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C*-algebraic generalization
of relative entropy and entropy (*)

by

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SUMMARY. — The concept of differentiability of a state with respect to a weight (state) on C*-algebra recently introduced by authors generalizes the notion of almost majorising introduced by Naudts in a von Neumann algebra context. It enables us to introduce the notions of entropy and relative entropy in the case of C*-algebraic description of a physical system. Our generalization of relative entropy leads to some modification of this notion concerning in the quantum case the effect of possible noncommutativity of the states.

1. INTRODUCTION

Let \( \mathcal{H} \) denote the Hilbert space corresponding to a quantum system and \( B(\mathcal{H}) \) denote the algebra of bounded operators on \( \mathcal{H} \). Observables are represented by selfadjoint elements of \( B(\mathcal{H}) \). In the most cases the statistical states of the system are described by normal states on \( B(\mathcal{H}) \). To each normal state \( \sigma \) on \( B(\mathcal{H}) \) corresponds a unique density operator \( \Sigma \) (semi-positive trace-class operator satisfying the condition \( \text{Tr} \Sigma = 1 \)) and \( \sigma(A) = \text{Tr} (\Sigma A) \), \( \forall A \in B(\mathcal{H}) \). The entropy of the normal state \( \sigma \) on \( B(\mathcal{H}) \) (called von Neumann entropy) is defined by the formula

\[
S^\sigma = - \text{Tr} (\Sigma \ln \Sigma).
\] (1.1)

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In the classical case the phase space of the system is a measure space $(\Omega, \mathcal{B}, \varphi)$. The macroscopic state of the system is described by a probability measure $\sigma$ (positive normalized measure) absolutely continuous with respect to $\varphi$. Then there exists a positive integrable function $f$ (Radon-Nikodym derivative) which satisfies $\int f \, d\varphi = 1$, $d\sigma = f \, d\varphi$ and the entropy of the state described by $\sigma$ (called generalized Boltzmann-Gibbs-Shannon entropy) is given by the formula

$$ S^\sigma = - \int f \ln f \, d\varphi = - \int \frac{d\sigma}{d\varphi} \ln \frac{d\sigma}{d\varphi} \, d\varphi. \quad (1.2) $$

Let us remind another notion of entropy, the so-called relative entropy, cf. for instance [5]. In the classical case consider two states described in terms of the probability measures $\sigma$ and $\varphi$ and assume $\sigma$ to be absolutely continuous with respect to $\varphi$. Denote $g = \frac{d\sigma}{d\varphi}$. The relative entropy of the state $\sigma$ with respect to $\varphi$ is defined by the formula

$$ S^{\sigma/\varphi} = \int g \ln g \, d\varphi = \int \frac{d\sigma}{d\varphi} \ln \frac{d\sigma}{d\varphi} \, d\varphi. \quad (1.3) $$

(This entropy is frequently called Kullback information or information gain).

The quantum analogue of (1.3) is usually written in the form [5]

$$ S^{\sigma/\varphi} = \text{Tr} \{ \Sigma (\ln \Sigma - \ln \Phi) \}, \quad (1.4) $$

where $\sigma(A) = \text{Tr} (\Sigma A)$, $\varphi(A) = \text{Tr} (\Phi A)$, $\forall A \in \mathcal{B}$. The aim of this paper is to generalize the notion of entropy and relative entropy in the case of a physical system described in terms of $C^*$-algebra. For this purpose we use the notion of differentiability of the state $\sigma$ with respect to a weight $\varphi$ on $C^*$-algebra $\mathcal{A}$ recently introduced by authors [2]. In the case of $\mathcal{A}$ being a von Neumann algebra, $\varphi$ — a faithful normal semi-finite weight on $\mathcal{A}$ and $\sigma$ — a normal state on $\mathcal{A}$, $\sigma$ differentiable with respect to $\varphi$ means that $\sigma$ is almost majorised by $\varphi$ in the sense of Naudts, [4]. Next, following Naudts, the $C^*$-algebraic generalization of entropy is defined (Sec. 3). It is verified that this expression for entropy in the cases $\mathcal{A} = \mathcal{B}(\mathcal{H})$ and $\mathcal{A} = \mathcal{F}(\Omega, \mathcal{B})$ takes the form (1.1) and (1.2), respectively. In Section 2 we consider the case of $\varphi$ being a state on $C^*$-algebra $\mathcal{A}$ and generalize the notion of relative entropy via the density operator of a state $\sigma$ differentiable with respect to the state $\varphi$. Our generalization of the relative entropy leads to the expression (1.3) in the classical case but in the quantum case $\mathcal{A} = \mathcal{B}(\mathcal{H})$ we obtain some modification of (1.4) concerning the effect of possible noncommutativity of the states $\sigma$ and $\varphi$. 

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Namely, if $\sigma(A) = \text{Tr} (\Sigma A)$, $\varphi(A) = \text{Tr} (\Phi A)$, assuming $\Phi$ to be strictly positive we obtain
\[ \mathcal{J}^{\sigma/\varphi} = \text{Tr} \left\{ \Sigma \ln (\Phi^{-1/2} \Sigma) \right\} = \text{Tr} \left\{ \Phi (\Phi^{-1/2} \Sigma \Phi^{-1/2}) \ln (\Phi^{-1/2} \Sigma \Phi^{-1/2}) \right\} \]
which differs from (1.4) except for the case of commuting $\Sigma$ and $\Phi$.

2. RELATIVE ENTROPY

Let $\varphi$ be a state on a C*-algebra $\mathcal{A}$ and let $\pi_\varphi : \mathcal{A} \to B(\mathcal{H}_\varphi)$ denote the cyclic representation of $\mathcal{A}$ with respect to $\varphi$. Let moreover $\langle \cdot , \cdot \rangle$ denote the inner product in $\mathcal{H}_\varphi$.

A state $\sigma$ on $\mathcal{A}$ will be called differentiable with respect to $\varphi$ ([2]) if it has the form
\[ \sigma(a) = \langle \xi | \pi_\varphi(a) \xi \rangle, \]
where $\xi \in \mathcal{H}_\varphi$ is the vector for which there exists a closable operator $\rho(\xi)$, densely defined in $\mathcal{H}_\varphi$ by the formula
\[ \rho(\xi) | a \rangle = \pi_\varphi(a) \xi, \quad \forall a \in \mathcal{A} \quad (*). \]

It is easy to verify that $\rho(\xi)$ is affiliated with $\pi_\varphi(\mathcal{A})'$. In this case there exists a unique vector $\xi$ for which $\check{\rho}(\xi)$ is positive and selfadjoint [4]. Such vector will be called the positive vector. An operator $P = \rho(\xi) + \check{\rho}(\xi)$ is called the density operator of the state $\sigma$ with respect to $\varphi$ and $\check{\rho}(\xi) = P^{1/2}$ for positive $\xi$, which we denote $\xi = (d\sigma/d\varphi)^{1/2}$. Following [4] we define the entropy $\mathcal{J}^{\sigma/\varphi}$ of the state $\sigma$ differentiable with respect to $\varphi$ in the following way
\[ \mathcal{J}^{\sigma/\varphi} = \lim_{\delta \downarrow 0} \langle \xi | \ln (PE_\delta) \xi \rangle, \]
whenever this limit exists, $E_\delta = E(\delta, \delta^{-1})$, where $E(d\lambda)$ stands for the spectral measure of $P$.

\[ a) \] Let us first consider the case $\mathcal{A} = B(\mathcal{H})$. Let $\varphi(A) = \text{Tr} (\Phi A)$, $A \in \mathcal{A}$, be a fixed state described by a density operator $\Phi$. Let $\mathcal{H}_\varphi$ denote the Hilbert space of cyclic representation of $\mathcal{A} = B(\mathcal{H})$ with respect to $\varphi$. The inner product in $\mathcal{H}_\varphi$ has the form
\[ \langle A | B \rangle = \varphi(A^+ B) = \text{Tr} (\Phi A^+ B). \]

Let
\[ \sigma(A) = \text{Tr} (\Sigma A) \]
and let moreover $\sigma$ fulfil the conditions
\[ \sigma(A) = \langle D | \pi_\varphi(A) D \rangle = \langle D | AD \rangle = \text{Tr} (\Phi D^+ AD) \]

(*) We employ the following notation: $| a \rangle$ stand for vectors belonging to the pre-Hilbert space obtained via the G. N. S. construction while $\xi$ can be an element of its $\langle \cdot , \cdot \rangle$-completion $\mathcal{H}_\varphi$.

and there exists an operator $\rho(D)$ such that

$$
\rho(D) |A\rangle = |AD\rangle \quad (2.7)
$$

$$
\rho(D)^+ |A\rangle = |AD^*\rangle \quad (2.8)
$$

(The involution $\ast$, conjugated to the involution $\dagger$, is defined in the following manner: if $\rho(D) = R((D_\ast)\dagger)$ then $\rho(D)^+ = R((D_\ast)^\dagger)$ and $|D_\ast\rangle^* = |D^\dagger\rangle$.

Hence, according to (2.1) and (2.2) the state $\sigma$ is differentiable with respect to the state $\phi$. From (2.5) and (2.6) one can easily obtain

$$
\Sigma = D\Phi D^+. \quad (2.9)
$$

Taking into account (2.7) and (2.8) we have

$$
\langle B | AD \rangle = \langle B | \rho(D)A \rangle = \langle \rho(D)^+B | A \rangle = \langle BD^* | A \rangle, \quad \forall A, B \in \mathcal{A}. \quad (2.10)
$$

From this condition and (2.4) assuming $\Phi$ to be strictly positive one can find

$$
D^* = \Phi D^* \Phi^{-1}. \quad (2.11)
$$

As mentioned above there is exactly one vector $|D\rangle$ for which $\rho(D)$ is selfadjoint and positive. Deriving the appropriate conditions from (2.10) we obtain with the help of (2.9) and (2.11) that $\rho(D)$ is selfadjoint and positive for

$$
D^* = D = (\Sigma \Phi^{-1})^{1/2} = \Phi^{1/2}(\Phi^{-1/2} \Sigma \Phi^{-1/2}) \Phi^{-1/2}. \quad (2.12)
$$

The operator $\Phi^{-1/2} \Sigma \Phi^{-1/2}$ is obviously selfadjoint and positive with respect to the initial inner product $(x, y)$ in $\mathcal{H}$. Let

$$
\Phi^{-1/2} \Sigma \Phi^{-1/2} = \sum_n \lambda_n E_n \quad (2.13)
$$

stand for its spectral decomposition. The operators $\Sigma \Phi^{-1}$ and $\Phi^{-1} \Sigma$ are not selfadjoint with respect to $(x, y)$ but they are selfadjoint and positive with respect to $(x, y)_\Phi^{-1} = (\Phi^{-1}x, y)$ and $(x, y)_\phi = (\Phi x, y)$, respectively. Denote by $\{E_n^{\Phi^{-1}}\}_{n \in \mathbb{N}}$ and $\{E_n^\Phi\}_{n \in \mathbb{N}}$ spectral families of the operators $\Sigma \Phi^{-1}$ and $\Phi^{-1} \Sigma$ (with respect to the inner products $(x, y)_{\Phi^{-1}}$ and $(x, y)_\phi$, resp.). Then

$$
\Sigma \Phi^{-1} = \sum_n \lambda_n E_n^{\Phi^{-1}}, \quad (2.14)
$$

$$
\Phi^{-1} \Sigma = \sum_n \lambda_n E_n^\Phi \quad (2.15)
$$

where

$$
E_n^{\Phi^{-1}} = \Phi^{1/2} E_n \Phi^{-1/2}, \quad (2.16)
$$

$$
E_n^\Phi = \Phi^{-1/2} E_n \Phi^{1/2}. \quad (2.17)
$$

From (2.14)-(2.17) we obtain

$$
f(\Sigma \Phi^{-1}) = \Phi^{1/2} f(\Phi^{-1/2} \Sigma \Phi^{-1/2}) \Phi^{-1/2}, \quad (2.18)
$$

$$
f(\Phi^{-1} \Sigma) = \Phi^{-1/2} f(\Phi^{-1/2} \Sigma \Phi^{-1/2}) \Phi^{1/2}. \quad (2.19)\)
Moreover one can easily verify the identity
\[ f(\Sigma \Phi^{-1}) \Phi = \Phi f(\Phi^{-1} \Sigma). \]  
(2.20)

Because
\[ P = [\rho \{ (\Sigma \Phi^{-1})^{1/2} \}]^2 \]  
(2.21)
we easily obtain
\[ P | A \rangle = | A \Sigma \Phi^{-1} \rangle. \]  
(2.22)

Taking into account (2.14) define bounded operators \( G_n^{\Phi^{-1}} \) by the formula
\[ G_n^{\Phi^{-1}} | A \rangle = | A E_n^{\Phi^{-1}} \rangle, \quad \forall A \in \mathcal{A}. \]  
(2.23)

The operators \( G_n^{\Phi^{-1}} \) are mutually orthogonal projectors in \( \pi_{\phi}(\mathcal{A})' \) with sum I. Then by (2.21) and (2.22) obviously \( \rho = \sum_n \lambda_n^{1/2} G_n^{\Phi^{-1}} \),
\[ P = \sum_n \lambda_n G_n^{\Phi^{-1}}. \]  
(2.24)

Now let us calculate from (2.3) the entropy of the state \( \sigma \) differentiable with respect to the state \( \phi \). Using (2.12), (2.24), (2.4) and (2.20) we obtain
\[
\mathcal{E}^{\sigma/\phi} = \lim_{n \to \infty} \left\langle (\Sigma \Phi^{-1})^{1/2} | \ln \left( \sum_{p=1}^{n} G_p^{\Phi^{-1}} \right)(\Sigma \phi^{-1})^{1/2} \right\rangle 
\]
\[
= \lim_{n \to \infty} \sum_{p=1}^{n} \ln \lambda_p \left\langle (\Sigma \Phi^{-1})^{1/2} | G_p^{\Phi^{-1}}(\Sigma \phi^{-1})^{1/2} \right\rangle 
\]
\[
= \lim_{n \to \infty} \sum_{p=1}^{n} \ln \lambda_p \left\langle (\Sigma \phi^{-1})^{1/2} | (\Sigma \phi^{-1})^{1/2} E_p^{\Phi^{-1}} \right\rangle 
\]
\[
= \lim_{n \to \infty} \sum_{p=1}^{n} \ln \lambda_p \left\langle (\Sigma \Phi^{-1}) \Phi(\Phi^{-1} \Sigma)^{1/2}(\Sigma \phi^{-1})^{1/2} E_p^{\Phi^{-1}} \right\rangle 
\]
\[
= \lim_{n \to \infty} \sum_{p=1}^{n} \ln \lambda_p \left\langle (\Sigma \Phi^{-1}) \Phi(\Sigma \phi^{-1})^{1/2} E_p^{\Phi^{-1}} \right\rangle 
\]
\[
= \lim_{n \to \infty} \sum_{p=1}^{n} \left( \frac{1}{2} \ln \lambda_p \left\langle \Phi(\Sigma \Phi^{-1}) \right\rangle \right) 
\]
\[
= \lim_{n \to \infty} \sum_{p=1}^{n} \lambda_p \ln \lambda_p \left\langle \Phi G_p^{\Phi^{-1}} \right\rangle 
\]
\[
= \left\langle \Phi(\Sigma \phi^{-1}) \ln (\Sigma \phi^{-1}) \right\rangle 
\]
\[
= \left\langle \Sigma \ln (\Sigma \Phi^{-1}) \right\rangle 
\]
\[
= \left\langle \Sigma \ln (\phi^{-1} \Sigma) \right\rangle . \]  
(2.25)
Let us generalize the expression (2.25) in the case of noninvertible $\Phi$. In this case our relative entropy is well defined by the formula

$$\mathcal{S}^{\sigma/\rho} = \text{Tr} \left( \Sigma \ln \Sigma \right)$$

(2.26)

for every differentiable $\sigma$ having the density operator $\Sigma = \Phi X$, where $X$ is an essentially selfadjoint operator with respect to the inner product $(x, y)_\Phi = (\Phi x, y)$, which for invertible $\Phi$ takes the form $X = \Phi^{-1} \Sigma$. It is obvious that $X$ is positive with respect to the inner product $(x, y)_\Phi; (x, Xx)_\Phi = (\Phi x, Xx) = (x, \Phi Xx) = (x, \Sigma x) \geq 0$ for all $x \in \mathcal{D}(X) \subseteq \mathcal{H}$ due to the positivity of $\Sigma$.

Note that $\ln X \neq \ln \Sigma - \ln \Phi$, except for the case of commuting $\Sigma$ and $\Phi$. Hence our relative entropy differs from the Araki's relative entropy [1], which is well defined only in the case of faithful states and in this example takes the form (1.4).

b) In the classical case consider $\mathcal{A} = \mathcal{F}(\Omega, \mathcal{B})$ the $C^*$-algebra of bounded measurable functions with the norm $\|g\|_\infty = \sup \{ |g(\omega)| : \omega \in \Omega \}$. Let $\rho$ and $\sigma$ be probability measures on $(\Omega, \mathcal{B})$ which define the states on $\mathcal{A}$

$$\varphi(g) = \int g d\varphi,$$

(2.27)

$$\sigma(g) = \int g d\sigma.$$  

(2.28)

Let $\mathcal{H}_\varphi = L^2(\Omega, \mathcal{B}, \varphi)$ and $\pi_\varphi : A \rightarrow B(\mathcal{H}_\varphi)$ be the cyclic $*$-representation with domain $\mathcal{D}(\pi_\varphi) = \mathcal{H}_\varphi$ defined by

$$[\pi(g)h](\omega) = g(\omega)h(\omega).$$

(2.29)

The inner product in $\mathcal{H}_\varphi$ has the form

$$\langle g | h \rangle = \int \bar{g} h d\varphi.$$  

(2.30)

Assume $\sigma$ to be differentiable with respect to $\varphi$. It is easy to verify that in this case $\rho(\xi) = \left( \frac{d\sigma}{d\varphi} \right)^{1/2}$, $P = \frac{d\sigma}{d\varphi} \equiv f$. Let $P = \int_0^\infty \lambda dE_\lambda$, where $E_\lambda = \chi(N_\lambda)$ denotes the characteristic function of $N_\lambda = \{ \omega \in \Omega : f(\omega) \leq \lambda \}$. From (2.3) we obtain

$$\mathcal{S}^{\sigma/\rho} = \lim_{\lambda \rightarrow 0+} \langle \xi | \ln (PE_\lambda) \xi \rangle = \lim_{\lambda \rightarrow \infty} \langle \xi | \ln P(E_\lambda - E_{1/\lambda}) \xi \rangle$$

$$= \lim_{\lambda \rightarrow \infty} \int f \ln f(E_\lambda - E_{1/\lambda}) d\varphi = \int f \ln f d\varphi$$

$$= \int \frac{d\sigma}{d\varphi} \ln \frac{d\sigma}{d\varphi} d\varphi.$$  

(2.31)
3. ENTROPY

We will briefly sketch some elements of the theory of weights on C*-algebras according to [3] and [4].

A weight on C*-algebra is a function \( \varphi : \mathcal{A}_+ \to \mathbb{R}^+ + \{ + \infty \} \) satisfying the conditions:

\[
\varphi(a + b) = \varphi(a) + \varphi(b), \quad \forall a, b \in \mathcal{A}_+
\]
\[
\varphi(\alpha a) = \alpha \varphi(a), \quad \forall \alpha \in \mathbb{R}^+, \quad \forall a \in \mathcal{A}_+
\]
(with the convention \( 0.(+ \infty) = 0 \)).

A trace on \( \mathcal{A} \) is a weight \( \varphi \) for which

\[
\varphi(aa^*) = \varphi(a^*a), \quad \forall a \in \mathcal{A}.
\]

Define

\[
\mathcal{L}_\varphi := \{ a \in \mathcal{A} : \varphi(a^*a) < + \infty \},
\]

then \( \mathcal{L}_\varphi \) is a left ideal in \( \mathcal{A} \). Let \( \mathcal{A}_\varphi = \mathcal{L}_\varphi^* / \mathcal{L}_\varphi \) that is the set of all complex linear combinations of elements \( a^*b, a, b \in \mathcal{L}_\varphi \). Then \( \mathcal{A}_\varphi \) is a \( * \)-subalgebra of \( \mathcal{A} \) and \( \mathcal{A}_\varphi + = \mathcal{A}_\varphi \cap \mathcal{A}_+ \) is exactly the set \( \{ a \in \mathcal{A}_+ : \varphi(a) < + \infty \} \) and \( \mathcal{A}_\varphi \) is the complex linear span of \( \mathcal{A}_\varphi + \). Moreover \( \mathcal{A}_\varphi + \) is a cone in \( \mathcal{A}_+ \) which is hereditary, i.e.,

\[
0 \leq a \leq b \in \mathcal{A}_\varphi + \Rightarrow a \in \mathcal{A}_\varphi +,
\]
hence \( \mathcal{A}_\varphi \) has the property that \( b^*ac \in \mathcal{A}_\varphi \) if \( b, c \in \mathcal{A}_\varphi \). The weight \( \varphi \) can be extended uniquely to a linear positive functional on \( \mathcal{A}_\varphi \) (again denoted by \( \varphi \)).

**Theorem (cf. [3], [4]).** — For each weight \( \varphi \) on \( \mathcal{A} \) there exists a Hilbert space \( \mathcal{H}_\varphi \) and two mappings: \( \Lambda_\varphi : \mathcal{L}_\varphi \to \mathcal{H}_\varphi \) and \( \pi_\varphi : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\varphi) \) such that \( \Lambda_\varphi \) is linear with the range dense in \( \mathcal{H}_\varphi \), \( \pi_\varphi \) is a representation of \( \mathcal{A} \), and

\[
\langle \Lambda_\varphi b | \pi_\varphi(a) \Lambda_\varphi c \rangle = \varphi(c^*ab)
\]

for all \( a \in \mathcal{A} \) and \( b, c \in \mathcal{L}_\varphi \).

Let \( \varphi \) be a weight on C*-algebra \( \mathcal{A} \) and let \( \pi_\varphi : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\varphi) \) denote the representation of \( \mathcal{A} \) corresponding to \( \varphi \). A state \( \sigma \) on \( \mathcal{A} \) will be called differentiable with respect to \( \varphi \) if it has the form

\[
\sigma(a) = \langle \xi | \pi_\varphi(a) \xi \rangle,
\]

where \( \xi \in \mathcal{H}_\varphi \) is the vector for which there exists a closable operator \( \rho(\xi) \), densely defined in \( \mathcal{H}_\varphi \) by the formula

\[
\rho(\xi) | \Lambda_\varphi a \rangle = \pi_\varphi(a)\xi, \quad \forall a \in \mathcal{A}.
\]

Again, there exists a unique vector \( \xi \) for which \( \overline{\rho(\xi)} \) is positive and selfadjoint and an operator \( P = \rho(\xi) + \rho(\xi)^* \) is called the density operator of the state \( \sigma \).
with respect to \( \varphi \). In the case of \( \mathcal{A} \) being a von Neumann algebra, \( \varphi \) — a faithful normal semi-finite weight on \( \mathcal{A} \) and \( \sigma \) — a normal state on \( \mathcal{A} \), \( \sigma \) differentiable with respect to \( \varphi \) means that \( \sigma \) is almost majorised by \( \varphi \) ([4]).

Analogously to [4] we define the entropy \( \mathcal{J}^{\sigma|\varphi} \) of a state \( \sigma \) differentiable with respect to a weight \( \varphi \) by the formula

\[
\mathcal{J}^{\sigma|\varphi} = - \lim_{\delta \to 0} \langle \xi \mid \ln (\mathcal{E}_\delta) \xi \rangle
\]

whenever this limit exists, \( \mathcal{E}_\delta = \mathcal{E}([\delta, \delta^{-1}]) \), where \( \mathcal{E}(d\lambda) \) stands for the spectral measure of \( \mathcal{P} \).

As in the previous section one can verify that (3.3) is a generalization of entropy.

a) Let \( \varphi(A^+A) = \text{Tr} (A^+A) \) for all \( A \in \mathcal{B(H)} \). Assuming that a state \( \sigma \) is differentiable with respect to \( \text{Tr} (\cdot) \) we can find that \( \rho(D) \) is positive and selfadjoint for \( D^* = D = \Sigma^{1/2}, \mathcal{P} = [\rho(\Sigma^{1/2})]^2 \) and consequently

\[
\mathcal{J}^{\sigma|\varphi} = - \text{Tr} (\Sigma \ln \Sigma).
\]

b) Let \( \varphi(gg) = \int gg d\varphi, \forall g \in \mathcal{F} (\Omega, \mathcal{B}) \), where \( \varphi \) is positive measure on \( (\Omega, \mathcal{B}) \). Assuming \( \sigma \) differentiable with respect to \( \varphi \) we obtain \( \rho(\xi) = \left( \frac{d\sigma}{d\varphi} \right)^{1/2} \), \( \mathcal{P} = \frac{d\sigma}{d\varphi} \equiv f \). Then from (3.3) we obtain

\[
\mathcal{J}^{\sigma|\varphi} = - \int \frac{d\sigma}{d\varphi} \ln \frac{d\sigma}{d\varphi} d\varphi.
\]

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REFERENCES


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