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On the supergroup $SU(m | n)$ and its superfield representations

by

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ABSTRACT. — The representations of the supergroup $SU(m | n)$ are studied on the basis of the generalized Gell-Mann's matrices and identities involving d - and f -tensors occurring in their commutation relations. The superfield representation in the space of functions of an anticommuting $m.n$ -component parameter is also considered.

1. INTRODUCTION

There have been recently several attempts to consider the unification model of strong, weak and electromagnetic interactions in the context of the gauge theory of the supergroup $SU(m | n)$ [1]-[7]. This formulation has many attractive features—it allows to obtain naturally the reasonable bare Weinberg angle, it also appears to lead naturally to a spectrum of particles in the theory, in which the Higgs mesons are various components of the gauge fields along with the usual vector mesons.

The aim of this paper is to discuss the properties of the representations of the supergroup $SU(m | n)$. In section 2 we study some general properties of the graded $SU(m | n)$ algebra, especially those of the generalized Gell-Mann's matrices realizing its basic matrix representation. Section 3 is devoted to spinor and tensor representations. Here too, some rules are derived for constructing scalar and vector representations from the product of two representations, and the eigenvalues of second-order Casimir

operators are found for some simple representations. In section 4 we consider the superfield representation which is realized in the space of functions of an anticommuting m, n -component parameter.

2. GENERALIZED GELL-MANN'S MATRICES, f - AND d -TENSORS

The graded algebra $SU(m|n)$ consists of the generators T_{AB}^B ,

$$A, B = 1, 2, \dots, m + n$$

satisfying the commutation and anticommutation relations [7]:

$$[T_A^B, T_C^D]_{-(AB, CD)} = \delta_A^D T_C^B - (AB, CD) \delta_C^B T_A^D \quad (2.1)$$

where the following notations are used:

$$\begin{aligned} (AB \dots, CD \dots) &\equiv (-1)^{((A)+(B)+\dots)((C)+(D)+\dots)} \\ (A), (B), \dots &= \begin{cases} 0 & \text{for even indices } i = 1, 2, \dots, m \\ 1 & \text{for odd indices } m + \alpha, \alpha = 1, 2, \dots, n \end{cases} \end{aligned} \quad (2.2)$$

Due to the relation

$$\sum_{A=1}^{m+n} T_A^A = 0 \quad (2.3)$$

the number of independent generators is $(m+n)^2 - 1$.

It is more convenient to use instead of T_{AB}^B the following generators:

$$\begin{aligned} F_a &\equiv \frac{1}{2} \sum_{i,j=1}^m (\lambda_a^{(m)})_i^j T_j^i \\ G_p &\equiv \frac{1}{2} \sum_{\alpha,\beta=1}^n (\lambda_p^{(n)})_\alpha^\beta T_{m+\alpha}^{m+\beta} \\ H &\equiv \frac{1}{2} \sum_{i=1}^m T_i^i = -\frac{1}{2} \sum_{\alpha=1}^n T_{m+\alpha}^{m+\alpha} \\ S_i^\alpha &\equiv \frac{1}{2} T_i^{m+\alpha} \\ R_\alpha^i &\equiv \frac{1}{2} T_{m+\alpha}^i \end{aligned} \quad (2.4)$$

where $\lambda_a^{(m)}$ and $\lambda_p^{(n)}$ are Gell-Mann's matrices for $SU(m)$ and $SU(n)$ groups, $a = 1, 2, \dots, m^2 - 1$; $p = 1, 2, \dots, n^2 - 1$.

In terms of these new generators the relations (2.1) read:

$$\begin{aligned}
 [F_a, F_b] &= if_{abc} F_c, & a, b, c &= 1, 2, \dots, m^2 - 1 \\
 [G_p, G_q] &= if_{pqr} G_r, & p, q, r &= 1, 2, \dots, n^2 - 1 \\
 [F_a, G_p] &= 0, & [H, F_a] &= 0, & [H, G_p] &= 0 \\
 [F_a, S_i^\alpha] &= -\frac{1}{2} (\lambda_a^{(m)})_i^\alpha S_j^\alpha \\
 [F_a, R_\alpha^i] &= \frac{1}{2} R_\alpha^K (\lambda_a^{(m)})_K^i \\
 [G_p, S_i^\alpha] &= \frac{1}{2} S_i^\beta (\lambda_p^{(n)})_\beta^\alpha \\
 [G_p, R_\alpha^i] &= -\frac{1}{2} (\lambda_p^{(n)})_\alpha^\beta R_\beta^i \\
 [H, S_i^\alpha] &= -\frac{1}{2} S_i^\alpha, & [H, R_\alpha^i] &= \frac{1}{2} R_\alpha^i \\
 \{S_i^\alpha, S_j^\beta\} &= 0, & \{R_\alpha^i, R_\beta^j\} &= 0 \\
 \{S_i^\alpha, R_\beta^j\} &= \frac{1}{4} \delta_\beta^\alpha (\lambda_a^{(m)})_i^\alpha F_a + \frac{1}{4} \delta_i^\alpha (\lambda_p^{(n)})_\beta^\alpha G_p - \frac{1}{2} \delta_\beta^\alpha \delta_i^\alpha \frac{m-n}{mn} H
 \end{aligned} \tag{2.5}$$

Here f_{abc} and f_{pqr} are the structure constants of $SU(m)$ and $SU(n)$. We see that the graded group $SU(m|n)$ contains $SU(m) \times SU(n) \times U(1)$ as its subgroup.

It is easy to find $(m+n)^2 - 1$ matrices $\frac{\beta_1}{2}$ of rank $m+n$ satisfying the same commutation relations as those for $F_a, G_p, H, S_i^\alpha, R_\alpha^i$ and therefore realizing the basic matrix representation of the algebra. They are:

$$\begin{aligned}
 \beta_{a^{(m)}} &\equiv M_a = \left(\begin{array}{c|c} \lambda_a^{(m)} & 0 \\ \hline 0 & 0 \end{array} \right) \\
 \beta_{p^{(n)}} &\equiv N_p = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \lambda_p^{(n)} \end{array} \right) \\
 \beta_h &\equiv \varphi = \frac{1}{n-m} \left(\begin{array}{c|c} n \cdots n & 0 \\ \hline 0 & m \cdots m \end{array} \right) \\
 \beta_{\binom{\alpha}{i}} &\equiv \zeta_i^\alpha, & (\zeta_i^\alpha)_A^B &= \delta_A^{m+\alpha} \delta_i^B \\
 \beta_{\binom{i}{\alpha}} &\equiv \eta_\alpha^i, & (\eta_\alpha^i)_A^B &= \delta_A^i \delta_{m+\alpha}^B
 \end{aligned} \tag{2.6}$$

The matrices β_I will be referred to as generalized $SU(m|n)$ Gell-Mann's matrices.

Let us denote the generator associated to the matrix β_I by $\mathcal{F}_I$, so that

$$\mathcal{F}_{a^{(m)}} \equiv F_a, \quad \mathcal{F}_{p^{(n)}} \equiv G_p, \quad \mathcal{F}_h \equiv H, \quad \mathcal{F}_{\binom{\alpha}{i}} \equiv S_i^\alpha, \quad \mathcal{F}_{\binom{i}{\alpha}} \equiv R_\alpha^i$$

We then have:

$$\begin{aligned} [\mathcal{F}_I, \mathcal{F}_J]_{-(I,J)} &= if_{IJK} \mathcal{F}_K \\ \left[\frac{\beta_I}{2}, \frac{\beta_J}{2} \right]_{-(I,J)} &= if_{IJK} \frac{\beta_K}{2} \end{aligned} \quad (2.7)$$

Here (I, J) is the same notation as in (2.2) with $(I) = 0$ for $I = a^{(m)}, p^{(n)}, h$ and $(I) = 1$ for $I = \binom{\alpha}{i}, \binom{i}{\alpha}$; f_{IJK} — the structure constants having the symmetry property followed from definition

$$f_{IJK} = - (I, J) f_{JKI} \quad (2.8)$$

their non-vanishing values are:

$$\begin{aligned} f_{a^{(m)}b^{(m)}c^{(m)}} &= f_{abc}, & f_{p^{(n)}q^{(n)}r^{(n)}} &= f_{pqr} \\ f_{a^{(m)}\binom{\alpha}{i}\binom{\alpha}{j}} &= \frac{i}{2} (\lambda_a^{(m)})_i^j, & f_{a^{(m)}\binom{i}{\alpha}\binom{j}{\alpha}} &= -\frac{i}{2} (\lambda_a^{(m)})_j^i \\ f_{p^{(n)}\binom{\alpha}{i}\binom{\beta}{i}} &= -\frac{i}{2} (\lambda_p^{(n)})_\beta^\alpha, & f_{p^{(n)}\binom{i}{\alpha}\binom{i}{\beta}} &= \frac{i}{2} (\lambda_p^{(n)})_\alpha^\beta \\ f_{\binom{\alpha}{i}\binom{j}{\alpha}a^{(m)}} &= -\frac{i}{4} (\lambda_a^{(m)})_i^j, & f_{\binom{\alpha}{i}\binom{i}{\beta}p^{(n)}} &= -\frac{i}{4} (\lambda_p^{(n)})_\beta^\alpha \\ f_{\binom{\alpha}{i}\binom{i}{\alpha}h} &= \frac{i}{2} \frac{m-n}{mn} \\ f_{\binom{\alpha}{i}h\binom{\alpha}{i}} &= -\frac{i}{2}, & f_{\binom{i}{\alpha}h\binom{i}{\alpha}} &= \frac{i}{2} \end{aligned} \quad (2.9)$$

In terms of β_I and $\mathcal{F}_I$ the equations (2.4) can be rewritten in a compact form:

$$\mathcal{F}_I = \frac{1}{2} \text{Tr } \beta_I T \equiv \frac{1}{2} \sum_{A,B} (\beta_I)_A^B T_B^A \quad (2.10)$$

Note some simple properties of the generalized Gell-Mann's matrices.

a) They are graded-traceless:

$$S \text{ Tr } \beta_I \equiv \sum_A [A] (\beta_I)_A^A = 0, \quad [A] \equiv (-1)^{(A)} \quad (2.11)$$

b) They satisfy the orthogonal relations:

$$S \operatorname{Tr} \beta_I \beta_J = \eta(I) \delta_{\tilde{I}J} \quad (2.12 \ a)$$

$$\operatorname{Tr} \beta_I \beta_J = \sigma(I) \delta_{\tilde{I}J} \quad (2.12 \ b)$$

where $\tilde{J} \equiv J$ for even indices

$$\begin{aligned} \overline{\begin{pmatrix} i \\ \alpha \end{pmatrix}} &\equiv \begin{pmatrix} \alpha \\ i \end{pmatrix}, \quad \overline{\begin{pmatrix} \alpha \\ i \end{pmatrix}} \equiv \begin{pmatrix} i \\ \alpha \end{pmatrix} \\ \eta(a^{(m)}) &= 2, \quad \eta(p^{(n)}) = -2, \quad \eta\left(\begin{pmatrix} \alpha \\ i \end{pmatrix}\right) = -1, \quad \eta\left(\begin{pmatrix} i \\ \alpha \end{pmatrix}\right) = 1, \quad \eta(h) = \frac{mn}{n-m} \quad (2.13) \\ \sigma(I) &= \begin{cases} |\eta(I)|, & I \neq h \\ \frac{mn(m+n)}{(m-n)^2}, & I = h \end{cases} \end{aligned}$$

As a consequence of (2.12) and (2.13) we have

$$\begin{aligned} \eta(I) &= [I] \eta(\tilde{I}) = S \operatorname{Tr} \beta_I \beta_{\tilde{I}} = [I] S \operatorname{Tr} \beta_{\tilde{I}} \beta_I \\ \sigma(I) &= \operatorname{Tr} \beta_I \beta_{\tilde{I}} \end{aligned} \quad (2.14)$$

From (2.6) and (2.13) it is evident that

$$\beta_I^+ = \beta_{\tilde{I}} \quad (2.15)$$

c) They satisfy the completeness identity:

$$\sum_I \frac{1}{\eta(I)} (\beta_{\tilde{I}}^B)_A (\beta_I)^D_C = [B] \delta_A^D \delta_C^B + \frac{1}{n-m} \delta_A^B \delta_C^D \quad (2.16)$$

d) They obey the multiplication law:

$$\beta_I \beta_J = (i f_{IJK} + d_{IJK}) \beta_K + \frac{\eta(I)}{m-n} \delta_{IJ} \quad (2.17)$$

where d_{IJK} are the constants appearing in the commutation relations

$$[\beta_I, \beta_J]_{+(I,J)} = 2d_{IJK} \beta_K + \frac{2\eta(I)}{m-n} \delta_{\tilde{I}J} \quad (2.18)$$

They have the symmetry property

$$d_{IJK} = (I, J) d_{JIK}, \quad (2.19)$$

their non-vanishing values are:

$$\begin{aligned}
 d_{a^{(m)}b^{(m)}c^{(m)}} &= d_{abc}, & d_{p^{(n)}q^{(n)}r^{(n)}} &= d_{pqr} \\
 d_{a^{(m)}a^{(m)}h} &= \frac{2}{m}, & d_{p^{(n)}p^{(n)}h} &= -\frac{2}{n} \\
 d_{a^{(m)}\binom{\alpha}{i}\binom{\alpha}{j}} &= \frac{1}{2}(\lambda_a^{(m)})_i^j, & d_{a^{(m)}\binom{i}{\alpha}\binom{j}{\alpha}} &= \frac{1}{2}(\lambda_a^{(m)})_j^i \\
 d_{a^{(m)}ha^{(m)}} &= \frac{n}{n-m}, & d_{p^{(n)}hp^{(n)}} &= \frac{m}{n-m} \\
 d_{\binom{\alpha}{i}h\binom{\alpha}{i}} &= \frac{1}{2}\frac{n+m}{n-m}, & d_{\binom{i}{\alpha}h\binom{i}{\alpha}} &= \frac{1}{2}\frac{n+m}{n-m} \\
 d_{hhh} &= \frac{n+m}{n-m} \\
 d_{p^{(n)}\binom{\alpha}{i}\binom{\beta}{i}} &= \frac{1}{2}(\lambda_p^{(n)})_{\beta}^{\alpha}, & d_{p^{(n)}\binom{i}{\alpha}\binom{i}{\beta}} &= \frac{1}{2}(\lambda_p^{(n)})_{\alpha}^{\beta} \\
 d_{\binom{\alpha}{i}\binom{j}{\alpha}a^{(m)}} &= -\frac{1}{4}(\lambda_a^{(m)})_i^j, & d_{\binom{\alpha}{i}\binom{i}{\beta}p^{(n)}} &= \frac{1}{4}(\lambda_p^{(n)})_{\beta}^{\alpha} \\
 d_{\binom{\alpha}{i}\binom{i}{\alpha}h} &= -\frac{1}{2}\frac{m+n}{mn}
 \end{aligned} \tag{2.20}$$

where d_{abc} and d_{pqr} are the usual d -tensors of $SU(m)$ and $SU(n)$.

It is useful for further purposes to quote here some simple identities involving f_{IJK} and d_{IJK} :

$$f_{IJK} = - (I, J) \frac{\eta(J)}{\eta(K)} f_{I\bar{K}\bar{J}} = - (J, K) \frac{\eta(I)}{\eta(K)} f_{\bar{K}\bar{J}\bar{I}} \tag{2.21}$$

$$d_{IJK} = (I, J) \frac{\eta(J)}{\eta(K)} d_{I\bar{K}\bar{J}} = (J, K) \frac{\eta(I)}{\eta(K)} d_{\bar{K}\bar{J}\bar{I}} \tag{2.22}$$

$$(I, K) f_{ILM} f_{JKL} + (J, I) f_{JLM} f_{KIL} + (K, J) f_{KLM} f_{IJL} = 0 \tag{2.23}$$

$$(I, K) f_{ILM} d_{JKL} + (J, I) f_{JLM} d_{KIL} + (K, J) f_{KLM} d_{IJL} = 0 \tag{2.24}$$

The validity of (2.21) and (2.22) can be checked immediately from the explicit expressions (2.9) and (2.20) for f and d . The equations (2.23) and (2.24) are consequences of the following identity for any three graded operators $\mathcal{M}_A, \mathcal{M}_B, \mathcal{M}_C$:

$$\begin{aligned}
 (A, C) [\mathcal{M}_A, [\mathcal{M}_B, \mathcal{M}_C]_{\pm(B,C)}]_{-(A,BC)} + (B, A) [\mathcal{M}_B, [\mathcal{M}_C, \mathcal{M}_A]_{\pm(C,A)}]_{-(B,CA)} \\
 + (C, B) [\mathcal{M}_C, [\mathcal{M}_A, \mathcal{M}_B]_{\pm(A,B)}]_{-(C,AB)} = 0
 \end{aligned} \tag{2.25}$$

when applied to the matrices $\beta_I, \beta_J, \beta_K$.

3. SPINOR AND TENSOR REPRESENTATIONS

The spinor representation of $SU(m|n)$ consists of $m+n$ operators ψ_A transforming in the following way:

$$[\mathcal{F}_b, \psi_A]_{-(I,A)} = -\frac{1}{2}(I, A)(\beta_I \psi)_A \quad (3.1)$$

Its conjugate representation $\bar{\psi}^A$ is defined by

$$[\mathcal{F}_b, \bar{\psi}^A]_{-(I,A)} = \frac{1}{2}(\bar{\psi} \beta_I)^A \quad (3.2)$$

The formulae (3.1) and (3.2) can be easily generalized for the spinors of arbitrary rank $\psi_{A_1 \dots A_r}$, $\bar{\psi}^{A_1 \dots A_r}$ and mixed spinors $\phi_{A_1 \dots A_r}^{B_1 \dots B_s}$. We have:

$$[\mathcal{F}_b, \psi_{A_1 \dots A_r}]_{-(I, A_1 \dots A_r)} = -\frac{1}{2} \sum_{K=1}^r (I, A_1 \dots A_K)(\beta_I)_{A_K}^B \psi_{A_1 \dots B \dots A_r} \quad (3.3)$$

$$[\mathcal{F}_b, \bar{\psi}^{A_1 \dots A_r}]_{-(I, A_1 \dots A_r)} = \frac{1}{2} \sum_{K=1}^r (I, A_1 \dots A_{K-1})(\beta_I)_{A_K}^{A_K} \bar{\psi}^{A_1 \dots B \dots A_r} \quad (3.4)$$

$$\begin{aligned} [\mathcal{F}_b, \phi_{A_1 \dots A_r}^{B_1 \dots B_s}]_{-(I, A_1 \dots A_r B_1 \dots B_s)} &= \frac{1}{2} \sum_{K=1}^s (I, B_1 \dots B_{K-1})(\beta_I)_{B_K}^{B_K} \phi_{A_1 \dots A_r}^{B_1 \dots C \dots B_s} \\ &\quad - \frac{1}{2} \sum_{K=1}^r (I, A_1 \dots A_K B_1 \dots B_s)(\beta_I)_{A_K}^C \phi_{A_1 \dots C \dots A_r}^{B_1 \dots B_s} \end{aligned} \quad (3.5)$$

The « vector » representation consists of $(m+n)^2 - 1$ operators φ_I transforming in a similar way as $\mathcal{F}_b$, namely:

$$[\mathcal{F}_b, \varphi_I]_{-(I,J)} = i f_{IJK} \varphi_K \quad (3.6)$$

In some cases it is more convenient to use instead of φ_I the traceless mixed spinor φ_A^B defined by:

$$\varphi_A^B \equiv \sum_I \frac{\sqrt{2}}{\eta(I)} [A](\beta_I)_A^B \varphi_I = \sum_I \frac{\sqrt{2}}{\eta(I)} [B](\beta_I)_A^B \varphi_I \quad (3.7)$$

The reciprocal formula of (3.7) is:

$$\varphi_I = \frac{1}{\sqrt{2}} \text{Tr } \beta_I \varphi \quad (3.8)$$

The generalization of the formula (3.6) to higher tensor representation is straightforward. We have:

$$[\mathcal{F}]_1, \varphi_{[1][2\dots p]}]_{-(1)[1][2\dots p]} = i \sum_{l=1}^p (1, J_1 \dots J_{l-1}) f_{IJ_l K} \varphi_{J_1 \dots K \dots J_p} \quad (3.9)$$

From (3.1)-(3.4) it follows that if ψ_A and χ_A are spinor operators then

$$\omega \equiv \bar{\psi}^A \chi_A \quad (3.10 a)$$

is $SU(m|n)$ invariant, and

$$\varphi_I \equiv \bar{\psi}^A (\beta_I)_A^B \chi_B \quad (3.11 a)$$

is a « vector » operator transforming according to (3.6). Similarly, if ψ_{AB} and χ_{CD} are second-rank spinor operators, then

$$\begin{aligned} & \bar{\psi}^{AB} \chi_{BA} \\ & (A, B) \bar{\psi}^{AB} \chi_{AB} \end{aligned} \quad (3.10 b)$$

are invariant, and

$$\begin{aligned} & \bar{\psi}^{AB} (\beta_I)_A^C \chi_{BC} \\ & (A, B) \bar{\psi}^{BA} (\beta_I)_A^C \chi_{BC} \\ & (B, C) \bar{\psi}^{AB} (\beta_I)_A^C \chi_{CB} \\ & (AC, B) \bar{\psi}^{BA} (\beta_I)_A^C \chi_{CB} \end{aligned} \quad (3.11 b)$$

are vector operators.

These rules can be easily generalized for higher-rank spinors.

From (3.6), using the identities (2.21)-(2.24) we can prove that if φ_I and ϕ_I are vector operators, then

$$\varphi \equiv \sum_I \frac{2}{\eta(I)} \phi_I \varphi_I \quad (3.12)$$

is $SU(m|n)$ invariant, and

$$\psi_K^{(F)} \equiv \sum_{I,J} (I, J) \frac{2\eta(\tilde{K})}{\eta(I)\eta(J)} f_{IJ\tilde{K}} \phi_I \varphi_J \quad (3.13)$$

$$\psi_K^{(D)} \equiv \sum_{I,J} (I, J) \frac{2\eta(\tilde{K})}{\eta(I)\eta(J)} d_{IJ\tilde{K}} \phi_I \varphi_J \quad (3.14)$$

are vector operators.

The expression (3.12) can be rewritten in terms of matrices ϕ_A^B and φ_A^B defined by (3.7) as follows:

$$\varphi = \sum_I \frac{2}{\eta(I)} \phi_I \varphi_I = S \operatorname{Tr} \phi \varphi = S \operatorname{Tr} \varphi \phi \quad (3.15)$$

In writing the last equation we have used the fact that the matrices ϕ and φ have graded elements satisfying the commutation relation:

$$[\phi_A^B, \varphi_C^D]_{-(AB, CD)} = 0 \quad (3.16)$$

More general, it is easy to prove that the supertrace of any product of the matrices with graded elements has the cyclic property:

$$S \operatorname{Tr} \phi \varphi \dots \chi = S \operatorname{Tr} \varphi \dots \chi \phi = \dots \quad (3.17)$$

In a similar way, with the help of (2.7), (2.16), (2.18), (2.21) and (2.22) the expressions (3.13) and (3.14) can be rewritten as follows:

$$\psi_K^{(F)} = \frac{i}{2} S \operatorname{Tr} (\phi \beta_K \varphi - \varphi \beta_K \phi) \quad (3.18)$$

$$\psi_K^{(D)} = \frac{1}{2} S \operatorname{Tr} (\phi \beta_K \varphi + \varphi \beta_K \phi) \quad (3.19)$$

According to (3.12) the second order Casimir operator is of the form:

$$\begin{aligned} C &\equiv \sum_I \frac{2}{\eta(I)} \mathcal{F}_I \bar{\mathcal{F}}_I \\ &= \sum_{a=1}^m F_a^2 - \sum_{p=1}^n G_p^2 + 2S_i^a R_a^i - 2R_a^i S_i^a - \frac{2(m-n)}{mn} H^2 \end{aligned} \quad (3.20)$$

From the transformation laws (3.1)-(3.6), using the above quoted properties of the matrices β_i we can find the eigenvalues of C for each irreducible representation. Thus, we have

$$C = \frac{(m-n)^2 - 1}{2(m-n)}$$

for the spinor representation ψ_A ,

$$C = m - n$$

for the vector representation ϕ_b ,

$$C = \frac{r(r+m-n)(m-n-1)}{2(m-n)}$$

for the graded totally symmetrized r -rank spinor

$$\psi_{C_1 C_2 \dots C_r}^{(+)} = (C_1, C_2) \psi_{C_2 C_1 \dots C_r}^{(+)} = \dots,$$

etc.

4. SUPERFIELD REPRESENTATION

Consider the space of functions of an anticommuting m, n -component parameter θ_α^i , $\alpha = 1, 2, \dots, m$; $i = 1, 2, \dots, n$. In this space the generators of the $SU(m|n)$ algebra can be realized as follows:

$$\begin{aligned} F_a &= \frac{1}{2} (\lambda_a^{(m)})_i^j \theta_\alpha^i \frac{\partial}{\partial \theta_\alpha^j} \\ G_p &= -\frac{1}{2} (\lambda_p^{(n)})_\alpha^\beta \theta_\beta^i \frac{\partial}{\partial \theta_\alpha^i} \\ H &= \frac{1}{2} \theta_\alpha^i \frac{\partial}{\partial \theta_\alpha^i} \\ S_i^z &= -\frac{i}{2} \frac{\partial}{\partial \theta_\alpha^i} \\ R_\alpha^i &= -\frac{i}{2} \theta_\beta^i \theta_\alpha^j \frac{\partial}{\partial \theta_\beta^j} \end{aligned} \quad (4.1)$$

Consider now the transformation laws of the superfield operators $\phi(\theta)$ defined in the space of the parameters θ_α^i . From (4.1) we note that the point $\theta_\alpha^i = 0$ remains unchanged under F_a , G_p , H and R_α^i transformations. So, these transformations form the little group of the $SU(m|n)$ group. According to any given representation of this little group we can define the entire action of the generators of the $SU(m|n)$ group on the field operators. This is done by the method of the theory of induced representations [8][10] in the following manner.

Let

$$\begin{aligned} [F_a, \phi_{\mathcal{A}}(0)]_- &= - (f_a^{(\phi)} \phi(0))_{\mathcal{A}} \\ [G_p, \phi_{\mathcal{A}}(0)]_- &= - (g_p^{(\phi)} \phi(0))_{\mathcal{A}} \\ [H, \phi_{\mathcal{A}}(0)]_- &= - (h^{(\phi)} \phi(0))_{\mathcal{A}} \\ [R_\alpha^i, \phi_{\mathcal{A}}(0)]_{-[\mathcal{A}]} &= - (r^{(\phi)i}{}_\alpha \phi(0))_{\mathcal{A}} \end{aligned} \quad (4.2)$$

where f_a , g_p , h , r_α^i are some matrices obeying the analogous commutation relations as those for F_a , G_p , H , R_α^i . We are to find the commutation rule $[\mathcal{F}_i, \phi_{\mathcal{A}}(\theta)]_{-(\mathbf{1}, \mathcal{A})}$. Choose the basis in index space $\mathcal{A}$ in such a way that the operators S_i^z do not act on the indices, i, e .

$$[S_i^z, \phi_{\mathcal{A}}(\theta)]_{-[\mathcal{A}]} = \frac{i}{2} \frac{\partial}{\partial \theta_\alpha^i} \phi_{\mathcal{A}}(\theta) \quad (4.3)$$

and, therefore,

$$\phi_{\mathcal{A}} \left(\theta_\gamma^K + \frac{1}{2} \eta_\gamma^K \right) = e^{-i\eta_\gamma^i S_i^z} \phi_{\mathcal{A}}(\theta) e^{i\eta_\gamma^i S_i^z} \quad (4.4)$$

With the help of (4.4) we can write

$$[\mathcal{F}_1, \phi_{\mathcal{A}}(\theta)]_{-(1, \mathcal{A})} = e^{-2i\theta_{\alpha}^i S_i^z} [\mathcal{F}'_1, \phi_{\mathcal{A}}(0)]_{-(1, \mathcal{A})} e^{2i\theta_{\alpha}^i S_i^z} \quad (4.5)$$

where we denote

$$\mathcal{F}'_1 \equiv e^{2i\theta_{\alpha}^i S_i^z} \mathcal{F}_1 e^{-2i\theta_{\alpha}^i S_i^z} \quad (4.6)$$

Using the commutation relations (2.5) we find:

$$\begin{aligned} F'_a &= F_a + i\theta_{\alpha}^i (\lambda_a^{(m)})_i S_j^z \\ G'_p &= G_p + iS_i^z (\lambda_p^{(n)})_{\alpha}^{\beta} \theta_{\beta}^i \\ H' &= H + i\theta_{\alpha}^i S_i^z \\ R_{\alpha}^{i'} &= R_{\alpha}^i + \frac{i}{2} \left[\theta_{\alpha}^j (\lambda_a^{(m)})_j^i F_a + \theta_{\beta}^j (\lambda_p^{(n)})_{\alpha}^{\beta} G_p - 2\theta_{\alpha}^i \frac{m-n}{mn} H \right] - \theta_{\alpha}^i \theta_{\beta}^j S_j^{\beta} \end{aligned} \quad (4.7)$$

By inserting (4.7) into (4.5) and taking into account (4.2) we get, after some manipulations:

$$\begin{aligned} [F_a, \phi_{\mathcal{A}}(\theta)]_- &= - \left\{ (f_a^{(\phi)} \phi(\theta))_{\mathcal{A}} + \frac{1}{2} (\lambda_a^{(m)})_i^j \theta_{\alpha}^i \frac{\partial \phi_{\mathcal{A}}(\theta)}{\partial \theta_{\alpha}^j} \right\} \\ [G_p, \phi_{\mathcal{A}}(\theta)]_- &= - \left\{ (g_p^{(\phi)} \phi(\theta))_{\mathcal{A}} - \frac{1}{2} (\lambda_p^{(n)})_{\alpha}^{\beta} \theta_{\beta}^i \frac{\partial \phi_{\mathcal{A}}(\theta)}{\partial \theta_{\alpha}^i} \right\} \\ [H, \phi_{\mathcal{A}}(\theta)]_- &= - \left\{ (h^{(\phi)} \phi(\theta))_{\mathcal{A}} + \frac{1}{2} \theta_{\alpha}^i \frac{\partial \phi_{\mathcal{A}}(\theta)}{\partial \theta_{\alpha}^i} \right\} \\ [R_{\alpha}^i, \phi_{\mathcal{A}}(\theta)]_{-[\mathcal{A}]} &= - \left\{ (r_{\alpha}^{(\phi)i} \phi(\theta))_{\mathcal{A}} + i\theta_{\alpha}^j \left(\frac{\lambda_a^{(m)}}{2} \right)_j^i (f_a^{(\phi)} \phi(\theta))_{\mathcal{A}} \right. \\ &\quad \left. + i\theta_{\beta}^i \left(\frac{\lambda_p^{(n)}}{2} \right)_{\alpha}^{\beta} (g_p^{(\phi)} \phi(\theta))_{\mathcal{A}} - i\theta_{\alpha}^i \frac{m-n}{mn} (h^{(\phi)} \phi(\theta))_{\mathcal{A}} + \frac{i}{2} \theta_{\alpha}^j \theta_{\beta}^i \frac{\partial \phi_{\mathcal{A}}(\theta)}{\partial \theta_{\beta}^j} \right\} \end{aligned} \quad (4.8)$$

Let us consider the simplest case when

$$f_a = 0, \quad g_p = 0, \quad r_{\alpha}^i = 0$$

and h is a number. Then the formulae (4.8) become:

$$\begin{aligned} [F_a, \phi(\theta)] &= -\frac{1}{2} (\lambda_a^{(m)})_i^j \theta_{\alpha}^i \frac{\partial \phi(\theta)}{\partial \theta_{\alpha}^j} \\ [G_p, \phi(\theta)] &= \frac{1}{2} (\lambda_p^{(n)})_{\alpha}^{\beta} \theta_{\beta}^i \frac{\partial \phi(\theta)}{\partial \theta_{\alpha}^i} \\ [H, \phi(\theta)] &= -h^{(\phi)} \phi(\theta) - \frac{1}{2} \theta_{\alpha}^i \frac{\partial \phi(\theta)}{\partial \theta_{\alpha}^i} \\ [R_{\alpha}^i, \phi(\theta)] &= i \frac{m-n}{mn} h \theta_{\alpha}^i \phi(\theta) - \frac{i}{2} \theta_{\alpha}^j \theta_{\beta}^i \frac{\partial \phi(\theta)}{\partial \theta_{\beta}^j} \end{aligned} \quad (4.9)$$

The superfield $\phi(\theta)$ can be expanded in a polynomial series of order 2^{mn} :

$$\phi(\theta) = \varphi + \theta_{\alpha}^i \varphi_i^{\alpha} + \theta_{\alpha_1}^{i_1} \theta_{\alpha_2}^{i_2} \varphi_{i_1 i_2}^{\alpha_1 \alpha_2} + \dots + \theta_{\alpha_1}^{i_1} \dots \theta_{\alpha_{mn}}^{i_{mn}} \varphi_{i_1 \dots i_{mn}}^{\alpha_1 \dots \alpha_{mn}} \quad (4.10)$$

Here the tensors $\varphi_{i_1 i_2 \dots i_K}^{\alpha_1 \alpha_2 \dots \alpha_K}$ are totally antisymmetric in the pairs of indices $\binom{\alpha_l}{i_l}$. Their infinitesimal change can be easily found from (4.3) and (4.9):

$$\delta^{(S)} \varphi_{i_1 i_2 \dots i_K}^{\alpha_1 \alpha_2 \dots \alpha_K} = \frac{K+1}{2} \eta_\alpha^i \varphi_{i_1 \dots i_K i}^{\alpha_1 \dots \alpha_K \alpha} \quad (4.11)$$

$$\delta^{(R)} \varphi_{i_1 i_2 \dots i_K}^{\alpha_1 \alpha_2 \dots \alpha_K} = (-1)^K \left\{ \frac{m-n}{mn} h \cdot [\sigma_{i_1}^{\alpha_1} \varphi_{i_2 \dots i_K}^{\alpha_2 \dots \alpha_K}] - \frac{K-1}{2} [\sigma_{i_2}^{\alpha_2} \varphi_{i_1 i_3 \dots i_K}^{\alpha_1 \alpha_3 \dots \alpha_K}] \right\} \quad (4.12)$$

$$\delta^{(H)} \varphi_{i_1 i_2 \dots i_K}^{\alpha_1 \alpha_2 \dots \alpha_K} = i\varepsilon \left(h + \frac{K}{2} \right) \varphi_{i_1 i_2 \dots i_K}^{\alpha_1 \alpha_2 \dots \alpha_K} \quad (4.13)$$

$$\delta^{(F)} \varphi_{i_1 i_2 \dots i_K}^{\alpha_1 \alpha_2 \dots \alpha_K} = \frac{i\omega_a}{2} \sum_{l=1}^K (\lambda_a^{(m)})_{i_l}^j \varphi_{i_1 \dots j \dots i_K}^{\alpha_1 \dots \alpha_l \dots \alpha_K} \quad (4.14)$$

$$\delta^{(G)} \varphi_{i_1 i_2 \dots i_K}^{\alpha_1 \alpha_2 \dots \alpha_K} = -\frac{i\omega_p}{2} \sum_{l=1}^K (\lambda_p^{(n)})_{\beta}^{\alpha_l} \varphi_{i_1 \dots i_l \dots i_K}^{\alpha_1 \dots \beta \dots \alpha_K} \quad (4.15)$$

Where η , σ , ε , ω are infinitesimal parameters, the symbol [...] in the r. h. s. of (4.12) means the antisymmetrization over all the pairs of indices $\binom{\alpha_l}{i_l}$.

The equations (4.11), (4.14), (4.15) show in particular that the highest field component $\varphi_{i_1 i_2 \dots i_{mn}}^{\alpha_1 \alpha_2 \dots \alpha_{mn}}$ is invariant under S, F and G transformations. In order to see in what condition it is R-invariant we write (see (4.10)):

$$\varphi_{i_1 i_2 \dots i_{mn}}^{\alpha_1 \alpha_2 \dots \alpha_{mn}} = \frac{1}{(mn)!} \frac{\partial^{(mn)}}{\partial \theta_{\alpha_{mn}}^{i_{mn}} \dots \partial \theta_{\alpha_2}^{i_2} \partial \theta_{\alpha_1}^{i_1}} \phi(\theta)$$

and therefore (using the last equation of (4.9)):

$$\delta^{(R)} \varphi_{i_1 i_2 \dots i_{mn}}^{\alpha_1 \alpha_2 \dots \alpha_{mn}} = \frac{1}{(mn)} \cdot \frac{m-n}{mn} \cdot \left(h + \frac{mn}{2} \right) \cdot \frac{\partial^{(mn)}}{\partial \theta_{\alpha_{mn}}^{i_{mn}} \dots \partial \theta_{\alpha_2}^{i_2} \partial \theta_{\alpha_1}^{i_1}} (\bar{\eta}_i^\alpha \theta_\alpha^i \phi(\theta))$$

From here we see that $\varphi_{i_1 i_2 \dots i_{mn}}^{\alpha_1 \alpha_2 \dots \alpha_{mn}}$ is R-invariant if $h = -\frac{mn}{2}$. It is obvious from (4.13) that with this value of h this component is also H-invariant.

Finally, we note that if $\phi_1(\theta)$ and $\phi_2(\theta)$ are superfields transforming according to (4.9) with h_1 and h_2 then their product $\psi \equiv \phi_1(\theta)\phi_2(\theta)$ is also a superfield transforming in the same way with

$$h = h_1 + h_2.$$

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