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Section A :
Physique théorique.

Relativistic Hamiltonian dynamics of singularities of the Liouville equation

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ABSTRACT. — A constraint Hamiltonian description of the dynamics of singularities (regarded as point particles) is given for a previously considered class of solutions of the Liouville equation in two space-time dimensions. Reparametrization invariant Newton like equations are written down for the N-particle motion. The corresponding phase space Hamiltonian approach is formulated in terms of asymptotic particle coordinates and momenta. In the 2-particle case interpolating canonical coordinates are introduced in the Markov-Yukawa gauge (in which $(q_1 - q_2)(p_1 + p_2) = 0$), thus making contact with current formulation of relativistic particle dynamics. The relation between (non-canonical) physical position variables and the corresponding velocities on one hand and asymptotic canonical coordinates and momenta on the other is also established in the 2-particle case.

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RÉSUMÉ. — On propose une description hamiltonienne avec contraintes de la dynamique des singularités (considérées comme particules ponctuelles) pour une classe de solutions de l'équation de Liouville à deux dimensions. On écrit des équations de type newtonien pour le mouvement de N particules sous une forme invariante par rapport à la reparamétrisation. L'image hamiltonienne correspondante est présentée dans l'espace des phases des coordonnées et des impulsions asymptotiques. Dans le cas de deux particules on introduit aussi des coordonnées interpolantes utilisant la jauge de Markov-Yukawa (dans lequel $(q_1 - q_2)(p_1 + p_2) = 0$), ce qui établit une connexion avec la formulation courante de la dynamique des particules relativistes. On établit aussi une relation entre les variables (non-canoniques) de positions physiques et leurs vitesses d'un côté et les coordonnées et les impulsions asymptotiques canoniques de l'autre dans le cas de deux particules.

1. INTRODUCTION

The time-like lines of singularities for a class of solutions of the Liouville equation

$$(\partial_0^2 - \partial_1^2)\varphi(x) + \frac{m^2}{2} \exp \varphi(x) = 0 \qquad \left(\partial_\mu = \frac{\partial}{\partial x^\mu}, \mu = 0,1\right) \tag{1.1}$$

have been interpreted as the particle world lines of a relativistic dynamical system (with a finite number of degrees of freedom) in two space-time dimensions [*1*] [*2*]. The objective of the present note is to provide a covariant Hamiltonian description of this particle system (without a reference to the underlying field). Thus we will end up with an action-at-a-distance description of the motion of singularities of the solution of Eq. (1.1) which is a perfectly causal field equation.

There is a fundamental difficulty on the way towards such an objective. It is related to the so called « no interaction theorem » [*3*] - [*5*] which has for a long while embarassed attemps to construct a canonical Hamiltonian dynamics for an interacting relativistic system of a finite number of degrees of freedom. We shall summarize here our present understanding of this problem (see [*6*]-[*8*]).

Gauge dependence of canonical world lines in the presence of interaction.

It is intuitively clear that if we regard the space-time properties of a mechanical system as fundamental, in particular, if we regard particle world lines as observables, we should demand that the dynamics and the world lines do not depend on the choice of an evolution parameter. We shall refer to such an invariance under the infinite parameter group of reparametrizations of individual world lines as *gauge invariance.*

There is a good justification for such a terminology. A set of N relativistic interacting particles can be defined as a constraint Hamiltonian system [*9*]. Its dynamics is specified by giving a 7N dimensional Poincaré invariant surface $\mathcal{M}$ in the 8N dimensional canonical phase space Γ, which defines locally particle energies as functions of the remaining variables. It is assumed that the restriction to $\mathcal{M}$ of the canonical symplectic form

$$\omega = \sum_{k=1}^{N} dq_k \wedge dp_k \left(\equiv \sum_{k=1}^{N} dq_k^\mu \wedge dp_{k\mu} \right) \tag{1.2}$$

vanishes on an N-dimensional kernel (*) $\mathcal{N}$ and that the foliation

$$\mathcal{M} \to \Gamma_* = \mathcal{M}/\mathcal{N} \tag{1.3}$$

is a locally trivial fibre bundle (with an N-dimensional fibre $\mathcal{N}$). Thus we arrive at a finite dimensional gauge theory as described by Faddeev (see Appendix to [*10*]); the specification of equal time surfaces and evolution parameters plays the role of a choice of gauge.

Roughly speaking, the above mentioned no-go theorem says that there are no gauge invariant world lines in the space of *canonical coordinates* unless the particles are free. More precisely, we have the following result valid for any number of particles N and an arbitrary space-time dimension D.

Let the projection of each fibre of the bundle (1.3) on the D-space of canonical coordinates q_k is 1-dimensional (for $k = 1, \ldots, N$) and let the canonical Hamiltonian

$$h(\mathbf{p}_1, \ldots, \mathbf{p}_N ; q_{12}, \ldots, q_{N-1N}) \approx \sum_{k=1}^{N} p_k^0 \qquad (q_{ij} = q_i - q_j) \tag{1.9}$$

be non-degenerate in the sense that the system

$$\frac{\partial h}{\partial \mathbf{p}_k} = \dot{\mathbf{q}}_k \qquad (k = 1, \ldots, N) \tag{1.5}$$

can be solved with respect to $\mathbf{p}_k$.

Then the canonical (q – space) trajectories of all particles are straight lines (see [*8*] Theorem 2).

(*) More precisely, we assume that there are exactly N linearly independent vectors X_k in the tangent space $T_{(p,q)}\mathcal{M}$ at each point $(p, q) \in \mathcal{M}$ such that $\omega(X_k, Y) = 0$ for any $Y \in T_{(p,q)}\mathcal{M}$. The vector fields X_k are assumed to be in involution, $[X_j, X_k] = 0$, and $\mathcal{N}$ is defined as their (N-dimensional) integral surface.

(Note that the above non-degeneracy condition excludes systems involving free zero mass particles.)

The way out of this difficulty has been indicated on more than one occasion [*11*] [*5*] [*12*]-[*15*]: the physical position variables x_k should not be identified with the canonical coordinates q_k; rather, they are vector valued functions of all the $q's$ and $p's$. An iterative procedure has been developed [*14*] [*15*] to construct x_k in the 2-particle case (in terms of products of derivatives of the interaction function) satisfying the « initial condition » (appropriate for a velocity independent potential in the centre-of-mass frame)

$$x_k = q_k, \qquad k = 1, 2,$$

for

$$qP = 0 = xP \qquad (q = q_1 - q_2, x = x_1 - x_2, P = p_1 + p_2). \quad (1.6)$$

One of our objectives in this work is to find a closed expression for x_k in the model we are studying and to get an idea of the ambiguity involved in the definition of the $x's$.

2. GAUGE INVARIANCE OF SINGULARITIES' WORLD LINES FOR THE LIOUVILLE EQUATION

The « particle world lines » of the singular solutions of the Liouville equation (1.1) can be defined by a set of N implicit parametric equations of the type

$$p_N * (x_k - q_N) = \sum_{j=1}^{N-1} \frac{f_j(p)}{p_j * (x_k - q_j)}, \qquad k = 1, \ldots, N, \qquad (2.1)$$

where

$$p * q = p^1 q^0 - p^0 q^1, \qquad (2.2)$$

and $f_j(p) = f_j(p_1, \ldots, p_N), j = 1, \ldots, N - 1$ are any set of $N - 1$ positive Lorentz invariant functions of the p's. The 2-vectors $p_k (= p_k^{as}), k = 1, \ldots, N$, which will be interpreted as asymptotic particle momenta satisfy the free mass-shell constraint equations

$$\begin{aligned} \varphi_j &\equiv \frac{1}{2} p_j^2 \approx 0 \qquad && j = 1, \ldots, N - 1, \\ \varphi_N &\equiv \frac{1}{2}(m^2 + p_N^2) \approx 0, \qquad && p_k^0 > 0, \qquad k = 1, \ldots, N. \end{aligned} \qquad (2.3)$$

Eqs. (2.1) do not depend on k; we have in fact a single equation for $x (= x_k)$.

Using the results of [1], it is not difficult to show that this equation has exactly N solutions satisfying the asymptotic condition

$$\lim_{\tau\to\infty}\left[x_k(\tau) - x_k^{\text{out}}(\tau)\right] = 0 = \lim_{\tau\to\infty}\left[\frac{\dot{x}_k(\tau)}{\dot{x}_k^0(\tau)} - \frac{\dot{x}_k^{\text{out}}(\tau)}{\dot{x}_k^{0\,\text{out}}(\tau)}\right], \qquad (2.4a)$$

where

$$x_k^{\text{out}}(\tau) = q_k(\tau), \qquad \dot{x}_k^{\text{out}}(\tau) = \lambda_k p_k, \qquad \lambda_k > 0, \qquad k = 1, \ldots, \mathrm{N}. \quad (2.4b)$$

Indeed, in the equal time gauge in which $q_k^0(0) = 0$, $\lambda_k = \frac{1}{p_k^0}$ $(\tau \equiv t = q_k^0)$ we fall into the setup of ref. [1] where the existence of N solutions has been established.

Eq. (2.1) involves 2N (gauge independent) parameters: the vectors p_k (which are determined, say, by their space components $\mathbf{p}_k$ because of the constraints (2.3)) and the skewsymmetric products $p_k * q_k$. The asymptotic conditions (2.4) involve, in addition, the gauge freedom which is reflected in the Lagrange multipliers λ_k and in the time components q_k^0 (for $\tau = 0$). We shall show in what follows that the world lines (in the 2-space of each x_k) are gauge independent.

Eqs. (2.4) demonstrate that q_k and p_k do indeed play the role of asymptotic particle coordinates and momenta. We shall regard the 4N-dimensional vector space spanned by the q's and the p's as a symplectic manifold with canonical 2-form (1.2) or equivalently with a Poisson bracket structure defined by

$$\{ q_j^\mu, p_{k\nu} \} = \delta_\nu^\mu \delta_{jk}, \qquad \mu, \nu = 0, 1; \qquad j, k = 1, \ldots, \mathrm{N}. \qquad (2.5)$$

Following the general framework of ref. [9] we define the Hamiltonian of the system whose world lines satisfy (2.1) as a linear combination of the constraints (2.3) with positive (variable) coefficients:

$$\mathrm{H} = \sum_{k=1}^{\mathrm{N}} \lambda_k \varphi_k (\approx 0), \qquad \lambda_k > 0. \qquad (2.6)$$

It gives rise to the free dynamics in the asymptotic variables q_k, p_k. Indeed,

$$\dot{q}_k = \{ q_k, \mathrm{H} \} \approx \lambda_k p_k, \qquad \dot{p}_k = \{ p_k, \mathrm{H} \} \approx 0, \qquad (2.7a)$$

so that the momenta, the velocities, and the pseudoscalars $p_k * q_k$ are constants of the motion:

$$\frac{d}{d\tau}\left(\frac{1}{\dot{q}_k^0} q_k\right) = \left\{ \frac{1}{p_k^0} p_k, \mathrm{H} \right\} \approx 0, \qquad \frac{d}{d\tau}(p_k * q_k) \approx \lambda_k p_k * p_k = 0 \qquad (2.7b)$$

Moreover, they are gauge independent, since they have zero Poisson brackets with each of the constraints (2.3) separately. (Eqs. (2.6) and

(2.7) involve the weak equality sign, $\approx$, which indicates that they are only valid on the mall-shell (2.3).)

The fact that coefficients of Eq. (2.1) are constants of the motion (as functions of p_k and $p_k * q_k$) and that the asymptotic condition (2.4) relates x_k^{out} to q_k demonstrates that

$$\dot{x}_k \equiv \frac{dx_k}{d\tau} = \{ x_k, \mathrm{H} \} \tag{2.8}$$

and

$$\left[p_{\mathrm{N}} + \sum_{j=1}^{\mathrm{N}-1} \frac{f_j(p)p_j}{[p_j * (x_k - q_j)]^2} \right] * \dot{x}_k = 0 \,. \tag{2.9}$$

As a consequence of (2.3) and of the assumed positivity of f_j the vectors in the square brackets in (2.9) are positive time like for all k's. Hence, the normalized velocities u_k are determined uniquely from (2.9):

$$u_k \equiv \frac{\dot{x}_k}{\sqrt{-\dot{x}_k^2}}$$

$$= \left\{ - \left[p_{\mathrm{N}} + \sum_{j=1}^{\mathrm{N}-1} \frac{f_j(p)p_j}{[p_j*(x_k - q_j)]^2} \right]^2 \right\}^{-1/2} \left[p_{\mathrm{N}} + \sum_{j=1}^{\mathrm{N}-1} \frac{f_j(p)p_j}{[p_j*(x_k - q_j)]^2} \right] . \tag{2.10}$$

Thus the tangent to the k-th particle world line at each point is gauge independent. In order to complete the proof of the gauge invariance of the world line in the Minkowski 2-space (x_k^0, x_k) it suffices, due to (2.4), to verify the invariance of the asymptotic line $q_k = q_k(\tau)$.

Eq. (2.4b) yields

$$\frac{dq_k}{dq_k^0} = \frac{\dot{q}_k}{\dot{q}_k^0} = \frac{p_k}{p_k^0} \,. \tag{2.11}$$

It follows that we can exclude the gauge dependent Lagrange multipliers λ_k from the equation of the asymptotic line; we find

$$q_k = -\frac{p_k * q_k}{p_k^0} + \frac{p_k}{p_k^0} q_k^0 \,. \tag{2.12}$$

3. A NEWTON-LIKE FORMULATION OF THE N-PARTICLE PROBLEM

Eqs. (2.1) and (2.10) can be regarded as 2N equations for the 2N independent gauge invariant variables p_k and $p_k * q_k$ (for $\varphi_k = 0$). For space-like x_{ij}, i.e. for

$$x_{ij}^2 \equiv (x_i - x_j)^2 > 0 \,, \qquad i, j = 1, \ldots, \mathrm{N} \tag{3.1}$$

the functions $f_k(p)$ can be chosen in such a way that p_k and $p_k * q_k$ could be expressed in terms of the x_i's and u_i's. (It can be shown-in the simplest case, for N = 2- that for a time like x_{12} eqs. (2.1) (2.10) admit no solution for p_k and $p_k * q_k$, whatever the choice of $f_1(p)$). The N independent components, say $\mathbf{p}_k$, of p_k provide N translation invariant integrals of motion that are in involution. In other words, as noted in [2], the dynamical system under consideration is completely integrable.

Differentiating (2.10) with respect to τ we arrive at the following reparameterization invariant Newton like equations

$$w_k \equiv (-\dot{x}_k^2)^{-1/2}\,\dot{u}_k = *\, u_k \phi_k \qquad \left(\dot{u}_k \equiv \frac{du_k}{d\tau}\right), \tag{3.2a}$$

where

$$(*\, u_k)^0 = u_k^1\,, \qquad (*\, u_k)^1 = u_k^0, \qquad k = 1, \ldots, \mathrm{N} \tag{3.2b}$$

and the scalar « forces » ϕ_k are given by

$$\phi_k(x, u) = -2\left\{-\left[p_{\mathrm{N}} + \sum_{j=1}^{\mathrm{N}-1} \frac{f_j(p)p_j}{[p_j*(x_k-q_j)]^2}\right]^2\right\}^{-1/2} \sum_{j=1}^{\mathrm{N}-1} \frac{f_j(p)(p_j*u_k)^2}{[p_j*(x_k-q_j)]^3} \tag{3.3}$$

with p_j and $p_j * q_j$ expressed in terms of x_i and u_i from (2.1), (2.10). In deriving (3.3) we have used the identity

$$p_j + (u_k p_j)u_k = (p_j * u_k) * u_k\,, \tag{3.4}$$

valid for $u_k^0 = \sqrt{1 + \mathbf{u}_k^2}$.

Finding the functions $p_k(x, u)$ and $(p_k * q_k)(x, u)$ is equivalent to finding the solution of a general N-th degree algebraic equation. We shall restrict our attention to the simplest case N = 2 in which everything can be evaluated explicitly. We have

$$\frac{p_1}{f_1(p)} = \frac{\mathrm{U} - *\,\mathrm{U}\varepsilon}{m} \frac{4\,|\,u_1 * u_2\,|\sqrt{|\,x\mathrm{U}\,|\,|\,x * \mathrm{U}\,|}}{[(x * \mathrm{U})^2(1 - u_1u_2) - (x\mathrm{U})^2(1 + u_1u_2)]^2} \left\{\frac{[\,|\,x * \mathrm{U}\,|(1 - u_1u_2) - |\,x\mathrm{U}\,|(1 + u_1u_2)]^5}{-(1 + u_1u_2)[\,|\,x * \mathrm{U}\,| + |\,x\mathrm{U}\,|\,]^3}\right\}^{1/2}$$

$$\begin{aligned} p_2 = m\Bigg\{ &\mathrm{U}\left[(x * \mathrm{U})^2 + 2\,|\,x\mathrm{U}\,|\,|\,x * \mathrm{U}\,| - (x\mathrm{U})^2 \frac{1 + u_1u_2}{1 - u_1u_2}\right] \\ &+ *\,\mathrm{U}\varepsilon\left[(x * \mathrm{U})^2 - (2\,|\,x\mathrm{U}\,|\,|\,x * \mathrm{U}\,| + (x\mathrm{U})^2)\frac{1 + u_1u_2}{1 - u_1u_2}\right]\Bigg\} \\ &\left\{\frac{4\,|\,x\mathrm{U}\,|\,|\,x * \mathrm{U}\,|\,[\,|\,x * \mathrm{U}\,| + |\,x\mathrm{U}\,|\,]}{[\,|\,x * \mathrm{U}\,|(1 - u_1u_2) - |\,x\mathrm{U}\,|(1 + u_1u_2)]\dfrac{1}{1 - u_1u_2}}\right\}^{-1/2} \end{aligned} \tag{3.5}$$

where

$$U = \frac{1}{2}(u_1 + u_2), \qquad \varepsilon = -\operatorname{sign}(x * U)(xU),$$

$$\phi_k(x, u) = 4(-1)^{k+1}\frac{(|x * U| - |xU|)^2(x * u_k)^2}{(1 - u_1 u_2)(x^2)^2(x * u_{3-k})}, \qquad k = 1, 2. \quad (3.6)$$

The « force » ϕ_k exhibits characteristic properties which are valid for arbitrary N. First, it is independent of the choice of the functions $f_j(p)$ in (2.1). Secondly, it satisfies the finite predictivity condition of ref. [*12*],

$$\frac{\partial \varphi_i}{\partial \tau_k} = 0 \qquad \text{for} \qquad i \neq k, \qquad i, k = 1, \ldots, N, \qquad (3.7)$$

where τ_k is the proper time of particle k, so that

$$\frac{\partial}{\partial \tau_k} = u_k \frac{\partial}{\partial x_k} + w_k \frac{\partial}{\partial u_k} = u_k \frac{\partial}{\partial x_k} - \phi_k(x, u) u_k * \frac{\partial}{\partial u_k}. \qquad (3.8)$$

Eq. (3.7) guarantees that the system under consideration is a second order differential system in the terminology of ref. [*15*]; it is another form of the gauge invariance condition.

It is instructive to point out why both properties take place in general.

Since the momenta p_j, $j = 1, \ldots, N - 1$ are light-like (because of (2.3)) the change of variables

$$p_j \to f_j(p) p_j \qquad (f_j > 0) \qquad j = 1, \ldots, N - 1 \qquad (3.9)$$

leaves the constraints (2.3) unaltered. This change allows to exclude the functions f_j from eqs. (2.1), (2.10) and (3.3) and hence from ϕ_k.

To prove the general validity of eq. (3.7) we first note that the variables p_k and $p_k * q_k$ are gauge invariant so that

$$\frac{\partial}{\partial \tau_i} p_k = 0 = \frac{\partial}{\partial \tau_i} p_k * q_k \qquad \text{for all} \qquad i, k = 1, \ldots, N. \qquad (3.10)$$

On the other hand, according to (3.3), ϕ_k depends on x_i and u_i through these variables and through x_k and u_k (with the same index k as ϕ_k). Thus, due to (3.8), they do satisfy (3.7) for arbitrary N.

4. INTERPOLATING CANONICAL VARIABLES AND PHYSICAL POSITIONS IN THE 2-PARTICLE CASE

We now turn to a more detailed study of the case N = 2. Noting that in- and out-coordinates and momenta are obtained from one another by a particle permutation, we shall introduce some interpolating canonical variables in a specific gauge and will then construct the physical position

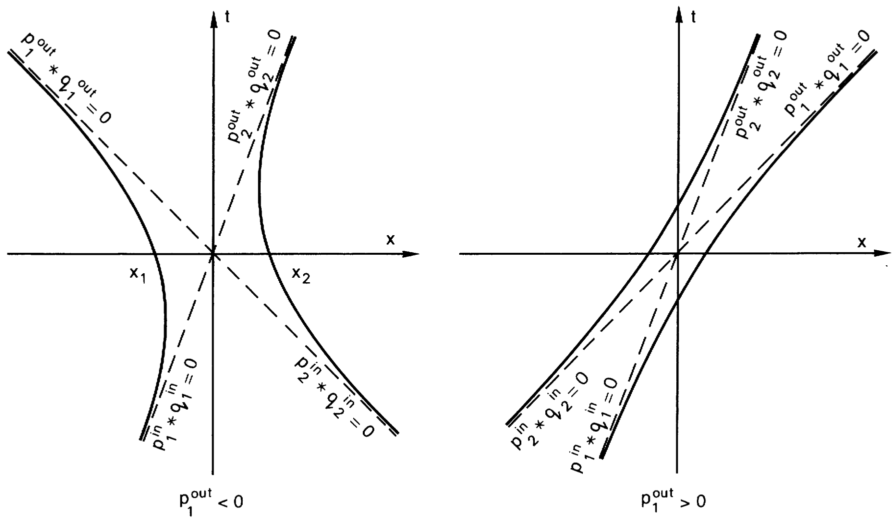

FIG. 1. — Two-particle world lines for $q_K^{as}(0) = 0$.

2-vectors as functions of these variables (and hence of the original asymptotic coordinates and momenta).

The general form of the world lines in the 2-particle case (cf. [*1*]) is displayed on Fig. 1. It is an essential feature of this picture that in- and out-coordinates and momenta exchange places:

$$p_k^{out} = p_{3-k}^{in}, \qquad q_k^{out} = q_{3-k}^{in}, \qquad k = 1, 2, \tag{4.1}$$

so that

$$(p_1^{out})^2 = (p_2^{in})^2 = 0, \ (p_2^{out})^2 = (p_1^{in})^2 = -m^2 . \tag{4.2}$$

Identifying the canonical variables p and q of the preceding sections with p^{out}, q^{out}, we shall introduce interpolating canonical coordinates p and q in the Markov-Yukawa gauge

$$qP = 0 = q^{out}P^{out}, \qquad \text{where} \qquad q = q_1 - q_2, \, P = p_1 + p_2 , \tag{4.3}$$

satisfying the following conditions:

$$Q \equiv \frac{1}{2}(q_1 + q_2) = \frac{1}{2}(q_1^{out} + q_2^{out}) \qquad \left(= \frac{1}{2}(q_1^{in} + q_2^{in})\right), \tag{4.4a}$$

$$P \equiv p_1 + p_2 = p_1^{out} + p_2^{out} \qquad (= p_1^{in} + p_2^{in}) . \tag{4.4b}$$

(« Interpolating » means that p and q tend to the out-variables for $t \to \infty$ and to the in-variables for $t \to -\infty$). Because of (4.4) we only need to express the relative coordinate and momentum.

$$q = q_1 - q_2, \, p = \frac{1}{2}(p_1 - p_2) \tag{4.5}$$

in terms of the asymptotic variables. A solution of this problem is given by

$$q = \sqrt{\kappa^2 + 1}\frac{q^{\text{out}}}{\kappa}, p = \frac{\kappa p^{\text{out}}}{\sqrt{1 + \kappa^2}} \tag{4.6}$$

where

$$\kappa = q^{\text{out}}p^{\text{out}} \,(= qp) . \tag{4.7}$$

(κ can be regarded as a Lorentz invariant evolution parameter). If we select q^{out} to be space-like or zero, then q is always space-like.

It is easily verified that the set (q, Q ; p, P) is canonical. We shall demonstrate that it satisfies an equation of the type (2.1) in the gauge (4.3). Indeed,

$$[p_1^{\text{out}} * (q_k - q_1^{\text{out}})][p_2^{\text{out}} * (q_k - q_2^{\text{out}})] = f \equiv \frac{(p_1^{\text{out}} * q^{\text{out}})(p_2^{\text{out}} * q^{\text{out}})}{4(p^{\text{out}}q^{\text{out}})^2}; \tag{4.8}$$

the right-hand side is independent of q^{out} in the gauge (4.3):

$$f = \frac{(p_1^{\text{out}}\mathrm{P})(p_2^{\text{out}}\mathrm{P})}{4(p^{\text{out}} * \mathrm{P})^2} (> 0) \quad \text{for} \quad q^{\text{out}}\mathrm{P} = 0 \quad (\mathrm{P} = \mathrm{P}^{\text{out}}). \tag{4.9}$$

We now turn to the problem of constructing physical position variables x_k (with gauge invariant world lines) that satisfy the « initial condition » (1.6) along with the property

$$\mathrm{X} \equiv \frac{1}{2}(x_1 + x_2) = \mathrm{Q} = \mathrm{Q}^{\text{out}} \qquad (= \mathrm{Q}^{\text{in}}). \tag{4.10}$$

(The consistency of this equation is verified in both the Markov-Yukawa gauge (4.3) and the equal-time gauge of ref. [*1*].) Eqs. (2.1), with $f_1 = f$ given by (4.9), and the supplementary condition (4.10) have a unique solution for the relative coordinate $x = x_1 - x_2$ given by

$$x = \rho q^{\text{out}},$$
$$\rho = (\operatorname{sign} q^{\text{out}}p^{\text{out}})\left[1 + \frac{(p_1^{\text{out}}\mathrm{P})(p_2^{\text{out}}\mathrm{P})}{(p^{\text{out}} * \mathrm{P})^2(p_1^{\text{out}} * q^{\text{out}})(p_2^{\text{out}} * q^{\text{out}})}\right]^{1/2}. \tag{4.11}$$

If we denote the constraints φ_k of Sec. 2 by φ_k^{out}, then the equivalent constraints

$$\begin{aligned} \varphi_1 &= p_2^{\text{out}} * q^{\text{out}}(\rho + 1)\varphi_1^{\text{out}} + p_1^{\text{out}} * q^{\text{out}}(\rho - 1)\varphi_2^{\text{out}}, \\ \varphi_2 &= p_2^{\text{out}} * q^{\text{out}}(\rho - 1)\varphi_1^{\text{out}} + p_1^{\text{out}} * q^{\text{out}}(\rho + 1)\varphi_2^{\text{out}}, \end{aligned} \tag{4.12}$$

satisfy the gauge invariance criterion of refs. [*7*] [*8*]:

$$\{x_1, \varphi_2\} = 0 = \{x_2, \varphi_1\}. \tag{4.13}$$

Solving eqs. (4.6), (4.7) with respect to p^{out} and q^{out}, we can express x_k and φ_k in terms of the interpolating variables p and q.

We end up with a final remark concerning the centre of mass variable and eq. (4.10).

Let

$$Q_c = Q + \frac{pP}{P^2} q , \qquad p_\perp = p - \frac{pP}{P^2} P ; \tag{4.14}$$

then it is easily seen that the generator of the Lorentz boosts can be written in the form

$$J = Q_c \wedge P + q \wedge p_\perp = Q_c \wedge P + q^{\text{out}} \wedge p^{\text{out}} , \tag{4.15}$$

and, furthermore

$$\{ Q_c^\mu, Q_c^\nu \} = \frac{(q \wedge p_\perp)^{\mu\nu}}{- P^2} , \tag{4.16}$$

where

$$(q \wedge p)^{\mu\nu} = q^\mu p^\nu - q^\nu p^\mu = (p * q)\varepsilon^{\mu\nu} . \tag{4.17}$$

It is argued in ref. [9] that eqs. (4.15), (4.16) should remain valid also for the physical centre of mass variable X_c and the corresponding relative momentum π orthogonal to P.

Setting

$$J = X_c \wedge P + x \wedge \pi , \qquad \{ X_c, X_c \} = \frac{x \wedge \pi}{- P^2} , \qquad \pi = \frac{1}{\rho} p_\perp^{\text{out}} \tag{4.18}$$

we find

$$X_c = Q_c = Q_c^{\text{out}} (= Q_c^{\text{in}}) . \tag{4.19}$$

Condition (4.19) (unlike (4.10)) can be imposed on the physical coordinates for more general relativistic 2-particle systems.

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