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LUKE HODGKIN

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Soliton Equations and Hyperbolic Maps

by

Luke HODGKIN
King's College, London

ABSTRACT. — A solution of the AKNS scattering equation [6] associated to a non-linear evolution equation determines an isometry from (\mathbb{R}^2, g) to the hyperbolic plane H , where g is the metric of curvature -1 defined by the scattering equation. This correspondence is (locally) 2-1 from solutions to isometries. For the modified KdV and sin-Gordon equations, the scattering equation can be seen as a flow on the space of constant-speed curves in H , with a simply-described curvature function. A geometrical interpretation of the Bäcklund transformation is given, together with a « soliton » example.

RÉSUMÉ. — Une solution de l'équation de diffusion AKNS [6] associée à une équation d'évolution non-linéaire donne une isométrie de (\mathbb{R}^2, g) dans le plan hyperbolique H , g étant la métrique de courbure -1 que définit l'équation de diffusion; cette correspondance des solutions aux isométries est (localement) 2-1. Pour les équations KdV modifiée et sinus-Gordon, l'équation de diffusion sera alors un flot sur l'espace des courbes à vitesse constante dans H , et la formule pour la courbure est simple. On donne une interprétation géométrique de la transformation de Bäcklund, ainsi qu'un exemple de type « soliton ».

1. INTRODUCTION

It has been recognized for some time that the non-linear partial differential equations which admit « soliton » type solutions are closely related to the group $SL(2, \mathbb{R})$ and its geometry (see in particular [1]-[4]); going

somewhat further, Sasaki and Bullough [5] described an explicit relation between the AKNS scattering scheme and metrics of constant curvature -1 on \mathbb{R}^2 . It turns out that an even simpler way of looking at the theory arises when we bring in the standard space of curvature -1 , i. e., the upper half plane $H = \mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2)$ with the hyperbolic metric. We find

(1) that solutions of the scattering equations correspond almost exactly to isometries from \mathbb{R}^2 (with the metric of [5]) to H ;

(2) that other features such as Bäcklund transformations have geometrical descriptions in terms of such isometries;

(3) that for the sin-Gordon and modified KdV equations the basic functions are natural geometrical ones.

The idea (which I shall only use as a suggestion here) is the following. A scattering scheme defines an $\mathrm{SL}(2, \mathbb{R})$ connection on \mathbb{R}^2 which is *integrable* iff the associated non-linear equation is satisfied [4]. Its integrability means that \mathbb{R}^2 can be isometrically « developed » on H , and such a development is the isometry we are looking for.

This note is concerned only with the general theory and not with the (multi) soliton solutions; but they also seem likely to correspond to objects with a geometrical meaning.

2. THE CORRESPONDENCE

We begin with the scattering equations themselves, in a formalism which is a mixture of [4] and [5] adapted for the present purposes. Let $\sigma^1, \sigma^2, \omega$ be 1-forms on \mathbb{R}^2 , defining a metric of constant curvature -1 ,

$$(1) \quad g = (\sigma^1)^2 + (\sigma^2)^2$$

via the structure equations

$$(2) \quad d\sigma^1 = \omega \wedge \sigma^2, \quad d\sigma^2 = -\omega \wedge \sigma^1, \quad d\omega = \sigma^1 \wedge \sigma^2$$

Then if Ω is the $\mathrm{SL}(2, \mathbb{R})$ -valued 1-form on \mathbb{R}^2

$$(3) \quad \Omega = \begin{pmatrix} \frac{1}{2}\sigma^2 & \frac{1}{2}(-\omega + \sigma^1) \\ \frac{1}{2}(\omega + \sigma^1) & -\frac{1}{2}\sigma^2 \end{pmatrix}$$

a *solution* of the scattering equation is a map $G: \mathbb{R}^2 \rightarrow \mathrm{SL}(2, \mathbb{R})$ such that

$$(4) \quad G^{-1}dG = \Omega$$

$$\text{or} \quad G^*(\omega^1) = \sigma^1, \quad G^*(\omega^2) = \sigma^2, \quad G^*(\omega^3) = \omega$$

where ω^i ($i = 1, 2, 3$) are Maurer-Cartan forms corresponding to the basis of the Lie algebra defined by (3).

Locally such solutions exist provided that Ω satisfies the integrability condition

$$(5) \quad d\Omega = \Omega \wedge \Omega$$

which in particular cases defines the non-linear equation in question. And if G, G' are two solutions defined on the same (connected) subset of \mathbb{R}^2 , they are related by $G'(x, t) = Q \cdot G(x, t)$, where $Q \in \text{SL}(2, \mathbb{R})$ is constant.

Note that our way of writing the scattering equation (4) is that of [4], although the basis of forms is different. The forms $\sigma^1, \sigma^2, \omega$ are essentially those of [5], given that the equation $d\bar{v} = \Omega \bar{v}$ has been replaced by its adjoint $d\bar{v}' = \bar{v}' \Omega'$; our Ω is therefore $\bar{\Omega}'$ in the more usual formalism.

Let $\pi: \text{SL}(2, \mathbb{R}) \rightarrow \text{SL}(2, \mathbb{R})/\text{SO}(2) = \text{H}$ be the canonical projection:

$$(6) \quad \pi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{ai + b}{ci + d}.$$

Write ξ, η for the coordinates on H . We choose for H the metric defined by the 1-forms $\sigma_{\text{H}}^1 = \frac{1}{\eta} d\xi$, $\sigma_{\text{H}}^2 = \frac{1}{\eta} d\eta$, $\omega_{\text{H}} = \frac{1}{\eta} d\xi$;

$$(7) \quad g_{\text{H}} = (\sigma_{\text{H}}^1)^2 + (\sigma_{\text{H}}^2)^2 = \frac{1}{\eta^2} (d\xi^2 + d\eta^2).$$

Our first observation is that if G is a solution of (4), then $f = \pi \circ G$ is an isometry from (\mathbb{R}^2, g) to (H, g_{H}) .

In fact, because $\text{SL}(2, \mathbb{R})$ acts by isometries on H , $\pi^*(g_{\text{H}})$ is a left invariant symmetric 2-form on $\text{SL}(2, \mathbb{R})$; by looking at the derivative of π at the identity this can be identified with $(\omega^1)^2 + (\omega^2)^2$. Hence

$$f^*(g_{\text{H}}) = G^*((\omega^1)^2 + (\omega^2)^2) = g.$$

Similarly, f^* takes the standard volume form $\sigma_{\text{H}}^1 \wedge \sigma_{\text{H}}^2 = \frac{1}{\eta^2} d\xi \wedge d\eta$ on H to $\sigma^1 \wedge \sigma^2$ on \mathbb{R}^2 . Hence f is orientation preserving from the orientation of \mathbb{R}^2 defined by $\sigma^1 \wedge \sigma^2$ (which may or may not be the standard one) to H .

In (one version of) the explicit AKNS scattering scheme we have [6]

$$(8) \quad \Omega = \begin{pmatrix} \lambda & r \\ q & -\lambda \end{pmatrix} dx + \begin{pmatrix} A & C \\ B & -A \end{pmatrix} dt.$$

(Recall that our Ω corresponds to the usual Ω' .) To keep everything in $\text{SL}(2, \mathbb{R})$, we specify that $\lambda (= -i\zeta)$ is a real constant, q, r are real-valued functions of x, t , and A, B, C are expressions involving λ and q, r and their derivatives, also real. (There is a corresponding theory for *complex* Ω and maps into $\text{SL}(2, \mathbb{C})$, which is certainly important — e. g., when λ is complex — but which we shall not deal with here.)

From (3) and (8) we have

$$(9) \quad \sigma^1 = (q + r)dx + (B + C)dt, \quad \sigma^2 = 2(\lambda dx + A dt), \\ \omega = (q - r)dx + (B - C)dt.$$

The « volume » form $\sigma^1 \wedge \sigma^2$ is $2(A(q + r) - \lambda(B + C))dxdt$. Where it vanishes — in general a 1-dimensional subset of \mathbb{R}^2 — the map f is singular. For example, in the sin-Gordon case [6], we can take $q = -r = -\frac{1}{2}u_x$, $B = C = \frac{1}{4\lambda} \sin u$; so $\sigma^1 \wedge \sigma^2 = -\sin u dxdt$. The orientation is determined by the sign of $\sin u$, while f is singular on the subset $\sin u = 0$.

3. THE INVERSE CORRESPONDENCE : LIFTING ISOMETRIES

We have seen that a solution G of (4) determines an isometry

$$f = \pi \circ G: \mathbb{R}^2 \rightarrow H.$$

(The « isometry » ceases to be a genuine isometry precisely when g ceases to be a proper metric on \mathbb{R}^2 , i. e., becomes indefinite.) Suppose now that we are given a map $f: \mathbb{R}^2 \rightarrow H$ satisfying

$$(10) \quad f^*(g_H) = g, \quad f^*(\sigma_H^1 \wedge \sigma_H^2) = \sigma^1 \wedge \sigma^2$$

— that is, an oriented isometry in the general sense. By topological considerations, f has a number of lifts to maps $G: \mathbb{R}^2 \rightarrow \text{SL}(2, \mathbb{R})$ such that $\pi \circ G = f$. It is a remarkable fact that we can specify geometrically those lifts which are solutions of the scattering problem — and that they are all but unique.

We can do this by looking at the tangents to x -parameter curves in H . From the formula

$$dG(\partial_x(x, t)) = G(x, t) \cdot \Omega(x, t)(\partial_x(x, t))$$

we find that if $f = \pi \circ G$ and G satisfies (4),

$$(11) \quad df(\partial_x(x, t)) = G(x, t) \cdot (i, \sigma^1(\partial_x) + i\sigma^2(\partial_x))$$

Here $(i, \sigma^1(\partial_x) + i\sigma^2(\partial_x)) \in T_i(H) = \{i\} \times \mathbb{C}$, and $G(x, t)$ acts as an isometry on H and so also on its tangent vectors. The effect of isometries on tangent vectors at $i \in H$ is not complicated: we find that if

$$(12) \quad G(x, t) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \\ df(\partial_x(x, t)) = \left(\frac{ai + b}{ci + d}, \frac{\sigma^1(\partial_x) + i\sigma^2(\partial_x)}{(ci + d)^2} \right).$$

Hence except in the « special » singular case where both σ^1 and σ^2 vanish on $\partial_x(x, t)$, we can find both $\frac{ai + b}{ci + d}$ and $\frac{1}{(ci + d)^2}$ from f by (12). (This case is explicitly excluded for sin-Gordon, where $\lambda \neq 0$, but could give trouble elsewhere.)

By a simple calculation, these two complex numbers determine

$$G(x, t) \in \mathrm{SL}(2, \mathbb{R})$$

up to a factor ± 1 — which is the most we could hope for, given that -1 acts trivially on H . Now if f is *any* isometry satisfying (10), define $G: \mathbb{R}^2 \rightarrow \mathrm{SL}(2, \mathbb{R})$ by (12) (we also, of course require G to be continuous). Then G is unique up to ± 1 ; we call the two maps the *canonical lifts* of f with respect to Ω . The essential fact is that *the canonical lifts of an isometry are solutions of the scattering equation* (4). To see this we first check from (12) that when G is a canonical lift, $G^*(\omega^i)(\partial_x(x, t)) = \sigma^i(\partial_x(x, t))$ for $i = 1, 2$; and then use the fact that

$$G^*((\omega^1)^2 + (\omega^2)^2) = (\sigma^1)^2 + (\sigma^2)^2, \quad G^*(\omega^1 \wedge \omega^2) = \sigma^1 \wedge \sigma^2$$

(since f is an isometry and G is a lift of f) to show that $G^*(\omega^i)$ and σ^i also agree on ∂_t for $i = 1, 2$. Now $G^*(\omega^3) = \omega$ follows from (2) and the Maurer-Cartan equations.

Schematically therefore we have a 2-1 correspondence

$$\left(\begin{array}{c} \text{solutions } G \text{ of the} \\ \text{scattering equation} \end{array} \right) \begin{array}{c} \xrightarrow{\text{compose with } \pi} \\ \xleftarrow{\text{canonical lift}} \end{array} \left(\begin{array}{c} \text{isometries} \\ f: (\mathbb{R}^2, g) \rightarrow (H, g_H) \end{array} \right)$$

Note 1. — If G were taken as mapping into the group of isometries of H , the projective group $\mathrm{PL}(2, \mathbb{R}) = \mathrm{SL}(2, \mathbb{R})/(\pm 1)$, we'd have a 1 – 1 correspondence; but it would be no easier to write down, so it seems best to stay in $\mathrm{SL}(2, \mathbb{R})$.

Note 2. — We can in fact define a canonical lift except where σ^1, σ^2 are identically zero. For if they are zero on ∂_x but not on ∂_t , we can replace the procedure above by one involving the t -curves; the same argument works.

4. SPEED AND CURVATURE

We now specialize to the case where Ω is defined by (8) and $q+r=0$; this will work for the sin-Gordon equation

$$(13) \quad u_{xt} = \sin u \left(q = -\frac{1}{2} u_x \right)$$

and for the modified KdV equation in the form

$$(14) \quad q_t + 6q^2q_x + q_{xxx} = 0.$$

(See [6]). Then $\sigma^1(\partial_x) = 0$ and $\sigma^2(\partial_x) = 2\lambda$. So the x -parameter curves in (\mathbb{R}^2, g) have constant speed $|2\lambda|$ — and hence so also do their images under f , the x -parameter curves in H . The scattering equation can therefore be regarded as a flow on the space of curves of speed $|2\lambda|$ in H .

Next, we have a very simple description of the canonical lift, from (12).

In fact $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{R})$ takes the standard tangent vector $(i, i) \in T_i(H)$

to $\left(\frac{ai+b}{ci+d}, \frac{i}{(ci+d)^2}\right)$. Hence if $f: \mathbb{R}^2 \rightarrow H$ is an isometry, the canonical

lift $G(x, t)$ is the unique (up to ± 1) isometry of H which takes (i, i) to

$$\frac{1}{2\lambda} df(\partial_x(x, t)) = \left(f(x, t), \frac{1}{2\lambda} f_x(x, t)\right) \in T_{f(x, t)}(H).$$

Note that this definition works precisely when $\lambda \neq 0$, which corresponds to the non-singular case.

Geometrically, $G(x, t)$ maps the standard unit tangent vector (i, i) to the unit vector along the x -curve in H at $f(x, t)$ pointing forwards (backwards) if λ is positive (negative).

The function q in its turn is described in terms of curvature. To see this, consider the standard basis vector fields $\underline{e}_1, \underline{e}_2$ on \mathbb{R}^2 corresponding to the forms σ^1, σ^2 [5],

$$(15) \quad \underline{e}_1 = -\frac{A}{\lambda(B+C)}\partial_x + \frac{1}{(B+C)}\partial_t; \quad \underline{e}_2 = \frac{1}{2\lambda}\partial_x.$$

From $\omega = 2qdx + (B-C)dt$ we deduce

$$(16) \quad \nabla_{\underline{e}_x}(\underline{e}_2) = -2q\underline{e}_1;$$

In other words, $2q$ is the covariant «rate of change of angle» along an x -parameter curve. To find the geodesic curvature κ_g of the curve we compute

$$\nabla_{\underline{e}_2}\underline{e}_2 \text{ (for } \lambda > 0) \text{ or } \nabla_{(-\underline{e}_2)}(-\underline{e}_2) \text{ (for } \lambda < 0), \text{ and find } -\frac{q}{\lambda}\underline{e}_1 \text{ in each case.}$$

Since $(\underline{e}_1, \underline{e}_2)$ are positively oriented this gives in general $\kappa_g = q/|\lambda|$.

Again, since f is an isometry, the same is true for the x -parameter curves in H .

To make clear what is meant by describing $2q$ as the covariant rate of change of angle, suppose q derived from a potential function u by the formula

$$(17) \quad q = -\frac{1}{2}u_x.$$

(This is standard for the sin-Gordon equation, of course.) Define the vector field \underline{v} by

$$(18) \quad \underline{v} = \underline{e}_1 \sin u + \underline{e}_2 \cos u.$$

A simple calculation then shows that \underline{v} is *parallel* along the x -parameter curves; while \underline{u} is the *clockwise* angle of rotation from \underline{e}_2 to \underline{v} . Hence the *anticlockwise* angle from \underline{v} to ∂_x is $-u$ (for $\lambda > 0$) and $\pi - u$ (for $\lambda < 0$); its rate of change is $-u_x = 2q$.

5. THE SIN-GORDON EQUATION

Here the situation is particularly simple — corresponding to the classical geometrical problem which the equation describes [7]. The equation is given by (13), and we have [6] [8].

$$(19) \quad B = C = \frac{1}{4\lambda} \sin u, \quad A = \frac{1}{4\lambda} \cos u$$

whence using (15), (18),

$$(20) \quad \partial_t = \frac{1}{2\lambda} (\underline{e}_1 \sin u + \underline{e}_2 \cos u) = \frac{1}{2\lambda} \underline{v}.$$

It follows that the t -curves have constant speed $\frac{1}{|2\lambda|}$ and that ∂_t is parallel along the x -curves; u is the clockwise angle of rotation from ∂_x to ∂_t whatever the sign of λ . We can state:

A solution of the scattering problem for sin-Gordon with given function $u(x, t)$ and parameter λ is (the canonical lift of) a map $f: \mathbb{R}^2 \rightarrow H$ such that in H

- i) the x -curves have constant speed $|2\lambda|$ and the t -curves have constant speed $1/|2\lambda|$.
- ii) the clockwise angle from $f_x(x, t)$ to $f_t(x, t)$ is $u(x, t)$.

6. BACKLUND TRANSFORMATIONS

Crampin in [4] gives a nice geometric description of a BT which corresponds to the « usual » one for particular choices of gauge. We shall investigate this only in the sin-Gordon case; unfortunately here as he points out his Ω differs from that of AKNS (and so from ours) by a gauge transformation. But this in itself deserves attention.

Let $P(x, t)$ be the matrix $\begin{pmatrix} \cos u/4 & -\sin u/4 \\ \sin u/4 & \cos u/4 \end{pmatrix} \in \text{SL}(2, \mathbb{R})$. Then $P(x, t)$ leaves $i \in H$ fixed and induces a rotation through $-u/2$ on $T_i(H)$. Hence if G is the *canonical* lift of f , $GP: \mathbb{R}^2 \rightarrow \text{SL}(2, \mathbb{R})$ where

$$(21) \quad GP(x, t) = G(x, t) \cdot P(x, t)$$

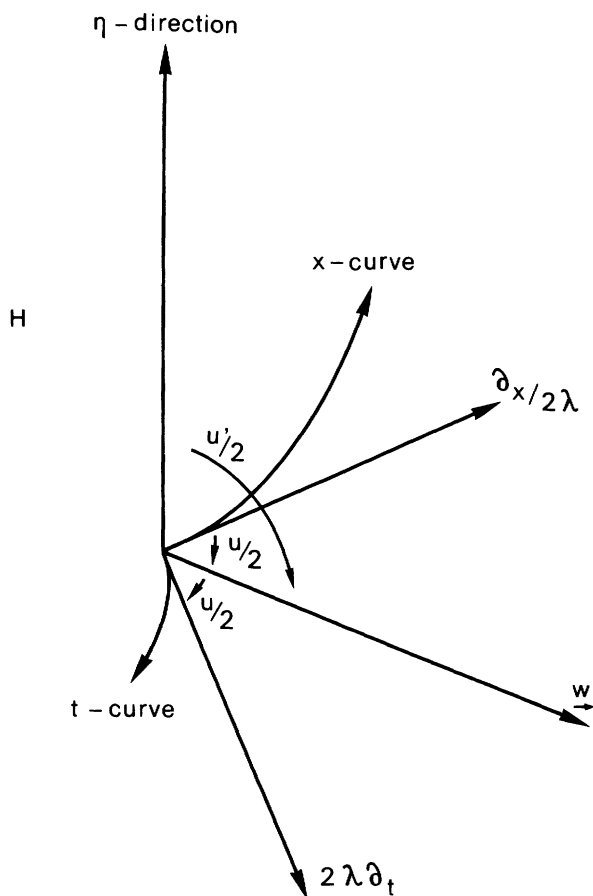
is another lift of f related by a gauge transformation; and

$$(22) \quad (GP)^{-1}d(GP) = P^{-1}dP + P^{-1}\Omega P$$

Comparing with Crampin's formula we see that his G is our GP , and his form Θ is given by (22).

The geometric meaning of this is as follows. G maps (i, i) to $\frac{1}{2\lambda} df(\partial_x(x, t))$;

P rotates through $-u(x, t)/2$. Since the angle from $\frac{1}{2\lambda} df(\partial_x)$ to $2\lambda df(\partial_t)$ is $-u$, GP maps (i, i) to the unit bisector of the angle between the two. And it is this lift that gives rise to the form Θ of [4] for scattering in the sin-Gordon equation.



The relations between u , u' etc. for $\lambda > 0$.

Write $GP = G'$; the method of [4] is to write

$$(23) \quad G' = TR^{-1}$$

where T is upper triangular and R is rotation. Then if R is rotation through $u'/2$, u' is a BT of u , and the equations can be derived in their standard form.

(Note that there is an error in the matrix representation of R in [4], which should, like P , contain *quarter* angles to give the BT as we shall see.)

Now T (a dilation + translation) does not change angles in the tangent space. So if $\underline{w}(x, t)$ is the unit bisector of the angle between $\frac{1}{2\lambda} df(\partial_x)$ and $2\lambda df(\partial_t)$, $u'/2$ is just the clockwise rotation from the vertical (the direction of (i, i)) to $\underline{w}(x, t)$. The diagram will perhaps make this relation clearer, as well as the geometrical nature of the angle u' .

Now we derive the formula for the BT in essentially the same way as [4] (not surprisingly). We have

$$(24) \quad G = G'P^{-1} = TR^{-1}P^{-1};$$

where

$$R'^{-1} = R^{-1}P^{-1} = \begin{pmatrix} \cos \frac{u' - u}{4} & -\sin \frac{u' - u}{4} \\ \sin \frac{u' - u}{4} & \cos \frac{u' - u}{4} \end{pmatrix}$$

and we require that

$$(25) \quad R'^{-1}dR' + R'^{-1}\Omega R'$$

should be upper triangular. The lower left corner of (25) is

$$(26) \quad \frac{du - du'}{4} + \frac{1}{2}\omega + \frac{1}{2}\left(\sigma^1 \cos \frac{u' - u}{2} + \sigma^2 \sin \frac{u' - u}{2}\right)$$

giving, when the values of ω , σ^1 , σ^2 are substituted in,

$$\begin{aligned} \frac{u'_x + u_x}{2} &= 2\lambda \sin \frac{u' - u}{2} \\ \frac{u'_t - u_t}{2} &= \frac{1}{2\lambda} \sin \frac{u' + u}{2} \end{aligned}$$

which is a standard form of the BT.

Conversely, let u' be a function satisfying (27). Then if R' is defined by (25), it is easy to see that $GR' = ST$, where T is upper triangular and S is constant. Hence, u' is a function which has the above geometric description for the solution $S^{-1}G$ of the scattering equation. So all Bäcklund transforms of u can be obtained geometrically; and the group of isometries of H acts on them (in a rather complicated way).

To end with a very simple example, set

$$(28) \quad f(x, t) = x - t + i \cosh(x + t).$$

It is easy to check that the x and t curves in H described by (28) have speed 1, so can be related to a sin-Gordon scattering problem with $\lambda = \frac{1}{2}$. To find $u(x, t)$, we have

$$(29) \quad f_x(x, t) = 1 + i \sinh(x + t), \quad f_t(x, t) = -1 + i \sinh(x + t).$$

So the clockwise angle u is given by

$$(30) \quad e^{iu(x, t)} = \frac{1 + i \sinh(x + t)}{-1 + i \sinh(x + t)} = (-\tanh(x + t) + i \operatorname{sech}(x + t))^2.$$

From this we can deduce that u is a simple soliton, $u(x, t) = 4 \tan^{-1} e^{x+t}$. The singular locus is $\sin u = 0$, which is simply $x + t = 0$ if we take $0 < u < 2\pi$.

It is immediate from (29) that f_x and f_t are symmetrical with respect to the imaginary axis in H . Hence the corresponding BT u' , using the geometrical definition, is trivial: $u'/2$ is $(2n + 1)\pi$ and

$$(31) \quad u' = (4n + 2)\pi.$$

However, non-trivial BT's can be obtained by applying an isometry to (28) and evaluating the corresponding angle u' .

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