq 2}} \int P(\Delta_{j_{i-1}}, \Delta_i) P(\Delta_{j_{i-2}}, \Delta_{i-1})$$

$$\cdots P(\Delta_{j_1}, \Delta_2) Q \left|_{\cup \Delta_i} ds_1, \dots, ds_{i-1} d\mu_{(s_1, \dots, s_{i-1}, s_i)} \right|_{s_i=0} \cdot S_{\sim \cup \Delta_i} \quad (1)$$

where $|_{\cup \Delta_i}$ means restricted to $\Delta_1 \cup \dots \cup \Delta_i$.

The formula giving $C(s_1, \dots, s_i)$ is the generalisation of the one giving $C(s_1, s_2)$:

$$C(s_1, \dots, s_i) = s_i C(s_1, \dots, s_{i-1}) + (1 - s_i) C(s_1, \dots, s_{i-1})_i$$

$$C(s_1, \dots, s_{i-1})_i = \chi_{\cup \Delta_i} C(s_1, \dots, s_{i-1}) \chi_{\cup \Delta_i} + (1 - \chi_{\cup \Delta_i}) C(s_1, \dots, s_{i-1}) (1 - \chi_{\cup \Delta_i})$$

with $\chi_{\cup \Delta_i} = \chi_{\Delta_1 \cup \dots \cup \Delta_i}$.

Then as noted above $C(s_1, \dots, s_i) |_{s_i=0}$ is a direct sum, so that $d\mu |_{s_i=0}$ factorizes and then in each term of formula (1) the Schwinger function in $\Delta_1 \cup \dots \cup \Delta_i$ is factorized out. We call such a term a tree, because the propagators $\frac{dC}{ds}$ form no cycle (by construction):

$$\frac{dC}{ds_k}(s_1, \dots, s_k)(x, y) \quad \text{is non zero only if} \quad \begin{array}{l} x \in \Delta_1 \cup \dots \cup \Delta_k \\ y \notin \Delta_1 \cup \dots \cup \Delta_k \end{array}$$

so that at each step a propagator connects the tree formed by the squares of the tree (in formation) with some new square.

If we choose $s_i = 0$ then the tree is formed by $\Delta_1, \dots, \Delta_i$ and is factorized out, as said.

Then in each term of formula (1) we expand in the same way $S_{\sim \cup \Delta_i}$ choosing a Δ'_1 in $\mathbb{R}^2 \setminus \cup \Delta_i$ which plays the role of $\Delta_1, \dots$. For given $\{j_1, \dots, j_{i-1}\}$ in formula (1), a vertex Δ_k is said of order n_k or

$$n(\Delta_k) = \# \{j_\alpha = k, \alpha \geq k\},$$

i. e. there are $n_k + 1$ propagators which have an extremity in Δ_k , except for Δ_1 where there are only n_1 .

Finally we obtain

$$S = \Sigma \Pi \text{ trees}$$

One tree contains Δf and Δg by parity.

For $\Delta_1, \dots, \Delta_i$ given we call $T(\Delta_1, \dots, \Delta_i)$ the sum of the trees whose vertices are $\Delta_1, \dots, \Delta_i$.

In the following we prove

PROPOSITION. —

$$\left| \sum_{\substack{\{\Delta_k\} \\ k \neq i, j}} T(\Delta_1, \dots, \Delta_n) \right| \leq 0(1) \frac{1}{(\text{dist}(\Delta_i, \Delta_j) + 1)^{3.5 + \varepsilon/4}} e^{-nK}$$

where K is as big as we want for λ small enough.

As a conclusion each tree is exponentially small like the number of squares that it contains. Moreover we can sum over the localization of the squares of each tree.

A standard argument of statistical mechanics [4, Chapter 6] gives then

Final result. — For λ small enough $\lim_{\lambda \rightarrow \infty} \frac{S_\Lambda(f, g)}{Z_\Lambda}$ converges and is bounded by

$$\text{Cst}(f) \text{Cst}(g) (\text{dist}(\Delta f, \Delta g) + 1)^{-(3.5 + \varepsilon/4)}$$

3. THE BOUNDS

The form of each tree is given by the $n(\Delta)$'s. The sum over each $n(\Delta)$ is controlled using $\sum_{n(\Delta)} 2^{-n(\Delta)} \leq 0(1)$ so that $\sum_{n(\Delta)} \cdot \leq 0(1) \sup_{n(\Delta)} 2^{n(\Delta)}$.

Now the sum over the order of the Δ 's in the tree is taken into account by summing over all the localizations of the vertices of the tree using $[d(\Delta_1, \Delta) + 1]^{-2 - \varepsilon/4}$ as a combinatorial factor. We use

$$\sum_{\Delta} (d(\Delta_1, \Delta) + 1)^{-2 - \varepsilon/4} \leq 0(1)$$

for the $n(\Delta_1)$ squares linked to Δ ; then we iterate the process.

A chain of propagators links Δf to Δg in the tree containing both. Then using:

$$\sum_{\Delta_2} \frac{1}{(\text{dist}(\Delta_1, \Delta_2) + 1)^{3.5 + \varepsilon/4}} \frac{1}{(\text{dist}(\Delta_2, \Delta_3) + 1)^{3.5 + \varepsilon/4}} \leq \frac{0(1)}{(\text{dist}(\Delta_1, \Delta_3) + 1)^{3.5 + \varepsilon/4}}$$

we obtain the decrease in the distance of Δf to Δg .

Now we prove the proposition on trees.

For each vertex (localized in a square Δ) we apply the bound:

$$\int_{\Delta} \prod_{i=1}^{\alpha} |C(x_i, y) \varphi^{4-\alpha}(y)| dy \leq \left(\prod_{i=1}^{\alpha} \sup_{y \in \Delta} |C(x_i, y)| \right) \int_{\Delta} |\varphi^{4-\alpha}(y)| dy$$

So that the contribution of a square Δ is [for $\Delta = \Delta_1$ the product is up to $n(\Delta_1)$]:

$$I(n(\Delta)) = \left| \left(\prod_{i=1}^{n(\Delta)+1} \int_{\Delta} \frac{\delta}{\delta |\varphi|(x)} dx \right) e^{-\lambda \int_{\Delta} |\varphi|^4} \right|$$

LEMMA 1. —

$$I(n(\Delta)) \leq 0(1) \lambda^{\frac{n(\Delta)}{4}} n(\Delta)^{\frac{3}{4}n(\Delta)} 0(1)^{n(\Delta)}$$

Proof. — The $\frac{\delta}{\delta |\varphi|}$'s derive either the exponentiel or already produced vertices. We use the Holder inequality:

$$\int_{\Delta} (\lambda^{1/4} |\varphi(x)|)^{\alpha} dx \leq \left(\lambda \int_{\Delta} \varphi(x)^4 \right)^{\alpha/4} \quad \alpha = 1, 2, 3,$$

then just by bookkeeping one obtains:

$$I(n) \leq 0(1) \lambda^{\frac{n}{4}} \sup_{n'} \left(\left[\lambda \int_{\Delta} |\varphi(x)|^4 \right]^{\frac{4n'-n}{4}} e^{-\lambda \int_{\Delta} \varphi^4} n^{(n-n')} \right)$$

where n' is the number of vertices created by the $\frac{\delta}{\delta \varphi}$'s and $n^{(n-n')}$ bounds the number of terms produced by the derivations not acting on the exponential.

Using $x^l e^{-x} \leq l^l$ ($x > 0$), one obtains the lemma.

— We have still a non used decrease of the propagators, and if there

is $n(\Delta)$ distinct cubes then there is at least $\left(1 - \frac{\varepsilon}{100}\right)n(\Delta)$ cubes which are such that their distance to Δ is bigger than $0(1)\varepsilon^{1/2}n(\Delta)^{1/2}$ so that:

LEMMA 2. — Let $\Delta_1, \dots, \Delta_{n(\Delta)}$ be the squares linked to Δ and $x_i \in \Delta_i$:

$$\prod_{i=1}^{n(\Delta)} (C(x_i, y) (\text{dist}(\Delta, \Delta_i) + 1)^{2+\varepsilon/4}) \leq \prod_{i=1}^{n(\Delta)} (\text{dist}(\Delta, \Delta_i) + 1)^{-3/2-3\varepsilon/4} \leq 0(1)\varepsilon^{-n(\Delta)} n(\Delta)^{-3/4n(\Delta)}$$

The two lemmas prove then the bound of the proposition.

Remark. — Such an analysis can be made on a lot of models using the positivity of the potential or its equivalent.

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