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C*-algebraic generalization of relative entropy and entropy (*)

by

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SUMMARY. — The concept of differentiability of a state with respect to a weight (state) on C*-algebra recently introduced by authors generalizes the notion of almost majorising introduced by Naudts in a von Neumann algebra context. In enables us to introduce the notions of entropy and relative entropy in the case of C*-algebraic description of a physical system. Our generalization of relative entropy leads to some modification of this notion concerning in the quantum case the effect of possible noncommutativity of the states.

1. INTRODUCTION

Let $\mathscr{H}$ denote the Hilbert space corresponding to a quantum system and $B(\mathscr{H})$ denote the algebra of bounded operators on $\mathscr{H}$. Observables are represented by selfadjoint elements of $B(\mathscr{H})$. In the most cases the statistical states of the system are described by normal states on $B(\mathscr{H})$. To each normal state σ on $B(\mathscr{H})$ corresponds a unique density operator Σ (semi-positive trace-class operator satisfying the condition $\mathrm{Tr}\,\Sigma = 1$) and $\sigma(A) = \mathrm{Tr}\,(\Sigma A)$, $\forall A \in B(\mathscr{H})$. The entropy of the normal state σ on $B(\mathscr{H})$ (called von Neumann entropy) is defined by the formula

$$\mathscr{S}^{\sigma} = -\,\mathrm{Tr}\,(\Sigma \ln \Sigma)\,. \tag{1.1}$$

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In the classical case the phase space of the system is a measure space $(\Omega, \mathscr{B}, \varphi)$. The macroscopic state of the system is described by a probability measure σ (positive normalized measure) absolutely continuous with respect to φ. Then there exists a positive integrable function f (Radon-Nikodym derivative) which satisfies $\int f d\varphi = 1, d\sigma = f d\varphi$ and the entropy of the state described by σ (called generalized Boltzmann-Gibbs-Shannon entropy) is given by the formula

$$\mathscr{S}^{\sigma} = - \int f \ln f d\varphi = - \int \frac{d\sigma}{d\varphi} \ln \frac{d\sigma}{d\varphi} d\varphi . \tag{1.2}$$

Let us remind another notion of entropy, the so-called relative entropy, cf. for instance [5]. In the classical case consider two states described in terms of the probability measures σ and φ and assume σ to be absolutely continuous with respect to φ. Denote $g = \frac{d\sigma}{d\varphi}$. The relative entropy of the state σ with respect to φ is defined by the formula

$$\mathscr{S}^{\sigma/\varphi} = \int g \ln g d\varphi = \int \frac{d\sigma}{d\varphi} \ln \frac{d\sigma}{d\varphi} d\varphi . \tag{1.3}$$

(This entropy is frequently called Kullback information or information gain).

The quantum analogue of (1.3) is usually written in the form [5]

$$\mathscr{S}^{\sigma/\varphi} = \operatorname{Tr} \{ \Sigma(\ln \Sigma - \ln \Phi) \} , \tag{1.4}$$

where $\sigma(A) = \operatorname{Tr}(\Sigma A)$, $\varphi(A) = \operatorname{Tr}(\Phi A)$, $\forall A \in B(\mathscr{H})$.

The aim of this paper is to generalize the notion of entropy and relative entropy in the case of a physical system described in terms of C*-algebra. For this purpose we use the notion of differentiability of the state σ with respect to a weight φ on C*-algebra $\mathscr{A}$ recently introduced by authors [2]. In the case of $\mathscr{A}$ being a von Neumann algebra, φ —a faithful normal semi-finite weight on $\mathscr{A}$ and σ— a normal state on $\mathscr{A}$, σ differentiable with respect to φ means that σ is almost majorised by φ in the sense of Naudts, [4]. Next, following Naudts, the C*-algebraic generalization of entropy is defined (Sec. 3). It is verified that this expression for entropy in the cases $\mathscr{A} = B(\mathscr{H})$ and $\mathscr{A} = \mathscr{F}(\Omega, \mathscr{B})$ takes the form (1.1) and (1.2), respectively. In Section 2 we consider the case of φ being a state on C*-algebra $\mathscr{A}$ and generalize the notion of relative entropy via the density operator of a state σ differentiable with respect to the state φ. Our generalization of the relative entropy leads to the expression (1.3) in the classical case but in the quantum case $\mathscr{A} = B(\mathscr{H})$ we obtain some modification of (1.4) concerning the effect of possible noncommutativity of the states σ and φ.

Namely, if $\sigma(A) = \mathrm{Tr}(\Sigma A)$, $\varphi(A) = \mathrm{Tr}(\Phi A)$, assuming Φ to be strictly positive we obtain

$$\mathcal{S}^{\sigma/\varphi} = \mathrm{Tr}\{\Sigma \ln(\Phi^{-1}\Sigma)\} = \mathrm{Tr}\{\Phi(\Phi^{-1/2}\Sigma\Phi^{-1/2})\ln(\Phi^{-1/2}\Sigma\Phi^{-1/2})\} \tag{1.5}$$

which differs from (1.4) except for the case of commuting Σ and Φ.

2. RELATIVE ENTROPY

Let φ be a state on a C*-algebra $\mathcal{A}$ and let $\pi_\varphi : \mathcal{A} \to B(\mathcal{H}_\varphi)$ denote the cyclic representation of $\mathcal{A}$ with respect to φ. Let moreover $\langle\,.\,|\,.\,\rangle$ denote the inner product in $\mathcal{H}_\varphi$.

A state σ on $\mathcal{A}$ will be called differentiable with respect to φ ([*2*]) if it has the form

$$\sigma(a) = \langle\,\xi\,|\,\pi_\varphi(a)\xi\,\rangle, \tag{2.1}$$

where $\xi \in \mathcal{H}_\varphi$ is the vector for which there exists a closable operator $\rho(\xi)$, densely defined in $\mathcal{H}_\varphi$ by the formula

$$\rho(\xi)\,|\,a\,\rangle = \pi_\varphi(a)\xi, \qquad \forall a \in \mathcal{A}\ (*). \tag{2.2}$$

It is easy to verify that $\rho(\xi)$ is affiliated with $\pi_\varphi(\mathcal{A})'$. In this case there exists a unique vector ξ for which $\overline{\rho(\xi)}$ is positive and selfadjoint [*4*]. Such vector will be called the positive vector. An operator $P = \rho(\xi)^+\overline{\rho(\xi)}$ is called the density operator of the state σ with respect to φ and $\overline{\rho(\xi)} = P^{1/2}$ for positive ξ, which we denote $\xi = (d\sigma/d\varphi)^{1/2}$. Following [*4*] we define the entropy $\mathcal{S}^{\sigma/\varphi}$ of the state σ differentiable with respect to φ in the following way

$$\mathcal{S}^{\sigma/\varphi} = \lim_{\delta \downarrow 0} \langle\,\xi\,|\,\ln(PE_\delta)\xi\,\rangle, \tag{2.3}$$

whenever this limit exists, $E_\delta = E([\delta, \delta^{-1}])$, where $E(d\lambda)$ stands for the spectral measure of P.

a) Let us first consider the case $\mathcal{A} = B(\mathcal{H})$. Let $\varphi(A) = \mathrm{Tr}(\Phi A)$, $A \in \mathcal{A}$, be a fixed state described by a density operator Φ. Let $\mathcal{H}_\varphi$ denote the Hilbert space of cyclic representation of $\mathcal{A} = B(\mathcal{H})$ with respect to φ. The inner product in $\mathcal{H}_\varphi$ has the form

$$\langle\,A\,|\,B\,\rangle = \varphi(A^+B) = \mathrm{Tr}(\Phi A^+ B). \tag{2.4}$$

Let

$$\sigma(A) = \mathrm{Tr}(\Sigma A) \tag{2.5}$$

and let moreover σ fulfil the conditions

$$\sigma(A) = \langle\,D\,|\,\pi_\varphi(A)D\,\rangle = \langle\,D\,|\,AD\,\rangle = \mathrm{Tr}(\Phi D^+ A D) \tag{2.6}$$

(*) We employ the following notation: $|\,a\,\rangle$ stand for vectors belonging to the pre-Hilbert space obtained via the G. N. S. construction while ξ can be an element of its $\langle\,.\,|\,.\,\rangle$-completion $\mathcal{H}_\varphi$.

and there exists an operator $\rho(D)$ such that

$$\rho(D)\,|\,A\rangle = |\,AD\rangle \tag{2.7}$$

$$\rho(D)^+\,|\,A\rangle = |\,AD^{\#}\rangle\,. \tag{2.8}$$

(The involution $^{\#}$, conjugated to the involution $^{+}$, is defined in the following manner: if $\rho(D) = R(|\,D\rangle)$ then $\rho(D)^+ = R(|\,D\rangle^{\#})$ and $|\,D\rangle^{\#} = |\,D^{\#}\rangle$). Hence, according to (2.1) and (2.2) the state σ is differentiable with respect to the state φ. From (2.5) and (2.6) one can easily obtain

$$\Sigma = D\Phi D^+\,. \tag{2.9}$$

Taking into account (2.7) and (2.8) we have

$$\langle B\,|\,AD\rangle = \langle B\,|\,\rho(D)A\rangle = \langle \rho(D)^+B\,|\,A\rangle = \langle BD^{\#}\,|\,A\rangle\,, \quad \forall A, B \in \mathscr{A}\,. \tag{2.10}$$

From this condition and (2.4) assuming Φ to be strictly positive one can find

$$D^{\#} = \Phi D^+\Phi^{-1}\,. \tag{2.11}$$

As mentioned above there is exactly one vector $|\,D\rangle$ for which $\rho(D)$ is selfadjoint and positive. Deriving the appropriate conditions from (2.10) we obtain with the help of (2.9) and (2.11) that $\rho(D)$ is selfadjoint and positive for

$$D^{\#} = D = (\Sigma\Phi^{-1})^{1/2} \equiv \Phi^{1/2}(\Phi^{-1/2}\Sigma\phi^{-1/2})\Phi^{-1/2}\,. \tag{2.12}$$

The operator $\Phi^{-1/2}\Sigma\Phi^{-1/2}$ is obviously selfadjoint and positive with respect to the initial inner product (x, y) in $\mathscr{H}$. Let

$$\Phi^{-1/2}\Sigma\Phi^{-1/2} = \sum_n \lambda_n E_n \tag{2.13}$$

stand for its spectral decomposition. The operators $\Sigma\Phi^{-1}$ and $\Phi^{-1}\Sigma$ are not selfadjoint with respect to (x, y) but they are selfadjoint and positive with respect to $(x, y)_{\Phi^{-1}} = (\Phi^{-1}x, y)$ and $(x, y)_\Phi = (\Phi x, y)$, respectively. Denote by $\{E_n^{\Phi^{-1}}\}_{n\in\mathbb{N}}$ and $\{E_n^{\Phi}\}_{n\in\mathbb{N}}$ spectral families of the operators $\Sigma\Phi^{-1}$ and $\Phi^{-1}\Sigma$ (with respect to the inner products $(x, y)_{\Phi^{-1}}$ and $(x, y)_\Phi$, resp.). Then

$$\Sigma\Phi^{-1} = \sum_n \lambda_n E_n^{\Phi^{-1}}\,, \tag{2.14}$$

$$\Phi^{-1}\Sigma = \sum_n \lambda_n E_n^{\Phi} \tag{2.15}$$

where

$$E_n^{\Phi^{-1}} = \Phi^{1/2}E_n\Phi^{-1/2}\,, \tag{2.16}$$

$$E_n^{\Phi} = \Phi^{-1/2}E_n\Phi^{1/2}\,. \tag{2.17}$$

From (2.14)-(2.17) we obtain

$$f(\Sigma\Phi^{-1}) = \Phi^{1/2}f(\Phi^{-1/2}\Sigma\Phi^{-1/2})\Phi^{-1/2}\,, \tag{2.18}$$

$$f(\Phi^{-1}\Sigma) = \Phi^{-1/2}f(\Phi^{-1/2}\Sigma\Phi^{-1/2})\Phi^{1/2}\,. \tag{2.19}$$

Moreover one can easily verify the identity

$$f(\Sigma\Phi^{-1})\Phi = \Phi f(\Phi^{-1}\Sigma). \tag{2.20}$$

Because

$$P = [\rho\{(\Sigma\Phi^{-1})^{1/2}\}]^2 \tag{2.21}$$

we easily obtain

$$P \,|\, A \rangle = |\, A\Sigma\Phi^{-1} \rangle. \tag{2.22}$$

Taking into account (2.14) define bounded operators $G_n^{\Phi^{-1}}$ by the formula

$$G_n^{\Phi^{-1}} \,|\, A \rangle = |\, AE_n^{\Phi^{-1}} \rangle, \qquad \forall A \in \mathscr{A}. \tag{2.23}$$

The operators $G_n^{\Phi^{-1}}$ are mutually orthogonal projectors in $\pi_\varphi(\mathscr{A})'$ with sum I. Then by (2.21) and (2.22) obviously $\rho = \sum_n \lambda_n^{1/2} G_n^{\Phi^{-1}}$,

$$P = \sum_n \lambda_n G_n^{\Phi^{-1}}. \tag{2.24}$$

Now let us calculate from (2.3) the entropy of the state σ differentiable with respect to the state φ. Using (2.12), (2.24), (2.4) and (2.20) we obtain

$$\begin{aligned}
\mathscr{S}^{\sigma/\varphi} &= \lim_{n\to\infty} \left\langle (\Sigma\Phi^{-1})^{1/2} \,\middle|\, \ln\left(P \sum_{\substack{p=1\\ \lambda_p \neq 0}}^{n} G_p^{\Phi^{-1}}\right)(\Sigma\phi^{-1})^{1/2} \right\rangle \\
&= \lim_{n\to\infty} \sum_{p=1}^{n} \ln \lambda_p \langle (\Sigma\Phi^{-1})^{1/2} \,|\, G_p^{\Phi^{-1}}(\Sigma\phi^{-1})^{1/2} \rangle \\
&= \lim_{n\to\infty} \sum_{p=1}^{n} \ln \lambda_p \langle (\Sigma\phi^{-1})^{1/2} \,|\, (\Sigma\phi^{-1})^{1/2} E_p^{\Phi^{-1}} \rangle \\
&= \lim_{n\to\infty} \sum_{p=1}^{n} \ln \lambda_p \operatorname{Tr}\{\Phi(\Phi^{-1}\Sigma)^{1/2}(\Sigma\phi^{-1})^{1/2} E_p^{\Phi^{-1}}\} \\
&= \lim_{n\to\infty} \sum_{p=1}^{n} \ln \lambda_p \operatorname{Tr}\{(\Sigma\Phi^{-1})\Phi(\Sigma\Phi^{-1})^{1/2} E_p^{\Phi^{-1}}\} \\
&= \lim_{n\to\infty} \sum_{p=1}^{n} \ln \lambda_p \operatorname{Tr}\{\Phi(\Sigma\Phi^{-1}) G_p^{\Phi^{-1}}\} \\
&= \lim_{n\to\infty} \sum_{p=1}^{n} \lambda_p \ln \lambda_p \operatorname{Tr}\{\Phi G_p^{\Phi^{-1}}\} \\
&= \operatorname{Tr}\{\Phi(\Sigma\phi^{-1}) \ln(\Sigma\phi^{-1})\} \\
&= \operatorname{Tr}\{\Sigma \ln(\Sigma\Phi^{-1})\} \\
&= \operatorname{Tr}\{\Sigma \ln(\phi^{-1}\Sigma)\}.
\end{aligned} \tag{2.25}$$

Let us generalize the expression (2.25) in the case of noninvertible Φ. In this case our relative entropy is well defined by the formula

$$\mathcal{S}^{\sigma/\varphi} = \mathrm{Tr}\,(\Sigma \ln \mathrm{X}) \tag{2.26}$$

for every differentiable σ having the density operator $\Sigma = \Phi \mathrm{X}$, where X is an essentially selfadjoint operator with respect to the inner product $(x, y)_\Phi = (\Phi x, y)$, which for invertible Φ takes the form $\mathrm{X} = \Phi^{-1}\Sigma$. It is obvious that X is positive with respect to the inner product $(x, y)_\Phi$; $(x, \mathrm{X}x)_\Phi = (\Phi x, \mathrm{X}x) = (x, \Phi \mathrm{X}x) = (x, \Sigma x) \geqslant 0$ for all $x \in \mathcal{D}(\mathrm{X}) \subseteq \mathcal{H}$ due to the positivity of Σ.

Note that $\ln \mathrm{X} \neq \ln \Sigma - \ln \Phi$, except for the case of commuting Σ and Φ. Hence our relative entropy differs from the Araki's relative entropy [1], which is well defined only in the case of faithful states and in this example takes the form (1.4).

b) In the classical case consider $\mathcal{A} = \mathcal{F}(\Omega, \mathcal{B})$ the C*-algebra of bounded measurable functions with the norm $\| g \|_\infty = \sup \{ | g(\omega) | : \omega \in \Omega \}$. Let φ and σ be probability measures on $(\Omega, \mathcal{B})$ which define the states on $\mathcal{A}$

$$\varphi(g) = \int g d\varphi\,, \tag{2.27}$$

$$\sigma(g) = \int g d\sigma\,. \tag{2.28}$$

Let $\mathcal{H}_\varphi = \mathrm{L}^2(\Omega, \mathcal{B}, \varphi)$ and $\pi_\varphi : \mathcal{A} \to \mathrm{B}(\mathcal{H}_\varphi)$ be the cyclic *-representation with domain $\mathcal{D}(\pi_\varphi) = \mathcal{H}_\varphi$ defined by

$$[\pi(g)h](\omega) = g(\omega)h(\omega)\,. \tag{2.29}$$

The inner product in $\mathcal{H}_\varphi$ has the form

$$\langle g \mid h \rangle = \int \bar{g} h d\varphi\,. \tag{2.30}$$

Assume σ to be differentiable with respect to φ. It is easy to verify that in this case $\rho(\xi) = \left(\dfrac{d\sigma}{d\varphi}\right)^{1/2}$, $\mathrm{P} = \dfrac{d\sigma}{d\varphi} \equiv f$. Let $\mathrm{P} = \int_0^\infty \lambda d\mathrm{E}_\lambda$, where $\mathrm{E}_\lambda = \chi(\mathrm{N}_\lambda)$ denotes the characteristic function of $\mathrm{N}_\lambda = \{ \omega \in \Omega : f(\omega) \leqslant \lambda \}$. From (2.3) we obtain

$$\begin{aligned} \mathcal{S}^{\sigma/\varphi} &= \lim_{\delta \downarrow 0} \langle \xi \mid \ln (\mathrm{P}\mathrm{E}_\delta)\xi \rangle = \lim_{\lambda \to \infty} \langle \xi \mid \ln \mathrm{P}(\mathrm{E}_\lambda - \mathrm{E}_{1/\lambda})\xi \rangle \\ &= \lim_{\lambda \to \infty} \int f \ln f (\mathrm{E}_\lambda - \mathrm{E}_{1/\lambda}) d\varphi = \int f \ln f d\varphi \\ &= \int \frac{d\sigma}{d\varphi} \ln \frac{d\sigma}{d\varphi}\, d\varphi\,. \end{aligned} \tag{2.31}$$

3. ENTROPY

We will briefly sketch some elements of the theory of weights on C*-algebra according to [3] and [4].

A weight on C*-algebra is a function $\varphi : \mathscr{A}_+ \to \mathbb{R}^+ + \{ +\infty \}$ satisfying the conditions:

$$\varphi(a + b) = \varphi(a) + \varphi(b), \qquad \forall a, b \in \mathscr{A}_+$$
$$\varphi(\alpha a) = \alpha\varphi(a), \qquad \forall \alpha \in \mathbb{R}^+, \quad \forall a \in \mathscr{A}_+$$

(with the convention $0.(+\infty) = 0$).

A trace on $\mathscr{A}$ is a weight φ for which

$$\varphi(a^*a) = \varphi(aa^*), \qquad \forall a \in \mathscr{A}.$$

Define

$$\mathscr{L}_\varphi := \{ a \in \mathscr{A} : \varphi(a^*a) < +\infty \},$$

then $\mathscr{L}_\varphi$ is a left ideal in $\mathscr{A}$. Let $\mathscr{A}_\varphi = \mathscr{L}_\varphi^* \mathscr{L}_\varphi$, that is the set of all complex linear combinations of elements a^*b, $a, b \in \mathscr{L}_\varphi$. Then $\mathscr{A}_\varphi$ is a *-subalgebra of $\mathscr{A}$ and $\mathscr{A}_{\varphi+} = \mathscr{A}_\varphi \cap \mathscr{A}_+$ is exactly the set $\{ a \in \mathscr{A}_+ : \varphi(a) < +\infty \}$ and $\mathscr{A}_\varphi$ is the complex linear span of $\mathscr{A}_{\varphi+}$. Moreover $\mathscr{A}_{\varphi+}$ is a cone in $\mathscr{A}_+$ which is hereditary, i. e.,

$$0 \leqslant a \leqslant b \in \mathscr{A}_{\varphi+} \Rightarrow a \in \mathscr{A}_{\varphi+},$$

hence $\mathscr{A}_\varphi$ has the property that $b^*ac \in \mathscr{A}_\varphi$ if $b, c \in \mathscr{A}_\varphi$. The weight φ can be extended uniquely to a linear positive functional on $\mathscr{A}_\varphi$ (again denoted by φ).

THEOREM (cf. [3], [4]). — For each weight φ on $\mathscr{A}$ there exists a Hilbert space $\mathscr{H}_\varphi$ and two mappings: $\Lambda_\varphi : \mathscr{L}_\varphi \to \mathscr{H}_\varphi$ and $\pi_\varphi : \mathscr{A} \to B(\mathscr{H}_\varphi)$ such that Λ_φ is linear with the range dense in $\mathscr{H}_\varphi$, π_φ is a representation of $\mathscr{A}$, and

$$\langle \Lambda_\varphi b \mid \pi_\varphi(a)\Lambda_\varphi c \rangle = \varphi(c^*ab)$$

for all $a \in \mathscr{A}$ and $b, c \in \mathscr{L}_\varphi$.

Let φ be a weight on C*-algebra $\mathscr{A}$ and let $\pi_\varphi : \mathscr{A} \to B(\mathscr{H}_\varphi)$ denote the representation of $\mathscr{A}$ corresponding to φ. A state σ on $\mathscr{A}$ will be called differentiable with respect to φ if it has the form

$$\sigma(a) = \langle \xi \mid \pi_\varphi(a)\xi \rangle, \tag{3.1}$$

where $\xi \in \mathscr{H}_\varphi$ is the vector for which there exists a closable operator $\rho(\xi)$, densely defined in $\mathscr{H}_\varphi$ by the formula

$$\rho(\xi) \mid \Lambda_\varphi a \rangle = \pi_\varphi(a)\xi, \qquad \forall a \in \mathscr{A}. \tag{3.2}$$

Again, there exists a unique vector ξ for which $\overline{\rho(\xi)}$ is positive and selfadjoint and an operator $P = \rho(\xi)^+\overline{\rho(\xi)}$ is called the density operator of the state σ

with respect to φ. In the case of $\mathscr{A}$ being a von Neumann algebra, φ —a faithful normal semi-finite weight on $\mathscr{A}$ and σ— a normal state on $\mathscr{A}$, σ differentiable with respect to φ means that σ is almost majorised by φ ([*4*]).

Analogously to [*4*] we define the entropy $\mathscr{S}^{\sigma/\varphi}$ of a state σ differentiable with respect to a weight φ by the formula

$$\mathscr{S}^{\sigma/\varphi} = - \lim_{\delta \downarrow 0} \langle \xi \mid \ln (\mathrm{PE}_\delta)\xi \rangle \tag{3.3}$$

whenever this limit exists, $\mathrm{E}_\delta = \mathrm{E}([\delta, \delta^{-1}])$, where $\mathrm{E}(d\lambda)$ stands for the spectral measure of P.

As in the previous section one can verify that (3.3) is a generalization of entropy.

a) Let $\varphi(\mathrm{A}^+\mathrm{A}) = \mathrm{Tr}\,(\mathrm{A}^+\mathrm{A})$ for all $\mathrm{A} \in \mathrm{B}(\mathscr{H})$. Assuming that a state σ is differentiable with respect to Tr (.) we can find that $\rho(\mathrm{D})$ is positive and selfadjoint for $\mathrm{D}^\# = \mathrm{D} = \Sigma^{1/2}$, $\mathrm{P} = [\rho(\Sigma^{1/2})]^2$ and consequently

$$\mathscr{S}^{\sigma/\varphi} = - \mathrm{Tr}\,(\Sigma \ln \Sigma)\,.$$

b) Let $\varphi(\bar{g}g) = \int \bar{g}g d\varphi$, $\forall g \in \mathscr{F}(\Omega, \mathscr{B})$, where φ is positive measure on $(\Omega, \mathscr{B})$. Assuming σ differentiable with respect to φ we obtain $\rho(\xi) = \left(\frac{d\sigma}{d\varphi}\right)^{1/2}$, $\mathrm{P} = \frac{d\sigma}{d\varphi} \equiv f$. Then from (3.3) we obtain

$$\mathscr{S}^{\sigma/\varphi} = - \int \frac{d\sigma}{d\varphi} \ln \frac{d\sigma}{d\varphi}\, d\varphi\,.$$

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