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Section A :
Physique théorique

Complete Mellin representation and asymptotic behaviours of Feynman amplitudes

by

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ABSTRACT. — Any Feynman amplitude is defined by an integral representation of the Mellin-Barnes type. The integrand is a product of Γ-functions, with linear arguments given by the topology of the graph, and depends on the invariants and masses in a completely factorized form. The integration path is the set of the imaginary axes.

These properties allow an easy geometrical determination of any asymptotic behaviour, giving explicitly the corresponding asymptotic expansion. Moreover in the formalism, the integrand is unaffected by the renormalization which is expressed by simple translations of the integration path.

RÉSUMÉ. — Toute amplitude de Feynman est définie par une représentation intégrale du type Mellin-Barnes. L'intégrand est un produit de fonctions d'Euler Γ, avec des arguments linéaires donnés par la topologie du graphe; les invariants et les masses sont complètement factorisés dans l'intégrand. Le chemin d'intégration est l'ensemble des axes imaginaires.

Ces propriétés permettent une détermination géométrique aisée de n'importe quel comportement asymptotique, et donnent explicitement le développement correspondant. De plus, dans ce formalisme, l'intégrand n'est pas modifié par la renormalisation, qui s'exprime par de simples translations du chemin d'intégration.

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I. INTRODUCTION

In a previous paper [*1*], we have considered the asymptotic behaviour of Feynman amplitudes under scaling of any subset of invariants or squared masses. We generally proved the existence of an asymptotic expansion with powers and powers of logarithms, of the scaling parameter. Examples of physical applications, and references, are quoted in [*1*]. Let us briefly recall the main features of the asymptotic behaviours of Feynman amplitudes.

For some peculiar asymptotic limits, arguments based on power counting are sufficient to determine the expansion. A useful technique is the Mellin transformation respective to the scaling parameter: the Mellin transform is easily desingularized if its integrand has what we called the « FINE » property. But in many other asymptotic regimes (scaling of a partial set of momenta, on-mass-shell infrared problem, etc.), this happens to be wrong, and power counting may lead to erroneous results. In the Schwinger parametric representation, « FINE » integrands are desingularized in each Hepp sector by the usual $\alpha \rightarrow \beta$ change of variables. If the FINE property fails to be true, one must find another adequate change of variables. But an alternative is to restore FINE integrands by introducing a « Multiple Mellin » representation, as we did in [*1*].

In this paper, we take the extreme point of view to use only the Multiple Mellin technique, by splitting all the polynomials of the integrand into their monomials. Then no change of variables is needed: not only the α variables provide a trivial desingularization, but the α integrations can be explicitly performed, and we are left with the pure geometrical study of convex polyhedrons in the Mellin variables. We prove the same results as in [*1*] in a simpler way, and asymptotic expansions are computed in a much more compact form, without any division of the integral into the $l!$ Hepp sectors. Furthermore, in contrast with ref. [*1*], renormalization of ultraviolet divergencies may be realized in a very simple way. On the other hand, we think that the representation we obtain could be very suited to the study of other problems, such as the determination of Landau singularities, or the dimensional renormalization.

For the sake of simplicity, we restrict ourselves to spinless particles. In section II, the CM representation is proved for any convergent amplitude, resulting in a simple integral of a product of Γ-functions with linear arguments, times factorized powers of the invariants. This form is used in section III to the determination of the expansion corresponding to an arbitrary asymptotic regime.

Finally we study in section IV the ultraviolet divergent amplitudes. It is shown how renormalization may be performed by simple translations of the integration path. The detailed organisation of the renormalization programm, in the CM representation, will be explained in a later paper: the aim here is mainly to show that the CM representation is preserved, and so the method for the determination of asymptotic expansions.

II. COMPLETE MELLIN REPRESENTATION FOR CONVERGENT GRAPHS

In this work, we shall treat simultaneously the case of euclidean and minkowskian metrics by performing a Wick rotation on the four-momenta, in the form:

$$p(\varepsilon) = \left(e^{i\left(\frac{\varepsilon}{2}-\frac{\pi}{4}\right)}p_0,\ e^{i\left(-\frac{\varepsilon}{2}+\frac{\pi}{4}\right)}\vec{p}\right)$$

and by using complex internal masses:

$$m(\varepsilon) = e^{i\left(-\frac{\varepsilon}{2}+\frac{\pi}{4}\right)}m_0$$

Thus the propagator, written as $\frac{1}{p^2(\varepsilon)+m^2(\varepsilon)}$, with the euclidean metrics $(++++)$ for $p^2(\varepsilon)$, equals

$$\frac{1}{-i\cos\varepsilon(p_0^2-\vec{p}^{\,2}-m_0^2)+\sin\varepsilon(p_0^2+\vec{p}^{\,2}+m_0^2)}$$

which is $\frac{1}{p_0^2+\vec{p}^{\,2}+m_0^2}$ for $\varepsilon=\frac{\pi}{2}$ and behaves like $\frac{1}{p_0^2-\vec{p}^{\,2}-m_0^2+i\varepsilon(p_0^2+\vec{p}^{\,2}+m_0^2)}$ in the limit $\varepsilon \to 0_+$.

The real part of the invariants $p^2(\varepsilon)$, $m^2(\varepsilon)$ is positive for $0<\varepsilon<\pi$. We omit in the following the ε dependence and therefore use the euclidean notation. The distributions in the minkowskian case are recovered in the limit $\varepsilon \to 0_+$.

Given a convergent Feynman graph with l lines, non vanishing internal masses m_i, L independent loops, the corresponding amplitude is written in the Schwinger representation:

$$F(s_k, m_i^2) = \int_0^\infty \prod_{i=1}^{l} d\alpha_i e^{-\sum_i \alpha_i m_i^2}\, U(\alpha_i)^{-\frac{D}{2}} e^{-\frac{N(\alpha_i, s_k)}{U(\alpha_i)}} \tag{1}$$

where D is the space-time dimension (we come back to the case of some vanishing masses at the end of section III).

$$U = \sum_{j=1}^{J} U_j = \sum_j \prod_{i=1}^{l} \alpha_i^{u_{ij}}$$

j is an index for the distinct « one-trees » (connected subgraphs, without loop, linking all vertices of the graph).

$$u_{ij} = \begin{cases} 0 & \text{if the line } i \text{ belongs to the one-tree } j \\ 1 & \text{otherwise} \end{cases}$$

$$\sum_{i=1}^{l} u_{ij} = \mathrm{L}$$

for every j

$$\mathrm{N} = \sum_k \mathrm{N}_k = \sum_k s_k \prod_{i=1}^{l} \alpha_i^{n_{ik}}$$

k is an index for the distinct « two-trees » (subgraphs without loop, with two connected components, linking all vertices of the graph).

$$n_{ik} = \begin{cases} 0 & \text{if the line } i \text{ belongs to the two-tree } k \\ 1 & \text{otherwise} \end{cases}$$

$$\sum_{i=1}^{l} n_{ik} = \mathrm{L} + 1 \qquad \text{for every } k$$

s_k is the invariant built by squaring the sum of the external momenta over one connected component of the two-tree (any one of then equivalently, by momentum conservation). For different k's, the corresponding invariants s_k may actually coincide.

For obtaining what we call the complete Mellin (CM) representation of the amplitude, we first write an integral Mellin representation of each term in $e^{-\frac{\mathrm{N}}{\mathrm{U}}}$:

$$e^{-\frac{\mathrm{N}_k}{\mathrm{U}}} = \int_{\tau_k} \Gamma(-y_k)\left(\frac{\mathrm{N}_k}{\mathrm{U}}\right)^{y_k} \tag{2}$$

which is true for

$$\mathrm{Re}\, y_k = \tau_k < 0 \quad , \quad \mathrm{Re}\,\frac{\mathrm{N}_k}{\mathrm{U}} > 0.$$

$\int_{\tau_k}$ is a short notation for $\int_{-\infty}^{+\infty} \frac{d\,\mathrm{Im}\, y_k}{2i\pi}$, Re y_k being fixed at the value τ_k.

Then $\mathrm{U}^{-\frac{\mathrm{D}}{2}-\sum_k y_k}$ is similarly represented by

$$\Gamma\left(\sum_k y_k + \frac{\mathrm{D}}{2}\right)\mathrm{U}^{-\sum_k y_k - \frac{\mathrm{D}}{2}} = \int_\sigma \prod_{j=1}^{\mathrm{J}} \Gamma(-x_j)\mathrm{U}_j^{x_j} \tag{3}$$

with

$$\operatorname{Re} x_j = \sigma_j < 0 \quad , \quad \operatorname{Re}\left(\sum_k y_k + \frac{D}{2}\right) = \sum_k \tau_k + \frac{D}{2} > 0$$

$$\int_\sigma \quad \text{means} \quad \int_{-\infty \atop \Sigma x_j + \Sigma y_k = -\frac{D}{2}}^{+\infty} \prod_{j=1}^{J-1} \frac{d \operatorname{Im} x_j}{2i\pi}$$

Thus we get:

$$F(s_k, m_i^2) = \int_0^\infty \prod_i d\alpha_i e^{-\sum_i \alpha_i m_i^2} \int_{\sigma,\tau<0 \atop \Sigma x_j + \Sigma y_k = -\frac{D}{2}} \frac{\prod_j \Gamma(-x_j)}{\Gamma(-\Sigma x_j)} \prod_k s_k^{y_k}\Gamma(-y_k) \prod_i \alpha_i^{\phi_i - 1} \tag{4}$$

where

$$\phi_i = \sum_j u_{ij} x_j + \sum_k n_{ik} y_k + 1.$$

Finally we may interchange the Im x, Im y and α integrations by absolute convergence of

$$\int_0^\infty d\alpha_i e^{-\alpha_i m_i^2} \alpha_i^{\operatorname{Re}\phi_i - 1} \quad \text{and} \quad \int_{\sigma,\tau} \frac{\prod_j |\Gamma(-x_j)|}{|\Gamma(-\Sigma x_j)|} \prod_k |s_k^{y_k}| . |\Gamma(-y_k)|$$

Indeed, the second integral is convergent by:

$$|\Gamma(z)| \underset{\operatorname{Im} z \to \pm\infty}{\sim} \sqrt{2\pi} e^{-\frac{\pi}{2}|\operatorname{Im} z|} |\operatorname{Im} z|^{\operatorname{Re} z - \frac{1}{2}}$$

provided $s_k = |s_k| e^{i\theta_k}$ with $|\theta_k| < \frac{\pi}{2}$, which is true for non vanishing Wick's angle ε.

The first integral is convergent when $\operatorname{Re} \phi_i > 0$. This condition may be realized for every i simultaneously, due to theorem 1.

THEOREM 1. — The following two propositions are equivalent:

1) $F(s_k, m_i^2) = \int_0^\infty \prod_i d\alpha_i e^{-\sum_i \alpha_i m_i^2} U^{-\frac{D}{2}}(\alpha_i)$ is convergent.

2) Convex domain

$$\Delta = \left\{ \sigma, \tau \;\middle|\; \begin{array}{l} \sigma_j < 0 \quad ; \quad \tau_k < 0 \quad ; \quad \Sigma x_j + \Sigma y_k = -\frac{D}{2} \, ; \\ \forall i, \ \operatorname{Re} \phi_i \equiv \sum_j u_{ij}\sigma_j + \sum_k n_{ik}\tau_k + 1 > 0 \end{array} \right\}$$

is not empty.

Proposition 1) expresses the absence of any ultraviolet divergency. It is implied by proposition 2) as directly proved by representation (4). The converse implication is proved in the appendix.

By theorem 1, we can perform first the α integrations:

$$\int_0^{\infty} d\alpha_i e^{-\alpha_i m_i^2} \alpha_i^{\phi_i - 1} = \Gamma(\phi_i) m_i^{2^{-\phi_i}} \tag{5}$$

and we obtain the CM representation of the Feynman amplitude:

$$F(s_k, m_i^2) = \int_\Delta \frac{\prod_j \Gamma(-x_j)}{\Gamma\left(-\sum_j x_j\right)} \prod_k s_k^{y_k} \Gamma(-y_k) \prod_i m_i^{2^{-\phi_i}} \Gamma(\phi_i) \tag{6}$$

where we recall the notations:

$$\phi_i = \sum_j u_{ij} x_j + \sum_k n_{ik} y_k + 1$$

$u_{ij}(n_{ik})$ is 0 or 1 following the line i belongs or not to the one-tree j (two-tree k).

$\int_\Delta$ means that we impose $\sum_j x_j + \sum_k y_k = -\frac{D}{2}$

and integrate over the remaining independent $\frac{\operatorname{Im} x_j}{2i\pi}, \frac{\operatorname{Im} y_k}{2i\pi}$, with Re x_j. Re y_k satisfying :

$$\operatorname{Re}(-x_j) > 0 \quad \forall j$$
$$\operatorname{Re}(-y_k) > 0 \quad \forall k$$
$$\operatorname{Re}(\phi_i) > 0 \quad \forall i$$

As discussed in section V of ref. [*1*], the minskowskian limit $\varepsilon \to 0$ cannot generally be taken directly in the integrand of (6), unless the relative phases of the complex numbers s_k, m_i^2, are bounded by $|\theta_{\max}| < \pi$, uniformly in ε. This is the case for example if the minkowskian imaginary parts of the invariants keep the same common sign. Otherwise we think that integrations by parts, or displacements of the integration path, may isolate the threshold—or, Landau-singularities and restore convergent integrals in the limit $\varepsilon \to 0$ as we shall discuss elsewhere.

III. ASYMPTOTIC BEHAVIOURS

The CM representation is particularly suited to the determination of an asymptotic expansion. A general asymptotic regime is defined by scaling the invariants s_k and squared masses m_i^2 by arbitrary powers of a real parameter λ:

$$s_k \to \lambda^{a_k} s_k$$
$$m_i^2 \to \lambda^{a_i} m_i^2 \tag{7}$$

(a_k, a_i positive, negative or null), and by letting λ go to infinite.

Since the invariants and masses are completely factorized in the integrand of the CM representation, we get the simple following form:

$$F(\lambda, s_k, m_i^2) = \int_\Delta \frac{\prod_j \Gamma(-x_j)}{\Gamma(-\Sigma x_j)} \prod_k s_k^{y_k}\Gamma(-y_k) \prod_i m_i^{2^{-\phi_i}}\Gamma(\phi_i)\lambda^{\psi_0} \tag{8}$$

where

$$\psi_0 = \sum_k a_k y_k - \sum_i a_i \phi_i(x_j, y_k)$$

Now the argument closely parallels that one in section IV of ref. [1] and leads to an asymptotic expansion:

$$F(\lambda, s_k, m_i^2) = \sum_{p=p_{\max}}^{-\infty} \sum_{q=0}^{q_{\max}(p)} F_{pq}(s_k, m_i^2)\lambda^p \ln^q \lambda \tag{9}$$

The main progress here, as compared to ref. [1], is that the coefficients F_{pq} will be given in a much simpler way, without any previous splitting of the Feynman integral into Hepp sectors.

We use the constraint $\sum_j x_j + \sum_k y_k = -\frac{D}{2}$ for eliminating any one of the integration variables. Let us relabel the remaining variables x_j, y_k as $z_m(\text{Re}\, z_m = \zeta_m)$, the linear functions $-x_j$, $-y_k$, $\phi_i(x_j, y_k)$ as $\psi_\nu(z_m)$, and the invariants as $s_\nu(s_\nu = 1$, s_k or m_i^2 for ν replacing j, k, i respectively). Then Δ is the convex domain $\{ \zeta \mid \psi_\nu(\zeta) > 0 \ \forall \nu \}$. A first bound on $F(\lambda)$ is thus given by

$$| F(\lambda) | < c^t \lambda^{p_{\max}+\eta} \tag{10}$$

where η is an arbitrarily small positive number

$$p_{\max} = \operatorname*{Inf}_\Delta \psi_0(\zeta)$$

From the definition of $p_{\max}$, the function $\psi_0(\zeta) - p_{\max}$, being positive in Δ and reaching 0 on its bound, must belong to the convex space generated by the ψ_ν's: there exist (generally non unique) non negative coefficients d_ν such that

$$\psi_0 - p_{\max} \equiv \sum_\nu d_\nu \psi_\nu$$

Therefore

$$\frac{1}{\prod_{\nu} \psi_{\nu}} \equiv \frac{1}{\psi_0 - p_{\max}} \sum_{\nu} \frac{d_{\nu}}{\prod_{\nu' \neq \nu} \psi_{\nu'}}$$

For a given ν, if the subset $\{ \psi_{\nu'}, \nu' \neq \nu \}$ again generates $\psi_0 - p_{\max}$ with non negative coefficients $d'_{\nu'}$, we repeat the procedure, which is iterated until we obtain:

$$\frac{1}{\prod_{\nu} \psi_{\nu}} \equiv \sum_{E} \frac{d_E}{(\psi_0 - p_{\max})^{q_E + 1}} \cdot \frac{1}{\prod_{\nu \in E} \psi_{\nu}} \tag{11}$$

Now for each E, $\psi_0 - p_{\max}$ does not belong to the convex space generated by the subset $\{ \psi_{\nu}, \nu \in E \}$. It becomes negative somewhere in

$$\Delta_E = \{ \zeta \mid \psi_{\nu} + \theta_{\nu E} > 0 \ \forall \nu \}$$

where $\theta_{\nu E} = 0$ if $\nu \in E$, 1 otherwise.

By

$$\Gamma(\psi_{\nu}) = \frac{1}{\psi_{\nu}} \Gamma(\psi_{\nu} + 1)$$

we write:

$$F(\lambda) = \sum_{E} d_E F_E(\lambda) \tag{12}$$

$$F_E(\lambda) = \int_{\substack{\Delta_E \\ \mathrm{Re}\,(\psi_0 - p_{\max}) > 0}} \lambda^{\psi_0} \frac{1}{(\psi_0 - p_{\max})^{q_E + 1}} M_E(z) \tag{13}$$

$$M_E(z) = \frac{1}{\Gamma(-\Sigma x_j)} \prod_{\nu} s_{\nu}^{-\psi_{\nu}} \Gamma(\psi_{\nu} + \theta_{\nu E})$$

is now analytical in Δ_E and we may move the integration path up to a point where $\psi_0 - p_{\max} < 0$ without crossing any other polar variety. This procedure is briefly sketched in the example of figure 1. By Cauchy theorem:

$$F_E(\lambda) = \lambda^{p_{\max}} \sum_{q=0}^{q_E} F^E_{p_{\max} q} \ln^q \lambda + G_E(\lambda) \tag{14}$$

$$F^E_{p_{\max} q} = \frac{1}{q\,!\,(q_E - q)\,!} \int_{\substack{\Delta_E \\ \psi_0 - p_{\max} = 0}} \nabla^{q_E - q} M_E(z) \tag{15}$$

where ∇ is the differential operator along any direction crossing the plane $\psi_0 = p_{\max}$.

$$G_E(\lambda) = \int_{\substack{\Delta_E \\ \mathrm{Re}\,(\psi_0 - p_{\max}) < 0}} \lambda^{\psi_0} \frac{1}{(\psi_0 - p_{\max})^{q_E + 1}} M_E(z) \tag{16}$$

Thus $\mid G_E(\lambda) \mid < c^t \lambda^{p_{\max} - a_E + \eta}$, where $p_{\max} - a_E = \operatorname*{Inf}_{\Delta_E} \psi_0(\zeta)$,
a_E is a strictly positive rational.

Therefore we have completely determined the $\lambda^{p_{\max}}$ part of the asymptotic expansion. By

$$\Gamma(\psi_\nu + \theta_{\nu E}) = \frac{1}{\psi_\nu + \theta_{\nu E}} \Gamma(\psi_\nu + \theta_{\nu E} + 1)$$

we can again determine the $\lambda^{p_{\max} - a_E}$ part, etc. We obtain similarly the complete asymptotic expansion.

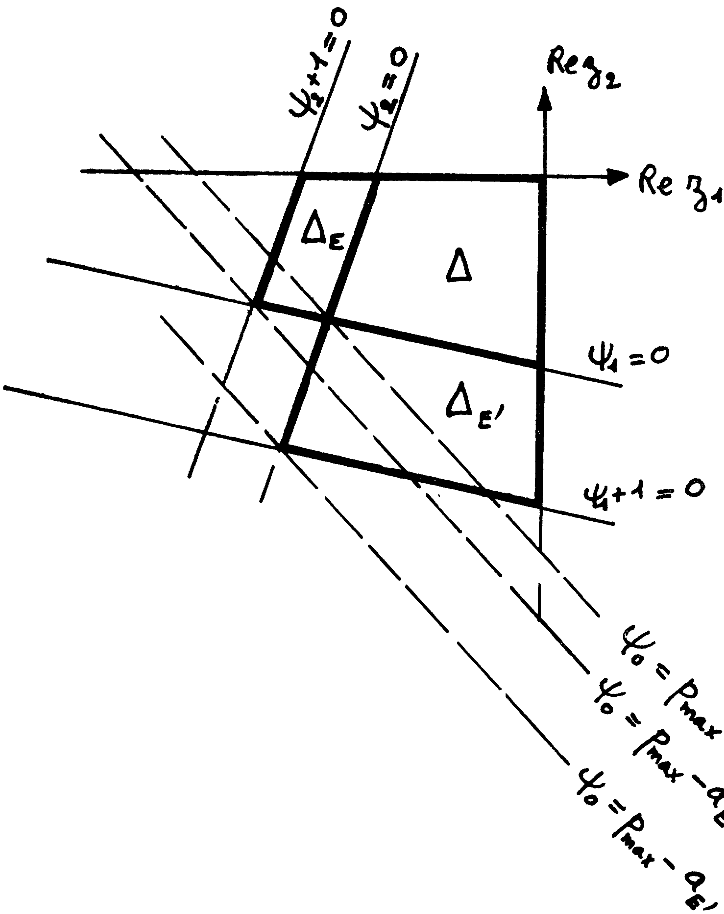

FIG. 1.

An equivalent way of determining this expansion is the following: starting from formula (8) one can move step by step the integration path from

$\zeta_m \in \Delta$ up to points where $\psi_0 < p_{\max}, p_{\max} - a_E$, etc., by crossing the various $\psi_v = 0, -1, \ldots$ polar varieties. We obtain integral with new domains, and integrals over the residues of these poles (generally multiple poles, since many such varieties may coincide). For both types of integrals the same procedure is then iterated, until one obtains integrals where $\psi_0 \equiv c^t$, and integrals with domains where ψ_0 becomes less than some p, determining the asymptotic expansion up to λ^p terms.

REMARK ON THE CASE OF VANISHING MASSES

If the Feynman integral (1) still converges for some vanishing mass, say m_1, the corresponding expansion in λ, where λ^{-1} scales m_1^2, must be of the type:

$$F(\lambda) = \lambda^0 F_0 + \mathcal{O}(\lambda^{-a}) \quad , \quad a > 0 \tag{17}$$

without any power of ln λ in the first term. Therefore $\psi_0 \equiv \phi_1$ must be a simple polar variety and the value of the Feynman integral is given by the following CM representation:

$$F_0 = \int_{\substack{\Delta \\ \phi_1 = 0}} \frac{\prod_j \Gamma(-x_j)}{\Gamma(-\Sigma x_j)} \prod_k s_k^{y_k}\Gamma(-y_k) \prod_{i \neq 1} m_i^{2-\phi_i}\Gamma(\phi_i) \tag{18}$$

The same argument can of course be repeated for any larger set of vanishing masses, for which the Feynman integral remains convergent.

IV. COMPLETE MELLIN REPRESENTATION FOR ULTRAVIOLET DIVERGENT GRAPHS

When ultraviolet divergences are present in a Feynman amplitude, the α integration in (4) cannot be performed. The integrand

$$I = U^{-\frac{D}{2}} e^{-\frac{N}{U}} = \int_{\substack{\sigma,\tau < 0 \\ \Sigma x_j + \Sigma y_k = -\frac{D}{2}}} \frac{\prod_j \Gamma(-x_j)}{\Gamma(-\Sigma x_j)} \prod_k s_k^{y_k}\Gamma(-y_k) \prod_i \alpha_i^{\phi_i - 1} \tag{19}$$

has first to be replaced by a renormalized integrand. One possible way of working is the use of analytic continuations. For example the generalized Feynnam amplitudes defined by Speer [2] correspond in our CM representation to the simple replacement $\phi_i \to \phi_i' = \phi_i + \lambda_i$. Then the new domain

$$\Delta' = \left\{ x, y \;\middle|\; \begin{array}{ll} \Sigma x_j + \Sigma y_k = -\dfrac{D}{2} \;\; ; & \operatorname{Re} \phi_i' > 0 \\ \operatorname{Re}(-x_j) > 0 \;\; ; & \operatorname{Re}(-y_k) > 0 \end{array} \right\} \tag{20}$$

is non empty for Re λ_i's high enough. The problem, then, is to define what Speer call « evaluators » and we think that an explicit study of such evaluators would be particularly simple in our representation.

An alternative would be the analytic continuation in the complex variable D [3]. Here too, Δ is trivially a non empty domain for Re D small enough and it would be also interesting to study with our formalism the dimensional renormalization.

In the following we shall use the $\mathcal{R}$ operation built from Taylor subtractions in the α space [4].

IV.1. Renormalization of individual divergent subgraphs.

Let us first renormalize one divergent subgraph S (l_S lines, L_S independent loops). Ultraviolet divergency of S is expressed by $\omega_S = l_S - \frac{D}{2} L_S \leqslant 0$. Then the domain Δ is trivially empty since

$$\phi_S = \sum_{i \in S} \phi_i = \sum_j \Big(\sum_{i \in S} u_{ij} \Big) x_j + \sum_k \Big(\sum_{i \in S} n_{ik} \Big) y_k + l_S \tag{21}$$

$$\sum_{i \in S} u_{ij} = L_S + a_{Sj}$$

$$\sum_{i \in S} n_{ik} = L_S + b_{Sk}$$

where a_{Sj}, b_{Sk} are non negative integers. By $\sum_j x_j + \sum_k y_k = -\frac{D}{2}$ we find:

$$\phi_S = \omega_S + \sum_j a_{Sj} x_j + \sum_k b_{Sk} y_k \tag{22}$$

which cannot have a strictly positive real part for Re $x_j < 0$, Re $y_k < 0$, $\omega_S \leqslant 0$.

Now the effect of $1 - \mathcal{T}_S$ acting on I given by (19), is to suppress the $\mathscr{E}(-\omega_S)$ first terms of its generalized Taylor expansion in ρ, where ρ scales the parameters α_i, $i \in S$. This is a problem quite similar to this one we studied in section III: we must find the asymptotic expansion of $I(\rho)$ when $\rho \to 0$. But at this stage, the only present singularities are those of the functions $\Gamma(-x_j)$, $\Gamma(-y_k)$. Since they are independent, we find only simple poles and no power of ln ρ, as expected since I has a Taylor expansion:

$$\rho^{l_S}\mathrm{I}(\rho) = \int_{\substack{\sigma,\tau<0 \\ \Sigma x_j + \Sigma y_k = -\frac{D}{2}}} \frac{\prod_j \Gamma(-x_j)}{\Gamma(-\Sigma x_j)} \prod_k s_k^{y_k}\Gamma(-y_k) \prod_i \alpha_i^{\phi_i - 1}\rho^{\phi_S}$$

$$= \rho^{\omega_S}(\mathrm{I}_0 + \rho \mathrm{I}_1 + \dots) \tag{23}$$

We determine this expansion in the way indicated at the end of section III. Let us call E_S the set of indices $m(z_m = x_j$ or $y_k)$ for which $a_{Sm}(a_{Sm} = a_{Sj}$ or $b_{Sk})$ is strictly positive:

$$\phi_S = \omega_S + \sum_{m \in E_S} a_{Sm} z_m \tag{24}$$

If this set is empty

$$\rho^{l_S}\mathrm{I}(\rho) = \rho^{\omega_S}\mathrm{I}_0 \qquad \text{and} \qquad (1 - \mathcal{T}_S)\mathrm{I} = 0$$

(this can happen only if S is the complete graph G, which is then superficially divergent, and if all external four—momenta vanish: N = 0).

Otherwise we displace the integration path by crossing the singularities of $\Gamma(-z_{m_1})$, $m_1 \in E_S$, until we reach the cell $n_1 < \operatorname{Re} z_{m_1} < n_1 + 1$ where $\operatorname{Re} \phi_S$ becomes positive. For each integral over the residue at $z_{m_1} = n$ we do the same by increasing $\operatorname{Re} z_{m_2}$, $m_2 \in E_S$, and we can finally rewrite

$$\int_{n_{12} < \operatorname{Re} z_{m_2} < n_{12}+1} [\text{residue at } z_{m_1} = n] = \int_{\substack{n-1 < \operatorname{Re} z_{m_1} < n \\ n_{12} < \operatorname{Re} z_{m_2} < n_{12}+1}} - \int_{\substack{n < \operatorname{Re} z_{m_1} < n+1 \\ n_{12} < \operatorname{Re} z_{m_2} < n_{12}+1}}$$

We similarly increase $\operatorname{Re} z_{m_3}$ for the integrals over the double residues at $z_{m_1} = n$ and $z_{m_2} = n'$, etc.

Let us consider the whole set of cells C_S we reach in this way:

$$C_S = \left\{ \operatorname{Re} z \;\middle|\; \begin{array}{ccc} \operatorname{Re} z_m < 0 \ \text{ for } \ m \notin E_S & ; & n^{(m)} < \operatorname{Re} z_m < n^{(m)} + 1 \ \text{ for } \ m \in E_S \\ \sup\limits_{C_S} \operatorname{Re} \phi_S > 0 & ; & \inf\limits_{C_S} \operatorname{Re} \phi_S \leq 0 \end{array} \right\}$$

At the end of the procedure, the total residues at $z_m = 0, 1, \dots, n_m$ give the terms in the expansion which are cancelled by the $1 - \mathcal{T}_S$ operation, and we are left with

$$(1 - \mathcal{T}_S)\mathrm{I} = \sum_{C_S} \mu_{C_S} \int_{C_S} \frac{\prod_j \Gamma(-x_j)}{\Gamma(-\Sigma x_j)} \prod_k s_k^{y_k}\Gamma(-y_k) \prod_i \alpha_i^{\phi_i - 1} \tag{25}$$

where the multiplicities μ_{C_S} are integers (positive, negative or null).

IV.2. Complete renormalization.

It is not trivial that the same technique can be iterated for other divergent subgraphs. Indeed, due to the constraint $\Sigma x_j + \Sigma y_k = -\frac{D}{2}$, some quantities Re ϕ_i may decrease when other increase. The relevant cells we must reach are those ones which are « tangent » to the convex space $\{ \phi_i > 0 \ \forall i \}$:

$$\mathscr{R}\mathrm{I} = \sum_c \mu_c \int_C \frac{\prod_j \Gamma(-x_j)}{\Gamma(-\Sigma x_j)} \prod_k s_k^{y_k}\Gamma(-y_k) \prod_i \alpha_i^{\phi_i - 1} \tag{26}$$

where the cells C are such that

$$\sup_C \mathrm{Re}\ \phi_i > 0 \ \forall i$$

$$\inf_S \left(\inf_C \sum_{i \in S} \mathrm{Re}\ \phi_i \right) \leqslant 0$$

The following theorem, proved in the appendix, prevents the relevant convex space from being empty.

THEOREM 2. — Given the integrand I of an arbitrary Feynman graph, and the corresponding linear functions $\phi_i(x_j, y_k)$, then:

i) either $\mathscr{R}\mathrm{I} = 0$ (for some exceptional momenta);

ii) or there exist points (x, y) where $\begin{cases} \mathrm{Re}\ \phi_i > 0 \ \forall i \\ \Sigma x_j + \Sigma y_k = -\frac{D}{2}. \end{cases}$

Provide we could generalize the procedure of the preceding paragraph, we would obtain:

$$\int_0^\infty \prod_i d\alpha_i e^{-\Sigma \alpha_i m_i^2} \mathscr{R}\mathrm{I} = \sum_c \mu_c \int_{\Delta_c} \frac{\prod_j \Gamma(-x_j)}{\Gamma(-\Sigma x_j)} \prod_k s_k^{y_k}\Gamma(-y_k) \prod_i m_i^{2-\phi_i}\Gamma(\phi_i) \tag{27}$$

where

$$\Delta_c = \left\{ x, y \ \middle| \ \begin{array}{l} \Sigma x_j + \Sigma y_k = -\frac{D}{2} \\ x, y \in \mathrm{C} \quad ; \quad \mathrm{Re}\ \phi_i > 0 \ \forall i \end{array} \right\}$$

expressing the renormalized amplitude by a simple translation of the integration path, without any change of the integrand.

IV.3. Examples.

For the graph of figure 2, in 4 space-time dimensions, we find

$$F_2 = \int_{\Delta_2^R} \frac{\Gamma(-x_1)\Gamma(-x_2)}{\Gamma(-x_1-x_2)} p^{2y}\Gamma(-y)$$
$$m_1^{2^{-x_1-y-1}}\Gamma(x_1+y+1)m_2^{2^{-x_2-y-1}}\Gamma(x_2+y+1)$$

with

$$\Delta_2^R = \left\{ x, y \,\middle|\, \begin{array}{c} x_1+x_2+y=-2 \quad ; \quad \mathrm{Re}\,(x_1+y+1)>0 \quad ; \quad \mathrm{Re}\,(x_2+y+1)>0 \\ \mathrm{Re}\,x_j<0 \quad ; \quad 0<\mathrm{Re}\,y<1 \end{array} \right\}$$

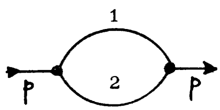

FIG. 2.

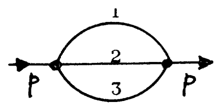

FIG. 3.

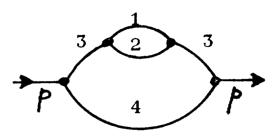

FIG. 4.

Similarly for the graph of figure 3, though it is quadratically divergent and contains overlapping logarithmically divergent subgraphs, we get the simple result:

$$F_3 = \int_{\Delta_3^R} \frac{\Gamma(-x_1)\Gamma(-x_2)\Gamma(-x_3)}{\Gamma(-x_1-x_2-x_3)} p^{2y}\Gamma(-y)\prod_{i=1}^{3} m_i^{2^{-\phi_i}}\Gamma(\phi_i)$$
$$\phi_1 = x_2 + x_3 + y + 1$$
$$\phi_2 = x_1 + x_3 + y + 1$$
$$\phi_3 = x_1 + x_2 + y + 1$$
$$\Delta_3^R = \left\{ x, y \,\middle|\, \begin{array}{lcl} x_1 + x_2 + x_3 + y = -2 & ; & \mathrm{Re}\,\phi_i > 0\ \forall i \\ \mathrm{Re}\,x_j < 0 & ; & 1 < \mathrm{Re}\,y < 2 \end{array} \right\}$$

As a further example, for the graph of figure 4, we get:

$$F_4 = \int_{\Delta'} M + \int_{\Delta''} M - \int_{\Delta'''} M$$

where

$$M = \frac{\prod_{j=1}^{5}\Gamma(-x_j)}{\Gamma(-\Sigma x_j)} p^{2\Sigma y_k}\prod_{k=1}^{3}\Gamma(-y_k)\prod_{i=1}^{4} m_i^{2-\phi_i}\Gamma(\phi_i)$$
$$\phi_1 = x_1 + x_2 + x_3 + y_1 + y_2 + 1$$
$$\phi_2 = x_1 + x_4 + x_5 + y_1 + y_3 + 1$$
$$\phi_3 = x_2 + x_4 + y_2 + y_3 + 2$$
$$\phi_4 = x_3 + x_5 + y_1 + y_2 + y_3 + 1$$
$$\Delta' = \left\{ x, y \,\middle|\, \begin{array}{c} \Sigma x_j + \Sigma y_k = -2 \quad ; \quad \mathrm{Re}\,\phi_i > 0\ \forall i \\ \mathrm{Re}\,x_j < 0\ \forall j \quad ; \quad 0 < \mathrm{Re}\,y_1 < 1 \quad ; \quad \mathrm{Re}\,y_k < 0\ \forall k \neq 1 \end{array} \right\}$$

and the same for the other domains, except that in Δ'':

$$0 < \operatorname{Re} x_1 < 1 \quad ; \quad \operatorname{Re} y_1 < 0$$

and in Δ''':

$$0 < \operatorname{Re} x_1 < 1 \quad ; \quad 0 < \operatorname{Re} y_1 < 1$$

IV.4. Comments.

We must be able to find the renormalized amplitudes as sums of terms, each of which is a finite quantity having a CM representation. Actually any integral like

$$\int_\delta \frac{\prod_j \Gamma(-x_j)}{\Gamma(-\Sigma x_j)} \prod_k s_k^{y_k}\Gamma(-y_k) \prod_i m_i^{2^{-\phi_i}}\Gamma(\phi_i)$$

is convergent, if in δ the real parts of the arguments of the Γ-functions are bounded between two consecutive integers. But of course this is not sufficient for providing an acceptable renormalization. What happens is that well choosen sums of such integrals differ from the unrenormalized one by a quantity implementable with a finite number of prescribed counter-terms (if the theory is renormalizable). From this point of view, we shall explore more extensively, and more explicitly, the effect of renormalization, with our CM Representation, in a later paper.

Now for determining any asymptotic expansion, the same method applies as well to any such integral. But looking at the translated domains, we see how deeply renormalization may change the asymptotic behaviour.

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APPENDIX

1. PROOF OF THEOREM 1

For achieving the proof, let us assume that 2) is wrong. Since each Re ϕ_i may be positive (for $|\sigma|, |\tau_k|$ small enough) if we forget the condition $\Sigma\sigma_j + \Sigma\tau_k = -\frac{D}{2}$, it means that the linear « diagonal » variety $\Sigma\sigma_j + \Sigma\tau_k + \frac{D}{2} = 0$ does not across the domain

$$\Delta_0 = \left\{ \sigma, \tau \left| \begin{array}{c} -\sigma_j > 0 \quad ; \quad -\tau_k > 0 \\ \mathrm{Re}\, \phi_i > 0 \end{array} \right. \right\}$$

Thus this linear variety, or a « higher » diagonal

$$\Sigma\sigma + \Sigma\tau_k + \frac{D}{2} - r_0 = 0 \quad , \quad r_0 \geqslant 0$$

must belong to the convex space generated by the bounds of the domain Δ_0: there must exist non negative r_0, r_i, r_j, r_k such that

$$\sum_i r_i \phi_i(\sigma_j, \tau_k) + \sum_j r_j(-\sigma_j) + \sum_k r_k(-\tau_k) \equiv \sum_j \sigma_j + \sum_k \tau_k + \frac{D}{2} - r_0$$

or:

$$\sum_i r_i u_{ij} = 1 + r_j \quad \forall j$$

$$\sum_i r_i n_{ik} = 1 + r_k \quad \forall k$$

$$\sum_i r_i = \frac{D}{2} - r_0$$

Moreover $\inf_k r_k \leqslant \inf_j r_j$ since $\frac{N}{U}$ never becomes infinite. Now if we define $\alpha_i = \gamma_i^{r_i}$ the radial power-counting in $F(s_k, m_i^2)$ with the γ variables, gives

$$\int_0 \rho^{\Sigma r_i - 1} \rho^{-\frac{D}{2}(1 + \inf_j r_j)} d\rho = \int_0 \rho^{-1 - r_0 - \frac{D}{2} \inf_j r_j} d\rho$$

It implies that the integral is divergent, that is 1) is wrong. This achieves the proof of the theorem, which is quite analogous to the theorem of appendix A in [*1*].

2. PROOF OF THEOREM 2

There exist points where all Re ϕ_i are positive if we forget the constraint $\Sigma x_j + \Sigma y_k = -\frac{D}{2}$. Thus, if *ii*) is wrong, the « diagonal » $\Sigma\sigma_j + \Sigma\tau_k + \frac{D}{2} - r_0$, $r_0 \geqslant 0$, must belong to the convex space generated by the ϕ's: there must exist non negative r_i, r_0 such that

$$\sum_i r_i u_{ij} = 1 \quad \forall j$$

$$\sum_k r_i n_{ik} = 1 \quad \forall k$$

$$\sum_i r_i + r_0 = \frac{D}{2}$$

But then the naive power-counting in the γ variables, with $\gamma_i^{r_i} = \alpha_i$, gives for the renormalized integral

$$\int_0 \rho^{\Sigma r_i - 1} \rho^{-\frac{D}{2}} \mathscr{R}\mathrm{I} d\rho = \int_0 \rho^{-1-s_0} \mathscr{R}\mathrm{I} d\rho$$

since $\mathscr{R}$I is nothing but I multiplied by a sum of products of terms like $\frac{U_j}{U}, \frac{N_k}{U}$. Unless $\mathscr{R}\mathrm{I} = 0$, this would lead to a divergent integral, in contradiction with the result of Bergère and Lam [4].

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