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Two charges in an external electromagnetic field: A generalized covariant Hamiltonian formulation

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ABSTRACT. — In a previous paper [*1*], we studied the non-isolated systems of two structureless point particles in the framework of Predictive Relativistic Mechanics, developing a perturbation technique which permits the recurrent calculation of the accelerations by assuming that these functions can be expanded into a power series of two characteristic parameters of the particles. We then applied this in the case of an electromagnetic external field and an electromagnetic interaction using causality as a subsidiary condition. In the present paper, the possibility of including the radiation reaction by means of a Lorentz-Dirac term is introduced. On the other hand, the possibility of such a dynamic system admitting a covariant Lagrangian formulation compatible with « predictivity » is dropped by a no-interaction theorem [*2*]. In spite of this, we construct a generalized covariant Hamiltonian formulation for fields satisfying certain weak conditions and give the expression of the two « hamiltonian »-like functions of the canonical coordinates to order three in charge expansions.

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I. INTRODUCTION

Predictive Relativistic Mechanics (PRM) is the only theory up to now that describes the relativistic $\underline{N}$ particle systems with the following apparently contradictory features: *i*) Newtonian causality, *ii*) Causal propagation and *iii*) Relativistic invariance (only for isolated systems).

« Newtonian causality » is understood here as: the evolution of a relativistic system (constituted by N structureless point particles) is governed by an ordinary second order differential system over $\mathbb{R}^{3N}$

$$\frac{dx_a^i}{dt} = v_a^i, \qquad \frac{dv_a^i}{dt} = \mu_a^i(t, x_b^j, v_c^k) \tag{1}$$

where the μ_a^i functions characterize the system. According to this principle, the motion of each particle is determined by knowing the positions and velocities of every particle at the same time $\underset{0}{t}$. We need $\underline{6N}$ initial data only in order to determine the future motion of the system. This property is usually remarked by introducing the word « predictivity » or the name « predictive relativistic systems ».

By « Causal propagation » we understand the following: the framework of the classical field theory, as is well-known, furnishes a scheme for interaction that, for at least a lineal field theory, can be represented (N = 2) in the form

$$a' \rightarrow \text{FIELD}_{a'} \rightarrow a$$

where fields are propagated with finite velocity in Special Relativity. This constitutes the causal propagation and yields, in general, to motion equations that are differential-difference-integral equations and not ordinary differential equations.

Finally, « Relativistic invariance » is understood as: if $\psi_a^i(\underset{0}{x}_b^j, \underset{0}{v}_c^k; t)$ denotes the general solution of (1) corresponding to initial data at $t = 0$,

$$\psi_a^i(\underset{0}{x}_b^j, \underset{0}{v}_c^k; 0) = \underset{0}{x}_a^i, \qquad \dot{\psi}_a^i(\underset{0}{x}_b^j, \underset{0}{v}_c^k; 0) = \underset{0}{v}_a^i \qquad \left(\dot{} \equiv \frac{d}{dt} \right), \tag{2}$$

we can associate the $\underline{N}$ curves of M_4 parametrized in the form

$$x_a^0 = t, \qquad x_a^i = \psi_a^i(\underset{0}{x}_b^j, \underset{0}{v}_c^k; t) \tag{3}$$

to each set of initial data $(\underset{0}{x}_a^i, \underset{0}{v}_b^j)$. Let us consider the $\underline{6N}$ parameter family Γ whose elements are those $\underline{N}$ curves. The dynamic system (1) is said to be relativistic invariant if the Poincaré group carries Γ into Γ.

Historically, the compatibility of « Newtonian causality » and « Relativistic invariance » was proved by D. G. Currie [*4*] and R. N. Hill [*5*]. They found some necessary conditions (later they were also proved to be sufficient by L. Bel [6]) that must be satisfied by the μ_a^i functions in order to have relativistic invariance

$$\frac{\partial \mu_a^i}{\partial t} = 0 , \qquad \varepsilon_b \frac{\partial \mu_a^i}{\partial x_b^j} = 0 , \tag{4}$$

$$\eta_{jk}^l \left(x_b^k \frac{\partial \mu_a^i}{\partial x_b^l} + v_b^k \frac{\partial \mu_a^i}{\partial v_b^l} \right) = \eta_{jk}^i \mu_a^k \tag{5}$$

$$c^{-2} v_b^k (x_{bj} - x_{aj}) \frac{\partial \mu_a^i}{\partial x_b^k} + \left[c^{-2} v_b^k v_{aj} + c^{-2} \mu_b^k (x_{bj} - x_{aj}) - \varepsilon_b \delta_j^k \right] \frac{\partial \mu_a^i}{\partial v_b^k} = c^{-2} (2\mu_a^i v_{aj} + \mu_{aj} v_a^i) . \tag{6}$$

These equations are usually known in the literature as the « Currie-Hill equations » and express the invariance of the system by time-space translations (4), space rotations (5) and pure Lorentz transformations (6), respectively. This constitutes the manifestly predictive formalism.

The compatibility between « Causal propagation », « Newtonian causality » and « Relativistic invariance » has been proved for some interactions (at least in a perturbative scheme) by many authors: L. Bel *et al.* [*7*], A. Salas and J. M. Sanchez [*8*], L. Bel and J. Martin [*9*], R. Lapiedra and L. Mas [*10*], for the electromagnetic interaction of two charges; L. Bel and J. Martin [*11*] for the scalar interaction of two particles. Recently L. Bel and X. Fustero [*12*] have studied the N particle systems in scalar or vectorial (short or long range) interaction.

The compatibility between « Causal propagation » and « Newtonian causality » has been recently proved by J. L. Sanz and J. Martin [*1*] for the electromagnetic interaction of two charges and the external electromagnetic field.

Actually, the majority of the previous results concerning the introduction of « Causal propagation » have been realized using another formalism, the so-called manifestly covariant formalism, that was developed by Ph. Droz-Vincent [*13*], J. Wray [*14*] and L. Bel [*15*] independently of the previous one. In this formalism, the « Newtonian causality » is understood as follows: the evolution of a relativistic system of N point-like particles is governed by an ordinary autonomous second order differential system over M_4

$$\frac{dx_a^\alpha}{d\tau} = u_a^\alpha , \qquad \frac{du_a^\alpha}{d\tau} = \xi_a^\alpha (x_b^\beta , u_c^\gamma) \tag{7}$$

where the ξ_a^α functions must satisfy

$$(u_a \xi_a) = 0 \tag{8}$$

$$u_{a'}^\rho \frac{\partial \xi_a^\alpha}{\partial x^{a'\rho}} + \xi_{a'}^\rho \frac{\partial \xi_a^\alpha}{\partial u^{a'\rho}} = 0 . \tag{9}$$

Condition (8) is common in Relativity and furnishes N first integrals: $u_a^2 \equiv -(u_a u_a)$. Condition (9) is new in the Relativity framework and expresses « predictivity », i. e., the condition must satisfy the ξ_a^α functions in order to determine the motion of each particle by knowing 6N initial data and the u_a^2 first integrals. This notation is usually employed by adopting a unitary point of view $(u_a^2 = 1)$ and identifying ξ_a^α as the 4-acceleration.

For isolated systems, the relativistic invariance is understood as follows: if $\varphi_a^\alpha(\underset{0}{x}{}_b^\beta, \underset{0}{u}{}_c^\gamma; \tau)$ denotes the general solution of (7) corresponding to initial data at $\tau = 0$

$$\varphi_a^\alpha(\underset{0}{x}{}_b^\beta, \underset{0}{u}{}_c^\gamma; 0) = \underset{0}{x}{}_a^\alpha, \qquad \dot{\varphi}_a^\alpha(\underset{0}{x}{}_b^\beta, \underset{0}{u}{}_c^\gamma; 0) = \underset{0}{u}{}_a^\alpha \qquad \left(\dot{} \equiv \frac{d}{d\tau}\right), \tag{10}$$

the dynamic system (7) is said to be relativistic invariant if the Poincaré group carries Δ into Δ, Δ being the family whose elements are the N curves in M_4 of the form $x_a^\alpha = \varphi_a^\alpha(\underset{0}{x}{}_b^\beta, \underset{0}{u}{}_c^\gamma; \tau)$. It is obvious that the conditions which are necessary and sufficient in order to have relativistic invariance in this formalism are

$$\varepsilon_b \frac{\partial \xi_a^\alpha}{\partial x_b^\beta} = 0 \tag{11.a}$$

$$(\delta_\lambda^\rho \eta_{\mu\sigma} - \delta_\mu^\rho \eta_{\lambda\sigma})\left(x_b^\sigma \frac{\partial \xi_a^\alpha}{\partial x_b^\rho} + u_b^\sigma \frac{\partial \xi_a^\alpha}{\partial u_b^\rho}\right) = \delta_\lambda^\alpha \xi_{a\mu} - \delta_\mu^\alpha \xi_{a\lambda} \tag{11.b}$$

and they express the invariance of the system under space-time translations (11.*a*) and Lorentz transformations (11.*b*), respectively.

The equivalence of the two formalisms for isolated and non-isolated systems has been proved by L. Bel [*15*] [*16*], by assuming an additional regularity condition on system (1). Its general solution satisfies

$$\exists t_a / \psi_a^i(\underset{0}{x}{}_b^j, \underset{0}{v}{}_c^k; t_a) = \psi_a^i(\underset{0}{\bar{x}}{}_b^j, \underset{0}{\bar{v}}{}_c^k; t_a),$$

$$\dot{\psi}_a^i(\underset{0}{x}{}_b^j, \underset{0}{v}{}_c^k; t_a) = \dot{\psi}_a^i(\underset{0}{\bar{x}}{}_b^j, \underset{0}{\bar{v}}{}_c^k; t_a) \Rightarrow (\underset{0}{x}{}_a^i, \underset{0}{v}{}_b^j) = (\underset{0}{\bar{x}}{}_a^i, \underset{0}{\bar{v}}{}_b^j) \tag{12}$$

the set $(\underset{0}{x}{}_a^i, \underset{0}{v}{}_b^j)$, which is obtained by inverting $x_a^i = \psi_a^i(\underset{0}{x}{}_b^j, \underset{0}{v}{}_c^k; t_a)$ and $v_a^i = \dot{\psi}_a^i(\underset{0}{x}{}_b^j, \underset{0}{v}{}_c^k; t_a)$ is unique for all t_b (i. e., if

$$\underset{0}{x}{}_a^i = f_a^i(x_b^j, v_c^k, t_d) \qquad \text{and} \qquad \underset{0}{v}{}_a^i = g_a^i(x_b^j, v_c^k, t_d),$$

then f_a^i and g_a^i are smooth functions).

It can be proved that in order to have this equivalence (in the sense that the trajectories of (1) coincide with the solutions of (7) associated to initial data, such that $u_a^2 = 1$) the μ's and ξ's must satisfy

$$\begin{cases} \mu_a^i(t, x_b^j, v_c^k) = (1 - v_a^2)(\delta_j^i - v_a^i v_{aj})\bar{\xi}_a^j, \\ \bar{\xi}_a^j(t, x_b^j, v_c^k) = \xi_a^j[x_b^0 \equiv t, x_c^k, u_d^0 \equiv (1 - v_d^2)^{-1/2}, u_e^l \equiv (1 - v_e^2)^{-1/2} v_e^l]. \end{cases} \tag{13}$$

In this paper we give, in the manifestly covariant formalism of P. R. M., a generalized covariant Hamiltonian formulation to the particular non-isolated system constituted by two charges affected by an external electromagnetic field whose dynamics has been recently studied in ref. [*1*]. However we leave open the possibility of including the radiation reaction by means of a Lorentz-Dirac term when we assume « Causal propagation ». We also give a new proof of a no-interaction theorem [*2*] which demonstrates the essential role played by two assumptions: *a*) the position coordinates x_a^α of the particles are canonical, and *b*) the predictive group acts like a set of canonical transformations.

II. A NO-INTERACTION THEOREM

It is useful to introduce a geometric point of view for masses in order to give a Hamiltonian formulation to the N particle systems. In this sense, we shall adopt the ordinary second order differential system over $(M_4)^N$ as motion equations

$$\frac{dx_a^\alpha}{d\lambda} = \pi_a^\alpha, \qquad \frac{d\pi_a^\alpha}{d\lambda} = \theta_a^\alpha(x_b^\beta, \pi_c^\gamma) \tag{14}$$

where the θ_a^α functions, that we shall call the *dynamics*, are related to the ξ_a^α 4-acceleration by

$$\theta_a^\alpha(x_b^\beta, \pi_c^\gamma) = \pi_a^2 \xi_a^\alpha(x_b^\beta, u_c^\gamma \rightarrow \pi_c^{-1}\pi_c^\gamma\,;\, m_d \rightarrow \pi_d), \qquad \pi_b \equiv +[-(\pi_b\pi_b)]^{1/2} \tag{15}$$

m_a being the mass of the particle a. Thus the dynamics must satisfy

$$(\pi_a\theta_a) = 0 \tag{16}$$

$$\pi_{a'}^\rho \frac{\partial\theta_a^\alpha}{\partial x^{a'\rho}} + \theta_{a'}^\rho \frac{\partial\theta_a^\alpha}{\partial\pi^{a'\rho}} = 0 \tag{17}$$

Assumptions (14), (16) and (17) constitute « Newtonian causality » in this manifestly covariant formalism.

Let us consider the $\underline{\mathrm{N}}$ vector fields on $(TM_4)^N$ [*17*]

$$\vec{\mathrm{H}}_a \equiv \pi_a^\rho \frac{\partial}{\partial x^{a\rho}} + \theta_a^\rho \frac{\partial}{\partial\pi^{a\rho}}. \tag{18}$$

It is then very easy to prove that conditions (16) and (17) are equivalent to the following:

$$£(\vec{\mathrm{H}}_a)\pi_b^2 = 0 \tag{19}$$

$$[\vec{\mathrm{H}}_a, \vec{\mathrm{H}}_b] = 0 \;\; [18] \tag{20}$$

The $\underline{N}$ vector fields $\vec{H}_a$ are the generators of an N-parametric abellian group of transformations acting on $(TM_4)^N$ that we shall call the predictive group.

Usually the dynamic system (14) is said to be Lagrangian if a function $L(x_a^\alpha, \pi_b^\beta)$ (without explicit dependence on λ) exists, such that

$$\pounds(\vec{H})\frac{\partial L}{\partial \pi_a^\alpha} - \frac{\partial L}{\partial x_a^\alpha} = 0, \qquad \det\left(\frac{\partial^2 L}{\partial \pi_a^\alpha \partial \pi_b^\beta}\right) \neq 0, \qquad \vec{H} \equiv \varepsilon^a \vec{H}_a. \tag{21}$$

As is well-known in the mathematics and physics literature (cf., for example, R. Abraham [*19*], chapt. III, C. Godbillon [*20*], chapt. VII, L. Bel [*21*], J. Martin and J. L. Sanz [*22*]) this definition is equivalent to the existence of a symplectic form Ω on $(TM_4)^N$, with the following properties

$$\begin{cases} i) & \Omega \wedge dx_1^0 \wedge \ldots \wedge dx_N^3 = 0, & (22) \\ ii) & \pounds(\vec{H})\Omega = 0, & (23) \end{cases}$$

which express the canonical character of the position variables [*23*] and the invariance under the one-parameter group generated by $\vec{H}$.

On the other hand, as the dynamic system is invariant under the predictive group (i. e., $[\vec{H}, \vec{H}_a] = 0$) it is logical to assume that the dynamic system admits a Lagrangian formulation compatible with its invariance under this group in the sense that

$$\pounds(\vec{H}_a)\Omega = 0. \tag{24}$$

These conditions mean that the predictive group acts as a canonical transformation group. Obviously, (23) is then identically verified. The relation between this covariant Lagrangian formulation and the predictive one can be seen in Appendix A.

Next, we will show the following theorem [*2*, *16*] : if a symplectic form Ω satisfies (22) and (24), then the θ_a^α functions can uniquely depend on the x_a^α and π_a^β variables (but not on the $x_{a'}^\alpha$ and $\pi_{a'}^\beta$ variables).

Physically, this means that the only Lagrangian dynamic systems which admit a Lagrangian predictive formulation (in the sense of (22) and (24)) are the non-interacting particles (only external forces acting on the particles are permitted).

The proof of the previous theorem is that (22) is equivalent to the existence of functions $p_\alpha^a(x_b^\beta, \pi_c^\gamma)$ with $\det\left(\frac{\partial p_\alpha^a}{\partial \pi_b^\beta}\right) \neq 0$ (defined except for the transformation $p_\alpha^a \to p_\alpha^a + \frac{\partial S}{\partial x_a^\alpha}(x_b^\beta)$) such that

$$\Omega = dx_a^\alpha \wedge dp_\alpha^a. \tag{25}$$

On the other hand, conditions (24), taking into account structure (25), lead straightforwardly to the following equations:

$$\delta_{ab}\frac{\partial p^b_\beta}{\partial \pi^\gamma_c} - \delta_{ac}\frac{\partial p^c_\gamma}{\partial \pi^\beta_b} = 0 \tag{26}$$

$$\frac{\partial}{\partial \pi^\gamma_c}\pounds(\vec{\mathrm{H}}_a)p^b_\beta - \frac{\partial p^c_\gamma}{\partial x^\beta_b}\delta_{ac} = 0 \tag{27}$$

$$\frac{\partial}{\partial x^\gamma_c}\pounds(\vec{\mathrm{H}}_a)p^b_\beta - \frac{\partial}{\partial x^\beta_b}\pounds(\vec{\mathrm{H}}_a)p^c_\gamma = 0 \tag{28}$$

We shall prove the theorem using only the sub-set of equations:

$$\frac{\partial p^a_\alpha}{\partial \pi^\beta_{a'}} = 0 \tag{29}$$

$$\frac{\partial}{\partial \pi^{a'\gamma}}\pounds(\vec{\mathrm{H}}_a)p_{a\beta} = 0 \tag{30}$$

$$\frac{\partial}{\partial \pi^{a\gamma}}\pounds(\vec{\mathrm{H}}_a)p_{a'\beta} - \frac{\partial p_{a\gamma}}{\partial x^{a'\beta}} = 0 \tag{31}$$

$$\frac{\partial}{\partial x^{a'\gamma}}\pounds(\vec{\mathrm{H}}_a)p_{a\beta} - \frac{\partial}{\partial x^{a\beta}}\pounds(\vec{\mathrm{H}}_a)p_{a'\gamma} = 0\,, \tag{32}$$

obtained from Eqs. (26)-(28) making $a = b$, $c = a'$ in (26); $b = a$, $c = a'$ and $b = a'$, $c = a$ in (27); $b = a$, $c = a'$ in (28).

The regularity condition $\det\left(\frac{\partial p^a_\alpha}{\partial \pi^\beta_b}\right) \neq 0$ obviously implies, taking into account (29),

$$\det\left(\frac{\partial p^a_\alpha}{\partial \pi^\beta_a}\right) \neq 0\,. \tag{33}$$

By developing (30) and using (29), one obtains

$$\frac{\partial \theta^\rho_a}{\partial \pi^{a'\gamma}}\frac{\partial p^\beta_a}{\partial \pi^{a\rho}} = 0$$

and thus (33) clearly implies

$$\frac{\partial \theta^\alpha_a}{\partial \pi^{a'}_\beta} = 0\,. \tag{34}$$

By developing (31) and using (29), one obtains

$$\frac{\partial p^a_\alpha}{\partial x^\beta_{a'}} - \frac{\partial p^{a'}_\beta}{\partial x^\alpha_a} = 0\,. \tag{35}$$

Finally, by developing (32) and using (29) and (35), we obtain

$$\frac{\partial \theta^\rho_a}{\partial x^{a'\gamma}}\frac{\partial p_{a\beta}}{\partial \pi^{a\rho}} = 0$$

and thus (33) implies

$$\frac{\partial \theta_a^\alpha}{\partial x_\beta^{a'}} = 0 . \tag{36}$$

Let us consider for a moment the assumptions that have led us to such a situation. Assumption (24) is reasonable because it seems quite natural to translate the symmetry possessed by the dynamic system to the Lagrangian scheme; (22) seems to be the essential assumption that inevitably yields the strong restriction $\theta_a^\alpha(x_a^\beta, \pi_a^\gamma)$.

Then the general case of interacting particles cannot be described by this Lagrangian predictive framework. However, we shall subsequently see that by dropping assumption (22) (i. e., that x_a^α are canonical) there is a possibility of constructing a Hamiltonian framework where interaction between the particles is permitted.

The situation here in this manifestly covariant formalism is analogous to the following: *a*) a charge whose dynamics must satisfy the Lorentz-Dirac equation [*24*], *b*) an isolated system (manifestly predictive formalism) which admits a Lagrangian formulation compatible with the Poincaré group. In both cases, we arrive at a no-interaction theorem: *a*) the only external electromagnetic fields that can act on the charge must be lineal, and *b*) only free particle systems ($\mu_a^i = 0$) are permitted. The latter is the Currie, Jordan and Sudarshan no-interaction theorem [*23*]. Some proofs of these theorems [*24*] [*26*] [*27*] make known the essential role played by the assumption that the position coordinates are canonical. In both cases, by dropping this assumption one can satisfactorily develop a Hamiltonian formulation in which the position coordinates are *not* canonical [*24*] [*28-30*].

III. TWO CHARGES IN AN EXTERNAL ELECTROMAGNETIC FIELD

III.1. Approximated dynamics.

By adopting the dynamic system (14) (N = 2) (the dynamics satisfying conditions (16) and (17)) to describe the non-isolated system constituted by two interacting structureless point charges and affected by an external electromagnetic field $F_{\alpha\beta}(x^\lambda)$ and assuming that the 4-accelerations ξ_a^α can be expanded into a power series of the two electrical charges e_b of the particles

$$\xi_a^\alpha = \sum_{r,s=0}^{\infty} e_a^r e_{a'}^s \xi_a^{(r,s)\alpha} = \xi_a^{(0,0)\alpha} + e_a \xi_a^{(1,0)\alpha} + e_{a'} \xi_a^{(0,1)\alpha} + e_a^2 \xi_a^{(2,0)\alpha} + e_a e_{a'} \xi_\alpha^{(1,1)\alpha}$$
$$+ e_{a'}^2 \xi_a^{(0,2)\alpha} + e_a^3 \xi_a^{(3,0)\alpha} + e_a^2 e_{a'} \xi_a^{(2,1)\alpha} + e_a e_{a'}^2 \xi_a^{(1,2)\alpha} + e_{a'}^3 \xi_a^{(0,3)\alpha} + \dots \tag{37}$$

where the $\xi_a^{(r,s)\alpha}$ functions are independent of e_b and satisfy

$$\xi_a^{(0,s)\alpha} \equiv 0 \qquad s \geqslant 0, \qquad\qquad \xi_a^{(r,0)\alpha} \equiv \xi_a^{(r)\alpha}(x_a^\beta, u_a^\gamma) \qquad r > 0,$$

we have obtained to order $r + s = 3$, the following [1]

$$\begin{aligned}\xi_a^\alpha = e_a\xi_a^{(1)\alpha} + e_a^2\xi_a^{(2)\alpha} + \xi_a^\alpha = e_a\overset{*}{\xi}{}_a^{(1)\alpha} + e_a^2\xi_a^{(2)\alpha} + e_ae_{a'}\overset{*}{\xi}{}_a^{(1,1)\alpha} \\ + e_ae_{a'}^2\left[\overset{*}{\xi}{}_a^{(1,2)\alpha} - \int_0^{\tau_{a'}} dy \left\{ \overset{(1)}{\xi}{}_{a'}^{\rho} \frac{\partial \overset{*}{\xi}{}_a^{(1,1)\alpha}}{\partial u^{a'\rho}} \right\}(x_a^\beta, x_{a'}^\gamma - yu_{a'}^\gamma, u_b^\lambda)\right] + \ldots . \end{aligned} \tag{38}$$

In this expression, $\overset{*}{\xi}{}_a^{(r,s)\alpha}$ must satisfy

$$\begin{cases} (u_a\overset{*}{\xi}{}_a^{(r,s)}) = 0, \\ \mathrm{D}_{a'}\overset{*}{\xi}{}_a^{(r,s)\alpha} = 0, \qquad \mathrm{D}_{a'} \equiv u_{a'}^\rho \dfrac{\partial}{\partial x^{a'\rho}}, \end{cases} \tag{39}$$

and $\tau_{a'}$ is defined by

$$\tau_{a'} \equiv (x_{aa'}u_{a'}) - \varepsilon r_{a'}, \qquad r_{a'} \equiv +\left[x_{aa'}^2 + (x_{aa'}u_{a'})^2\right]^{1/2}, \quad x_{aa'}^\alpha \equiv x_a^\alpha - x_{a'}^\alpha. \tag{40}$$

III.2. **Causal propagation.**

If $\mathrm{F}_{\alpha\beta}(x^\lambda)$ is the external electromagnetic field acting on the two charges e_a with mass m_a, we shall use the Causality Principle in the following sense: the motion $x_a^\alpha = \varphi_a^\alpha(\tau)$ of the charge e_a must be the solution of the Lorentz motion equations corresponding to the addition of two terms: the first is related to the external electromagnetic field $\mathrm{F}_{\alpha\beta}$ and the second is related to the electromagnetic field whose source is $e_{a'}$ (calculated with the retarded Lienard-Wiechert potentials, $\varepsilon = -1$). The possibility of including a third term of the Lorentz-Dirac type, taking into account radiation effects, is opened by introducing a parameter γ whose value is $\frac{2}{3}$ in this case or 0 when these effects are not included.

This implies the following equations [1]:

$$\frac{dx_a^\alpha}{d\tau} = u_a^\alpha, \qquad \frac{du_a^\alpha}{d\tau} = \mathrm{W}_a^\alpha(x_a^\beta, \hat{x}_{a'}^\gamma, u_a^\rho, \hat{u}_{a'}^\sigma; \xi_a^\lambda, \zeta_a^\mu; \hat{\xi}_{a'}^\delta), \quad \left(\varepsilon = \pm 1; \gamma = 0, \frac{2}{3}\right) \tag{41}$$

with

$$\begin{aligned}\mathrm{W}_a^\alpha \equiv e_am_a^{-1}\mathrm{F}^{\alpha\rho}(x_a^\sigma)u_{a\rho} + \gamma e_a^2m_a^{-1}(\zeta_a^\alpha - \xi_a^2u_a^\alpha) \\ + \varepsilon e_ae_{a'}m_a^{-1}(\hat{x}_{aa'}\hat{u}_{a'})^{-2}\left\{ (u_a\hat{\xi}_{a'})\hat{x}_{aa'}^\alpha - (\hat{x}_{aa'}u_a)\hat{\xi}_{a'}^\alpha + (\hat{x}_{aa'}\hat{u}_{a'})^{-1}[1 + (\hat{x}_{aa'}\hat{\xi}_{a'})] \right. \\ \left. \cdot\ [(\hat{x}_{aa'}u_a)\hat{u}_{a'}^\alpha - (u_a\hat{u}_{a'})\hat{x}_{aa'}^\alpha] \right\} \end{aligned} \tag{42}$$

where $\hat{x}_{aa'}^\alpha \equiv x_a^\alpha - \hat{x}_{a'}^\alpha$, $\hat{x}_{a'}^\alpha$ being the intersection of the future ($\varepsilon = +1$)

or past ($\varepsilon = -1$) cone of vertex x_a^α with the word line of the charge a' : $\hat{u}_{a'}^\alpha$ is the unitary tangent vector to this line on the point $\hat{x}_{a'}^\alpha$; $\hat{\xi}_{a'}^\alpha$ the 4-acceleration on the same point and ξ_a^α, ζ_a^α the 4-acceleration and derivative of the 4-acceleration of the particle a on the point x_a^α. The possibility of using advanced Lienard-Wiechert potentials is opened with the parameter ε; in this case its value is $\varepsilon = +1$.

Equations (41) are not ordinary differential equations but differential-difference equations (with a difference depending on time). However, they can be considered supplementary conditions contributing to the determination of the dynamics of the system given by (38) and (39). The development of the compatibility between Newtonian Causality and Causal Propagation is analogous to the development followed in reference 1 (identical, except for the term regarding Lorentz-Dirac which is not included in said reference). Then, we obtain:

$$\begin{aligned}
\xi_a^\alpha = {} & e_a m_a^{-1} F^{\alpha\rho}(x_a^\sigma) u_{a\rho} + e_a e_{a'} m_a^{-1} r_{a'}^{-3}\left[k x_{aa'}^\alpha + (x_{aa'} u_a) u_{a'}^\alpha\right] \\
& + \gamma e_a^3 m_a^{-2} u_{a\rho} u_{a\nu} \frac{\partial F^{\alpha\rho}}{\partial x_\nu^\alpha}(x_a^\sigma) \\
& + e_a e_{a'}^2 m_a^{-1} m_{a'}^{-1} r_{a'}^{-3} \left\langle \left\{ F^{\rho\sigma}(\tilde{x}_{a'}^\lambda) + 3 r_{a'}^{-2} \int_0^{\tau_{a'}} dy\,[(x_{aa'} u_a) - y]\bar{F}^{\rho\sigma} \right\} x_{aa'\rho} u_{a'\sigma} \right. \\
& \quad . \left[k x_{aa'}^\alpha + (x_{aa'} u_a) u_{a'}^\alpha\right] + \\
& + \left\{ \varepsilon r_{a'} F^{\rho\sigma}(\tilde{x}_{a'}^\lambda)(x_{aa'}^\alpha + \tau_{a'} u_{a'}^\alpha) + \int_0^{\tau_{a'}} dy \bar{F}^{\rho\sigma}(x_{aa'}^\alpha + y u_{a'}^\alpha) \right\} u_{a\rho} u_{a'\sigma} \\
& \left. + \left\{ \varepsilon r_{a'}[k\tau_{a'} - (x_{aa'} u_a)] F^{\alpha\sigma}(\tilde{x}_{a'}^\lambda) + \int_0^{\tau_{a'}} dy [ky - (x_{aa'} u_a)] \bar{F}^{\alpha\sigma} \right\} u_{a'\sigma} \right\rangle
\end{aligned} \tag{43}$$

where $k \equiv -(u_1 u_2)$, $\bar{F}_{\alpha\beta}(x_{a'}^\rho, u_{a'}^\sigma ; y) \equiv F_{\alpha\beta}(x^\rho \to x_{a'}^\rho - y u_{a'}^\rho)$, $\tilde{x}_{a'}^\alpha \equiv x_{a'}^\alpha - \tau_{a'} u_{a'}^\alpha$. We remark that the e_a-term is the typical Lorentz force that acts on a test charge, as no other charge exists. The $e_a e_{a'}$-term represents a charge-charge interaction, as no external field exists. The e_a^3-term does not exists when $\gamma = 0$ and it represents a typical self-interaction when $\gamma = \frac{2}{3}$, as no other charge exists. The $e_a e_{a'}^2$-term represents a field—charge 1—charge 2 interaction. If $e_{a'} = 0$ we obtain from (43):

$$\xi_{a'}^\alpha = 0\,, \qquad \xi_a^\alpha = e_a m_a^{-1} u_{a\rho}\left[F^{\alpha\rho}(x_a^\sigma) + \gamma e_a^2 m_a^{-1} u_{a\nu} \frac{\partial F^{\alpha\rho}}{\partial x_\nu^a}(x_a^\sigma) + \dots \right] \tag{44}$$

i. e., one charge is free and the other is only affected by the external field $F_{\alpha\beta}$ [*24*]. For a more detailed discussion of all terms, as well as the study of particular cases of external fields and the results, see reference 1.

The dynamics, according to (15), is:

$$\left\{\begin{aligned}
&\theta^\alpha_a = e_a\theta_a^{(1)\alpha} + e_a e_{a'}\theta_a^{(1,1)\alpha} + \gamma e_a^3\theta_a^{(3)\alpha} + e_a e_{a'}^2\theta_a^{(1,2)\alpha} + \dots \\
&\theta_a^{(1)\alpha} = \mathrm{F}^{\alpha\rho}(x^\sigma_a)\pi_{a\rho}, \qquad \theta_a^{(1,1)\alpha} = \pi_{a'}^{-1} r_{a'}^{-3}\left[\mathrm{K}x^\alpha_{aa'} + (x_{aa'}\pi_a)\pi^\alpha_{a'}\right], \\
&\theta_a^{(3)\alpha} = \pi_a^{-2}\pi_{a\rho}\pi_{a\nu}\frac{\partial \mathrm{F}^{\alpha\rho}}{\partial \pi^a_\nu}(x^\sigma_a), \\
&\theta_a^{(1,1)\alpha} = \pi_{a'}^{-2} r_{a'}^{-3}\left\langle \left\{ \mathrm{F}^{\rho\sigma}(\tilde{x}^\lambda_{a'}) + 3r_{a'}^{-2}\int_0^{\tau_{a'}} dy\left[\pi_a^{-1}(x_{aa'}\pi_a) - y\right]\bar{\mathrm{F}}^{\rho\sigma}\right\} x_{aa'\rho}\pi_{a'\sigma}\right. \\
&\quad \cdot \left[\mathrm{K}x^\alpha_{aa'} + (x_{aa'}\pi_a)\pi^\alpha_{a'}\right] + \\
&\quad + \left\{ \varepsilon r_{a'}\mathrm{F}^{\rho\sigma}(\tilde{x}^\lambda_{a'})\left[x^\alpha_{aa'} + \tau_{a'}\pi_{a'}^{-1}\pi^\alpha_{a'}\right] + \int_0^{\tau_{a'}} dy\bar{\mathrm{F}}^{\rho\sigma}\left[x^\alpha_{aa'} + y\pi_{a'}^{-1}\pi^\alpha_{a'}\right]\right\} \pi_{a\rho}\pi_{a'\sigma} \\
&\quad + \left.\left\{ \varepsilon r_{a'}\left[\pi_{a'}^{-1}\mathrm{K}\tau_{a'} - (x_{aa'}\pi_a)\right]\mathrm{F}^{\alpha\sigma}(\tilde{x}^\lambda_{a'}) + \int_0^{\tau_{a'}} dy\bar{\mathrm{F}}^{\alpha\sigma}\left[\pi_{a'}^{-1}\mathrm{K}y - (x_{aa'}\pi_a)\right]\right\}\pi_{a'\sigma}\right\rangle
\end{aligned}\right. \tag{45}$$

where

$$\left\{\begin{aligned}
&\mathrm{K} \equiv -(\pi_1\pi_2), \qquad r_{a'} \equiv +\left[x^2_{aa'} + \pi_{a'}^{-2}(x_{aa'}\pi_{a'})^2\right]^{1/2}, \qquad \tau_{a'} \equiv \pi_{a'}^{-1}(x_{aa'}\pi_{a'}) - \varepsilon r_{a'}, \\
&\tilde{x}^\alpha_{a'} \equiv x^\alpha_{a'} - \tau_{a'}\pi_{a'}^{-1}\pi^\alpha_{a'}, \qquad \bar{\mathrm{F}}_{\alpha\beta}(x^\rho_{a'}, \pi^\sigma_{a'}; y) \equiv \mathrm{F}_{\alpha\beta}(x^\rho \to x^\rho_{a'} - y\pi_{a'}^{-1}\pi^\alpha_{a'}).
\end{aligned}\right. \tag{46}$$

IV. A GENERALIZED COVARIANT HAMILTONIAN FORMULATION

Before introducing the generalized covariant Hamiltonian formulation, we will define some previous concepts of great interest and prove a lemma whose use will be subsequently demonstrated.

IV.1. The separability condition.

Let us consider a scalar or tensor function $f(x^\alpha_a, \pi^\beta_b)$ on $(\mathrm{TM}_4)^2$. We shall say that f tends to zero at the infinite past (resp., future) and we shall write

$$\lim_{x\to\nu\infty} f = 0, \qquad \nu = -1 \quad \text{past} \quad (\text{resp. } \nu = +1 \text{ future}) \tag{47.a}$$

if

$$\lim_{\mu\to\nu\infty} f(x^\rho_a + \mu n^\rho_a, \pi^\sigma_b) = 0 \left\{\begin{aligned} &\forall (x^\rho_a, \pi^\sigma_b) \in (\mathrm{TM}_4)^2, \\ &\forall n^\rho_a : n^\rho_1 \neq n^\rho_2, n^2_a \equiv -(n_a n_a) = +1, \qquad 0 < n^0_a < +\infty. \end{aligned}\right. \tag{47.b}$$

Consider a 2-form on $(\mathrm{TM}_4)^2$

$$\sigma = \frac{1}{2}\sigma^{ab}_{\alpha\beta}dx^\alpha_a \wedge dx^\beta_b + \sigma^{ab}_{\alpha\underline{\beta}}dx^\alpha_a \wedge d\pi^\beta_b + \frac{1}{2}\sigma^{ab}_{\underline{\alpha\beta}}d\pi^\alpha_a \wedge d\pi^\beta_b \tag{48.a}$$

where $\sigma^{ab}_{\cdot\cdot}(x^\rho_c, \pi^\sigma_d)$ are functions on $(\mathrm{TM}_4)^2$ and $\sigma^{ab}_{\alpha\beta} = -\sigma^{ba}_{\beta\alpha}$, $\sigma^{ab}_{\underline{\alpha\beta}} = -\sigma^{ba}_{\underline{\beta\alpha}}$.

We shall say that σ is regular in the past (resp. future) if

$$\lim_{x\to v\infty} \sigma = 0 \tag{48.b}$$

in the sense that

$$\lim_{x\to v\infty} \sigma^{ab}_{\alpha\beta} = \lim_{x\to v\infty} \sigma^{ab}_{\underline{\alpha}\beta} = \lim_{x\to v\infty} \sigma^{ab}_{\underline{\alpha}\underline{\beta}} = 0\,, \tag{48.c}$$

i. e., each tensor component of σ with respect to the co-basis $(dx^{\alpha}_{a}, d\pi^{\beta}_{b})$ tends to zero at the infinite past (resp. future).

LEMMA. — *i*) Consider the differential system

$$D_a\Phi = 0\,, \qquad D_a \equiv \pi^{\rho}_{a}\frac{\partial}{\partial x^{\alpha\rho}} \tag{49.a}$$

where Φ is a scalar or tensor function on $(TM_4)^2$. The general solution of the system satisfying the condition

$$\lim_{x\to v\infty} \Phi = 0 \tag{49.b}$$

is $\Phi = 0$.

ii) Consider the differential system

$$\pounds(\vec{H}^{(0,0)}_{a})\sigma = 0\,, \qquad \vec{H}^{(0,0)}_{a} \equiv D_a \tag{50}$$

where σ is a regular 2-form on $(TM_4)^2$ in the past (resp. future). Its general solution is $\sigma = 0$.

Proof. — *i*) (49.*a*) is a linear homogeneous system whose general solution is an arbitrary function or a tensor of 14 independent variables, for example, $(h^{i}_{a}, \pi^{\sigma}_{b})$

$$h^{i}_{a} \equiv x^{i}_{a} - z_a\pi^{i}_{a}\,, \qquad z_a \equiv \eta_a\Lambda^{-2}\left[\pi^{2}_{a'}(x\pi_a) - K(x\pi_{a'})\right]$$
$$\left(x^{\rho} \equiv x^{\rho}_{1} - x^{\rho}_{2},\ \eta_a = \eta_1 = +1,\ \eta_2 = -1\right)$$

because $D_a z_b = \delta_{ab}D_a h^{i}_{b} = 0$. We shall then write $\Phi(h^{i}_{a}, \pi^{\sigma}_{b})$. By imposing the limit condition (49.*b*) we get $\Phi \equiv 0$ and thus, the first part of the lemma is proven.

ii) Adopting the general form (48.*a*) for σ, we have

$$\pounds(\vec{H}^{(0,0)}_{c})\sigma = \frac{1}{2}(D_c\sigma^{ab}_{\alpha\beta})dx^{\alpha}_{a}\wedge dx^{\beta}_{b} + (D_c\sigma^{ab}_{\alpha\underline{\beta}} + \delta^{b}\sigma^{ab}_{\alpha\beta})dx^{\alpha}_{a}\wedge d\pi^{\beta}_{b}$$
$$+ \frac{1}{2}(D_c\sigma^{ab}_{\underline{\alpha}\underline{\beta}} + \delta^{a}_{c}\sigma^{ab}_{\alpha\underline{\beta}} - \delta^{b}_{a}\sigma^{ba}_{\beta\underline{\alpha}})d\pi^{\alpha}_{a}\wedge d\pi^{\beta}_{b}$$

and the differential system (50) is explicitly

$$\begin{cases} D_c\sigma^{ab}_{\alpha\beta} = 0 \\ D_c\sigma^{db}_{\alpha\underline{\beta}} = -\,\delta^{b}_{c}\sigma^{ab}_{\alpha\beta} \\ D_c\sigma^{ab}_{\underline{\alpha}\underline{\beta}} = -\,\delta^{a}_{c}\sigma^{ab}_{\alpha\underline{\beta}} + \delta^{b}_{c}\sigma^{ba}_{\beta\underline{\alpha}} \end{cases}$$

Moreover, as σ is regular, we can apply the first part of the lemma repeatedly and conclude that $\sigma = 0$. Thus, the lemma has been proven.

IV.2. The hamiltonian form up to the third order.

The Hamiltonian formulation that we shall adopt is characterized by the following fundamental properties: a symplectic form Ω exists on $(TM_4)^2$, such that

i) It is invariant under the predictive group

$$\pounds(\vec{H}_a)\Omega = 0 . \tag{51}$$

ii) Ω can be expanded into a power series of the two charges e_a, $e_{a'}$

$$\Omega = \sum_{r,s=0}^{\infty} e_a^r e_{a'}^s \Omega^{(r,s)} \tag{52}$$

where $\Omega^{(r,s)}$ is independent of e_b.

iii) Ω satisfies the limit condition

$$\lim_{x\to\vee\infty} (\Omega - \sigma + \varepsilon^a e_a F_a) = 0 \begin{cases} \sigma = dx_a^\alpha \wedge d\pi_\alpha^a , \\ F_a = \frac{1}{2} F_{\alpha\beta}(x_a^\rho) dx_a^\alpha \wedge dx_a^\beta \end{cases} \tag{53}$$

where $F_{\alpha\beta}(x^\rho)$ is the external electromagnetic field.

We have clearly dropped the assumption that the position variables x_a^α are canonical, because if this assumption is not dropped we would be dealing with assumptions implying no-interaction theorem. This assumption is substituted by two regularity conditions: (52), which expresses that the Ω tensor components are regular functions of charges e_b and (53), which expresses that the 2-form

$$\Omega - \{ dx_1^\varrho \wedge d[\pi_{1\alpha} + e_1 A_\alpha(x_1^\rho)] + dx_2^\alpha \wedge d[\pi_{2\alpha} + e_2 A_\alpha(x_2^\varrho)] \} ,$$

A_α being the electromagnetic 4-potential, is regular in the past (resp. future). This latter assumption is based on the following; when we consider a single e affected by an external electromagnetic field $F_{\alpha\beta}(x^\rho)$, the symplectic form $\Delta = dx^\alpha \wedge d(\pi_\alpha + eA_\alpha)$ can be adopted when the evolution is governed by the Lorentz equation [*24*].

Taking into account that Ω is a closed 2-form, (51) yields the existence (almost locally) of two generating functions H_a on $(TM_4)^2$, such that

$$i(\vec{H}_a)\Omega = - dH_a .$$

On the other hand, as Ω is a symplectic form (according to the Darboux theorem [19]) coordinates (q^α_a, p^b_β) exist such that Ω can be written in the form

$$\Omega = dq^\alpha_a \wedge dp^a_\alpha .$$

Then we can equivalently express (54) as follows

$$£(\vec{H}_b)q^\alpha_a = -\frac{\partial H_b}{\partial p^a_\alpha}, \qquad £(\vec{H}_b)p^a_\alpha = \frac{\partial H_b}{\partial q^\alpha_a}.$$

These equations recall, because of their form, the Hamilton equations; but in this case there are two generalized covariant generating functions, H_a and $q^\alpha_a \neq x^\alpha_a$. We shall call this formulation a generalized covariant Hamiltonian formulation; generalized, because there are two generating functions H_a that we shall call the « Hamiltonians » when there is no possibility of confusion, and covariant because we are dealing with M_4 as the geometric framework. The relation between the covariant formulation and the predictive one can be seen in Appendix B.

Next we will prove that Ω which verifies properties (51)-(53), exists and is unique. In the past case (resp. future) we shall call Ω the Hamiltonian form in the past and write $\Omega_{(-1)}$ (resp. the Hamiltonian form in the future and write $\Omega_{(+1)}$). In general, $\Omega_{(-1)} \neq \Omega_{(+1)}$, but if they are equal we shall say that the system is conservative.

By introducing the developments of $\vec{H}_a$ and Ω into (51), we obtain up to order $r + s = 3$:

$$\left\{\begin{array}{l} \vec{H}_a = \vec{H}^{(0,0)}_a + e_a\vec{H}^{(1,0)}_a + e_a e_{a'}\vec{H}^{(1,1)}_a + \gamma e^3_a\vec{H}^{(3,9)}_a + e_a e^2_{a'}\vec{H}^{(1,2)}_a + \dots \\ \vec{H}^{(0,0)}_a \equiv D_a = \pi^\rho_a \dfrac{\partial}{\partial x^{a\rho}}, \qquad \vec{H}^{(r,s)}_a \equiv \theta^{(r,s)\rho}_a \dfrac{\partial}{\partial \pi^{a\sigma}}, \end{array}\right.$$

$$£(\vec{H}^{(0,0)}_a)\Omega^{(0,0)} = 0, \tag{54}$$

$$£(\vec{H}^{(0,0)}_a)\Omega^{(1,0)} = -£(\vec{H}^{(1,0)}_a)\Omega^{(0,0)}, \qquad £(\vec{H}^{(0,0)}_{a'})\Omega^{(1,0)} = 0, \tag{55}$$

$$£(\vec{H}^{(0,0)}_a)\Omega^{(0,1)} = 0, \qquad £(\vec{H}^{(0,0)}_{a'})\Omega^{(0,1)} = -£(\vec{H}^{(1,0)}_{a'})\Omega^{(0,0)}, \tag{56}$$

$$£(\vec{H}^{(0,0)}_a)\Omega^{(2,0)} = -£(\vec{H}^{(1,0)}_a)\Omega^{(1,0)}, \qquad £(\vec{H}^{(0,0)}_{a'})\Omega^{(2,0)} = 0, \tag{57}$$

$$£(\vec{H}^{(0,0)}_a)\Omega^{(0,2)} = 0, \qquad £(\vec{H}^{(0,0)}_{a'})\Omega^{(0,2)} = -£(\vec{H}^{(1,0)}_{a'})\Omega^{(0,1)}, \tag{58}$$

$$\left\{\begin{array}{l} £(\vec{H}^{(0,0)}_a)\Omega^{(1,1)} = -£(\vec{H}^{(1,1)}_a)\Omega^{(0,0)} - £(\vec{H}^{(1,0)}_a)\Omega^{(0,1)}, \\ £(\vec{H}^{(0,0)}_{a'})\Omega^{(1,1)} = -£(\vec{H}^{(1,1)}_{a'})\Omega^{(0,0)} - £(\vec{H}^{(1,0)}_{a'})\Omega^{(1,0)}, \end{array}\right. \tag{59}$$

$$\left\{\begin{array}{l} £(\vec{H}^{(0,0)}_a)\Omega^{(3,0)} = -\gamma£(\vec{H}^{(3,0)}_a)\Omega^{(0,0)} - £(\vec{H}^{(1,0)}_a)\Omega^{(2,0)}, \\ £(\vec{H}^{(0,0)}_{a'})\Omega^{(3,0)} = 0, \end{array}\right. \tag{60}$$

$$\left\{\begin{array}{l} £(\vec{H}^{(0,0)}_a)\Omega^{(0,3)} = 0, \\ £(\vec{H}^{(0,0)}_{a'})\Omega^{(0,3)} = -\gamma£(\vec{H}^{(3,0)}_{a'})\Omega^{(0,0)} - £(\vec{H}^{(1,0)}_{a'})\Omega^{(0,2)}, \end{array}\right. \tag{61}$$

$$\left\{\begin{array}{l} £(\vec{H}^{(0,0)}_a)\Omega^{(1,2)} = -£(\vec{H}^{(1,2)}_a)\Omega^{(0,0)} - £(\vec{H}^{(1,0)}_a)\Omega^{(0,2)} - £(\vec{H}^{(1,1)}_a)\Omega^{(0,1)}, \\ £(\vec{H}^{(0,0)}_{a'})\Omega^{(1,2)} = -£(\vec{H}^{(1,0)}_{a'})\Omega^{(1,1)} - £(\vec{H}^{(1,1)}_{a'})\Omega^{(0,1)}, \end{array}\right. \tag{62}$$

$$\left\{\begin{array}{l} £(\vec{H}^{(0,0)}_a)\Omega^{(2,1)} = -£(\vec{H}^{(1,0)}_a)\Omega^{(1,1)} - £(\vec{H}^{(1,1)}_a)\Omega^{(1,0)}, \\ £(\vec{H}^{(0,0)}_{a'})\Omega^{(2,1)} = -£(\vec{H}^{(1,2)}_{a'})\Omega^{(0,0)} - £(\vec{H}^{(1,0)}_{a'})\Omega^{(2,0)} - £(\vec{H}^{(1,1)}_{a'})\Omega^{(1,0)}. \end{array}\right. \tag{63}$$

Moreover, condition (53) implies that

$$\lim_{x\to v\infty} (\Omega^{(0,0)} - \sigma) = 0\,, \qquad \sigma \equiv dx_a^\alpha \wedge d\pi_\alpha^a\,, \tag{64}$$

$$\lim_{x\to v\infty} (\Omega^{(1,0)} + \mathrm{F}_a) = \lim_{x\to v\infty} (\Omega^{(0,1)} + \mathrm{F}_{a'}) = 0\,, \tag{65}$$

$$\lim_{x\to v\infty} \Omega^{(r,s)} = 0\,, \qquad \forall (r,s) \neq (0,0)\,,\ (0,1)\,,\ (1,0)\,. \tag{66}$$

ORDER (0, 0)

Taking into account the structure for σ, we deduce

$$\pounds(\vec{\mathrm{H}}_a^{(0,0)})(\Omega^{(0,0)} - \sigma) = 0$$

that together with condition (64) furnishes, by applying the lemma,

$$\Omega^{(0,0)} = \sigma\,. \tag{67}$$

We expected this result because if $e_1 = e_2 = 0$, there is no interaction between the particles and the external field does not act on each particle.

ORDERS (1, 0) AND (0, 1)

By defining $\mathrm{J} \equiv \Omega^{(1,0)} + \mathrm{F}_a$, and taking into account (55), we get

$$\pounds(\vec{\mathrm{H}}_a^{0,0)})\mathrm{J} = 0\,, \qquad \pounds(\vec{\mathrm{H}}_{a'}^{(0,0)})\mathrm{J} = 0\,.$$

On the other hand, (55) is now written as

$$\lim_{x\to v\infty} \mathrm{J} = 0\,.$$

By applying the lemma, we obtain $\mathrm{J} = 0$, i. e.,

$$\Omega^{(1,0)} = -\,\mathrm{F}_a\,. \tag{68}$$

Analogously, it can be calculated that

$$\Omega^{(0,1)} = -\,\mathrm{F}_{a'}\,. \tag{69}$$

ORDERS (2, 0) AND (0, 2)

Taking into account result (68), we obtain for (57)

$$\pounds(\vec{\mathrm{H}}_a^{(0,0)})\Omega^{(2,0)} = \pounds(\vec{\mathrm{H}}_a^{(1,0)})\mathrm{F}_a = 0\,,$$

that together with the limit condition (66), yields by application of the lemma

$$\Omega^{(2,0)} = 0\,. \tag{70.a}$$

Analogously, it can be calculated that

$$\Omega^{(0,2)} = 0\,. \tag{70.b}$$

ORDERS (3, 0) AND (0, 3)

By defining $\mathrm{K} \equiv \Omega^{(3,0)} - \gamma\delta_{a(v)}$ with

$$\delta_{a(v)}(x_a^\rho,\, \pi_a^\sigma) \equiv \delta_{(v)}(x^\rho \to x_a^\rho,\, \pi^\rho \to \pi_a^\rho)$$

where $\delta_{(v)}$ is the 2-form given by

$$\delta_v \equiv dx^\alpha \wedge d(\pi^{-2}\pi^\rho F_{\alpha\rho}) + d\left[\pi^{-2}\pi^\rho \int_{v\infty}^{0} dy F_{\alpha\rho}(x^\sigma + y\pi^\sigma)\right] \wedge d\pi^\alpha \qquad (71)$$

and taking into account (60) and the preceding results (67) and (70), we obtain

$$£(\vec{H}_a^{(0,0)})K = £(\vec{H}_{a'}^{(0,0)})K = 0 .$$

On the other hand, we can prove that $\lim_{x\to v\infty} \delta_{a(v)} = 0$ so that $\lim_{x\to v\infty} K = 0$. By applying the lemma, we conclude that $K = 0$, i. e.,

$$\Omega^{(3,0)} = \gamma\delta_{a(v)} . \qquad (72)$$

Analogously, it can be calculated that

$$\Omega^{(0,3)} = \gamma\delta_{a'(v)} . \qquad (73)$$

Summing up, taking into account the preceding results (67)-(73), equations (59), (62) and (63) can be written:

$$£(\vec{H}_c^{(0,0)})\Omega^{(1,1)} = -£(\vec{H}_c^{(1,1)})\Omega^{(0,0)}, \qquad (74)$$

$$£(\vec{H}_a^{(0,0)})\Omega^{(1,2)} = -£(\vec{H}_a^{(1,2)})\Omega^{(0,0)}, \qquad £(\vec{H}_{a'}^{(0,0)})\Omega^{(1,2)} = -£(\vec{H}_{a'}^{(1,0)})\Omega^{(1,1)}, \qquad (75)$$

$$£(\vec{H}_a^{(0,0)})\Omega^{(2,1)} = -£(\vec{H}_a^{(1,0)})\Omega^{(1,1)}, \qquad £(\vec{H}_{a'}^{(0,0)})\Omega^{(2,1)} = -£(\vec{H}_{a'}^{(1,2)})\Omega^{(0,0)}. \qquad (76)$$

These equations, together with the limit conditions

$$\lim_{x\to v\infty} \Omega^{(1,1)} = \lim_{x\to v\infty} \Omega^{(1,2)} = \lim_{x\to v\infty} \Omega^{(2,1)} = 0 \qquad (77)$$

must be verified by the 2-forms $\Omega^{(1,1)}$, $\Omega^{(1,2)}$ and $\Omega^{(2,1)}$, respectively.

ORDER (1, 1)

From (74) we decuce:

$$£(\vec{H}_c^{(0,0)})\Omega^{(1,1)} = -\,\delta_c^a dx_a^\alpha \wedge d\theta_\alpha^{(1,1)a} . \qquad (78)$$

Let us consider the 2-form $\rho_{(v)}$ defined by

$$\rho_{(v)} \equiv dx_a^\alpha \wedge dp_{\alpha(v)}^{(1,1)a} + dq_{a(v)}^{(1,1)\alpha} \wedge d\pi_\alpha^a \qquad (79.a)$$

$p_{a(v)}^{(1,1)\alpha}$ and $q_{a(v)}^{(1,1)\alpha}$ being the following functions defined on $(TM_4)^2$

$$\left\{\begin{aligned} p_{a(v)}^{(1,1)\alpha} &\equiv -\int_{v\infty}^{0} dy\theta_a^{(1,1)\alpha}(x_b^\rho + y\pi_b^\rho, \pi_c^\sigma), \\ q_{a(v)}^{(1,1)\alpha} &\equiv \int_{-z_a}^{0} dy p_{a(v)}^{(1,1)\alpha}(x_b^\rho + y\pi_b^\rho, \pi_c^\sigma), \\ z_a &= \eta_a\Lambda^{-2}[\pi_{a'}^2(x\pi_a) - K(x\pi_{a'})], \qquad \Lambda \equiv (K^2 - 1)^{1/2} . \end{aligned}\right. \qquad (79.b)$$

Next, we summarize some properties of functions $p_{a(v)}^{(1,1)\alpha}$ and $q_{a(v)}^{(1,1)\alpha}$:

i) Taking into account that $D_c z_a = \delta_{ca}$, $D_{a'}\theta_a^{(1,1)\alpha} = 0$ and $\lim_{x\to v\infty} \theta_a^{(1,1)\alpha} = 0$, we obtain

$$D_c p_{\alpha(v)}^{(1,1)a} = -\delta_c^a \theta_\alpha^{(1,1)a}, \qquad D_c q_{a(v)}^{(1,1)\alpha} = \delta_{ac} p_{a(v)}^{(1,1)\alpha}. \tag{80.a}$$

ii) By introducing the vectors on $(TM_4)^2$

$$h^\alpha \equiv x^\alpha - z_1\pi_1^\alpha + z_2\pi_2^\alpha, \qquad t_a^\alpha \equiv \pi_{a'}^2\pi_a^\alpha - K\pi_{a'}^\alpha$$

we can easily calculate the integrals (79.*b*), thus furnishing

$$\begin{cases} p_{a(v)}^{(1,1)\alpha} = -\left[\Lambda^{-1}h^{-2}K(\Lambda\pi_{a'}^{-1}z_a r_{a'}^{-1} - v)\eta_a h^\alpha + \Lambda^{-2}r_{a'}^{-1}\pi_{a'}t_{a'}^\alpha\right] \\ q_{a(v)}^{(1,1)\alpha} = -\{\Lambda^{-1}h^{-2}K[\Lambda^{-1}\pi_{a'}(r_{a'} - h) - vz_a]\eta_a h^\alpha \\ \qquad\qquad + \Lambda^{-3}\pi_{a'}^2 \ln\left[h^{-1}(r_{a'} + \Lambda\pi_{a'}^{-1}z_a)\right]t_{a'}^\alpha\} \end{cases} \tag{80.b}$$

where $h \equiv +(hh)^{1/2}$.

These functions, p's and q's, are a particular case from those obtained in ref. 30 for the problems of an isolated system constituted by two charges (to first order in the coupling constant $g \equiv e_1e_2$).

iii) Taking into account expressions (80.*b*), we can deduce

$$\lim_{x\to v\infty} p_{a(v)}^{(1,1)\alpha} = \lim_{x\to v\infty} [x_{aa'}^2]^{-1/2} q_{a(v)}^{(1,1)\alpha} = 0 \tag{80.c}$$

It can also be proven that:

$$\lim_{x\to v\infty}\left(\frac{\partial p_{\alpha(v)}^{(1,1)a}}{\partial x_b^\beta} - \frac{\partial p_{\beta(v)}^{(1,1)b}}{\partial x_a^\alpha}\right) = \lim_{x\to v\infty}\left(\frac{\partial p_{\alpha(v)}^{(1,1)a}}{\partial \pi_b^\beta} + \frac{\partial q_{\beta(v)}^{(1,1)b}}{\partial x_a^\alpha}\right)$$
$$= \lim_{x\to v\infty}\left(\frac{\partial q_{\alpha(v)}^{(1,1)a}}{\partial \pi_b^\beta} - \frac{\partial q_{\beta(v)}^{(1,1)b}}{\partial \pi_a^\alpha}\right) = 0 \tag{80.d}$$

and taking into account the definition of $\rho_{(v)}$, (79), we can conclude the following limit condition

$$\lim_{x\to v\infty} \rho_{(v)} = 0. \tag{80.e}$$

Now by defining $J \equiv \Omega^{(1,1)} - \rho_{(v)}$ and taking into account (79.*a*), (80.*a*) and (80.*e*), we obtain

$$\pounds(\vec{H}_c^{(0,0)})J = 0, \qquad \lim_{x\to v\infty} J = 0.$$

Thus, by applying the lemma, we conclude that $J \equiv 0$, i. e.,

$$\Omega^{(1,1)} = \rho_{(v)}. \tag{81}$$

ORDERS (1, 2) AND (2, 1)

$\Omega^{(1,2)}$ must satisfy (75), i. e.,

$$\pounds(\vec{H}_a^{(0,0)})\Omega^{(1,2)} = -dx_a^\alpha \wedge d\theta_\alpha^{(1,2)a}, \qquad E_{a'} \equiv \theta_{a'}^{(1)\rho}\frac{\partial}{\partial \pi^{a'\rho}},$$
$$\pounds(\vec{H}_{a'}^{(0,0)})\Omega^{(1,2)} = -\left[dx_b^\alpha \wedge d(E_{a'}p_{\alpha(v)}^{(1,1)b}) + d(E_{a'}q_{b(v)}^{(1,1)\alpha}) \wedge d\pi_\alpha^b + dq_{a'(v)}^{(1,1)\alpha} \wedge d\theta_{a'\alpha}^{(1)}\right]. \tag{82}$$

Let us consider the 2-form $\Gamma_{a(v)}$ defined by:

$$\Gamma_{a(v)} = dq^{(1,1)\rho}_{a'(v)} \wedge dA_{a'\rho} + dx^\rho_a \wedge dp^{(1,2)}_{a(v)} + dq^{(1,2)\rho}_{a(v)} \wedge d\pi_{a\rho} + dx^\rho_{a'} \wedge dp^{(2,1)}_{a'\rho(v)} + dq^{(2,1)\rho}_{a'(v)} \wedge d\pi_{a'\rho} \quad (83.a)$$

$p^{(1,2)\alpha}_{a(v)}$, $q^{(1,2)\alpha}_{a(v)}$, $p^{(2,1)\alpha}_{a(v)}$ and $q^{(2,1)\alpha}_{a(v)}$ being the following functions defined on $(TM_4)^2$

$$\left\{\begin{aligned} p^{(1,2)\alpha}_{a(v)} &\equiv -\int_{v\infty}^{0} dy \left[\theta^{(1,2)\alpha}_a + E_{a'} p^{(1,1)\alpha}_{a(v)} - \frac{\partial(\pi_{a'}A_{a'})}{\partial x^{a'\rho}} \frac{\partial q^{(1,1)\rho}_{a'(v)}}{\partial x^a_\alpha}\right](x^\rho_b + y\pi^\rho_b, \pi^\sigma_c), \\ q^{(1,2)\alpha}_{a(v)} &\equiv -\int_{-z_a}^{0} dy \left[p^{(1,2)\alpha}_{a(v)} - E_{a'} q^{(1,1)\alpha}_{a(v)} + \frac{\partial(\pi_{a'}A_{a'})}{\partial x^{a'\rho}} \frac{\partial q^{(1,1)\rho}_{a'(v)}}{\partial \pi^a_\alpha}\right](x^\rho_b + y\pi^\rho_b, \pi^\sigma_c), \end{aligned}\right. \quad (83.b)$$

$$\left\{\begin{aligned} p^{(2,1)\alpha}_{a(v)} &\equiv -\int_{v\infty}^{0} dy \left[E_a p^{(1,1)\alpha}_{a(v)} + \frac{\partial(\pi_a A_a)}{\partial x^{a\rho}} \frac{\partial q^{(1,1)\rho}_{a(v)}}{\partial x^a_\alpha} - \frac{\partial A_{a\rho}}{\partial x^a_\alpha} p^{(1,1)\rho}_{a(v)}\right](x^\rho_b + y\pi^\rho_b, \pi^\sigma_c), \\ q^{(2,1)\alpha}_{a(v)} &\equiv \int_{-z_a}^{0} dy \left[p^{(2,1)\alpha}_{a(v)} - E_a q^{(1,1)\alpha}_{a(v)} + \frac{\partial(\pi_a A_a)}{\partial x^{a\rho}} \frac{\partial q^{(1,1)\rho}_{a(v)}}{\partial \pi^a_\alpha}\right](x^\rho_b + y\pi^\rho_b, \pi^\sigma_c), \end{aligned}\right. \quad (83.c)$$

where $A^\alpha_a(x^\rho_a) \equiv A^\alpha(x^\rho_a)$, A^α is the electromagnetic 4-potential, and $p^{(1,1)\alpha}_{a(v)}$, $q^{(1,1)\alpha}_{a(v)}$ are given by (79.*b*). We shall assume that $A^\alpha(x^\rho)$ satisfies the necessary conditions in order to warrant the existence of the preceding integrals and implying the following limit condition on $\Gamma_{a(v)}$

$$\lim_{x\to v\infty} {}^{*}\Gamma_{a(v)} = 0. \quad (84)$$

Then it can be verified that the p's and q's satisfy

$$\left\{\begin{aligned} &D_a p^{(1,2)\alpha}_{a(v)} = -\theta^{(1,2)\alpha}_a, \qquad D_{a'} p^{(1,2)\alpha}_{a(v)} = -E_{a'} p^{(1,1)\alpha}_{a(v)} - \frac{\partial(\pi_{a'}A_{a'})}{\partial x^{a'\rho}} \frac{\partial q^{(1,1)\rho}_{a'(v)}}{\partial x^a_\alpha}, \\ &D_a q^{(1,2)\alpha}_{a(v)} = p^{(1,2)\alpha}_{a(v)}, \qquad D_{a'} q^{(1,2)\alpha}_{a(v)} = -E_{a'} q^{(1,1)\alpha}_{a(v)} + \frac{\partial(\pi_{a'}A_{a'})}{\partial x^{a'\rho}} \frac{\partial q^{(1,1)\rho}_{a'(v)}}{\partial \pi^a_\alpha}, \end{aligned}\right. \quad (85.a)$$

$$\left\{\begin{aligned} &D_a p^{(2,1)\alpha}_{a(v)} \\ &\quad = -E_a p^{(1,1)\alpha}_{a(v)} - \frac{\partial(\pi_a A_a)}{\partial x^{a\rho}} \frac{\partial q^{(1,1)\rho}_{a(v)}}{\partial x^a_\alpha} + \frac{\partial A_{a\rho}}{\partial x^a_\alpha} p^{(1,1)\rho}_{a(v)}, \qquad D_{a'} p^{(2,1)\alpha}_{a(v)} = 0, \\ &D_a q^{(2,1)\alpha}_{a(v)} \\ &\quad = p^{(2,1)\alpha}_{a(v)} - E_a q^{(1,1)\alpha}_{a(v)} + \frac{\partial(\pi_a A_a)}{\partial x^{a\rho}} \frac{\partial q^{(1,1)\rho}_{a(v)}}{\partial \pi^a_\alpha}, \qquad D_{a'} q^{(2,1)\alpha}_{a(v)} = 0. \end{aligned}\right. \quad (85.b)$$

By defining $M \equiv \Omega^{(1,2)} - \Gamma_{a(v)}$ and taking into account (82), (83.*a*), (85.*a*, *b*) and (84), we obtain

$$\pounds(\vec{H}^{(0,0)}_c)M = 0, \qquad \lim_{x\to v\infty} M = 0.$$

Thus, by applying the lemma we conclude that $M \equiv 0$, i. e.,

$$\Omega^{(1,2)} = \Gamma_{a(v)}. \quad (86.a)$$

Analogously, it can be calculated that

$$\Omega^{(2,1)} = \Gamma_{a'(v)}\,. \tag{86.b}$$

$\Gamma_{a'(v)}$ having been obtained from $\Gamma_{a(v)}$, when the $a \leftrightarrow a'$ exchange is carried out.

Summing up, we have proven that the Hamiltonian form exists in the past (resp. future) and is unique to order $r + s = 3$ (if the electromagnetic external field and the 4-potential verify certain weak conditions) and is given by

$$\Omega_{(v)} = \varepsilon^a \left\{ \frac{1}{2}\sigma - e_a \mathrm{F}_a + \frac{1}{2} e_a e_{a'} \rho_{(v)} - \gamma e_a^3 \delta_{a(v)} + e_a e_{a'}^2 \Gamma_{a(v)} + \dots \right\} \tag{87}$$

where σ, F_a, $\rho_{(v)}$, $\delta_{a(v)}$ and $\Gamma_{a(v)}$ are given by (53), (79), (71) and (83), respectively.

We remark that $\Omega_{(v)}$ is S_2-invariant (S_2 being the permutation group of two elements)

$$\Omega_{(v)}\left(x_a^\rho, x_{a'}^\sigma, \pi_a^\lambda, \pi_{a'}^\mu ; e_a, e_{a'}\right) = \Omega_{(v)}\left(x_{a'}^\rho, x_a^\sigma, \pi_{a'}^\lambda, \pi_a^\mu ; e_{a'}, e_a\right). \tag{88}$$

We expect this result because the dynamic system is constituted by 2 particles of the same type.

The system is not conservative $\Omega_{(-1)} \neq \Omega_{(+1)}$, because

$$\Omega_{(+1)} - \Omega_{(-1)} = e_1 e_2 d\,(\ln h^2) \wedge d(\mathrm{K}\Lambda^{-1}) + \mathrm{O}(e_1^r e_2^s)\,, \qquad r+s>2\,. \tag{89}$$

This result, appearing to this order, is tied to the long range character of the electromagnetic interaction [*30*].

Obviously, if $\mathrm{F}_{\alpha\beta} \equiv 0$

$$\Omega_{(v)} = g\rho_{(v)} + \mathrm{O}(g^2)\,, \qquad g \equiv e_1 e_2\,, \tag{90}$$

which is the result obtained in ref. [*30*] for the isolated system constituted by two charges in electromagnetic interaction. If $e_{a'} = 0$

$$\left\{ \begin{aligned} &\Omega_{(v)} = dx_{a'}^\alpha \wedge d\pi_{a'\alpha} + \Delta_{(v)}\left(x_a^\rho, \pi_a^\sigma ; e_a\right), \\ &\qquad\qquad \Delta_{(v)}\left(x^\rho, \pi^\sigma ; e\right) \equiv \Omega^{(0)} - e\mathrm{F} - \gamma e^3 \mathrm{G}_{(v)} + \mathrm{O}(e^4) \\ &\Omega^{(0)} \equiv dx^\alpha \wedge d\pi_\alpha, \qquad \mathrm{F} \equiv \frac{1}{2}\mathrm{F}_{\alpha\beta}(x^\rho) dx^\alpha \wedge dx^\beta, \\ &\mathrm{G}_{(v)} \equiv dx^\alpha \wedge d(\pi^{-2}\pi^\rho \mathrm{F}_{\alpha\rho}) + d\left[\pi^{-2}\pi^\rho \int_{v\infty}^{0} dy \mathrm{F}_{\alpha\rho}(x^\lambda + y\pi^\lambda)\right] \wedge d\pi^\alpha. \end{aligned} \right. \tag{91}$$

$\Delta_{(v)}$ coincide with the 2-form obtained in ref. [*24*] for the case of a single charge affected by an external field.

On the other hand, although the convergence of the series obtained by this method remains to be proven, it can be proven (at least, formally) that the Hamiltonian form, if it exists at any order, is unique. For this, we shall suppose that two symplectic forms, Ω_1 and Ω_2, exist such that properties (51)-(53) are verified. It is clear that $\Xi \equiv \Omega_1 - \Omega_2$ must satisfy

$$\pounds(\vec{H}_a)\Xi = 0 \tag{92.a}$$

$$\Xi = \sum_{r,s=0}^{\infty} e_a^r e_{a'}^s \Xi^{(r,s)} \tag{92.b}$$

$$\lim_{x\to v\infty} \Xi = 0. \tag{92.c}$$

By introducing the developments for $\vec{H}_a$ and Ξ into (92.*a*) and proceeding inductively, i. e., supposing $\Xi^{(r,s)} \equiv 0 \ \forall (r, s) : r + s < n$, we arrive at

$$\pounds(\vec{H}_a^{(0,0)})\Xi^{(p,q)} = 0\,, \qquad \forall (p, q) : p + q = n\,.$$

Then taking into account the limit condition (92.*c*), we obtain $\Xi^{(p,q)} = 0$, $p + q = n$, and it is therefore proven that if the Hamiltonian form exists in the past (resp. future), it is unique.

IV.3. Canonical coordinates and « Hamiltonians ».

Beginning with result (87), it can be directly proven up to $r + s = 3$ order that the H_a functions associated to the $\vec{H}_a$ generators of the predictive group, which we have called the Hamiltonians, making a language abuse, can be written as

$$H_a = \frac{1}{2}\pi_a^2 + O(e_1^r e_2^s)\,, \qquad r + s > 3\,. \tag{93}$$

On the other hand, $\Omega_{(v)}$ given by (87), can be expressed in the form

$$\begin{aligned}\Omega_{(v)} = d\big[x_a^\rho + e_a e_{a'} q_{a(v)}^{(1,1)\rho} + e_a^3 q_{a(v)}^{(3)\rho} + e_a e_{a'}^2 q_{a(v)}^{(1,2)\rho} + e_a^2 e_{a'} q_{a(v)}^{(2,1)\rho} + \dots\big] \\ \wedge\, d\big[\pi_\rho^a + e_a A_\rho^a + e_a e_{a'} p_{\rho(v)}^{(1,1)a} + e_a^3 p_\rho^{(3)a} + e_a e_{a'}^2 p_{\rho(v)}^{(1,2)a} + e_a^2 e_{a'} p_{\rho(v)}^{(2,1)a} + \dots\big]\end{aligned} \tag{94}$$

where $q_{a(v)}^{(1,1)\alpha}$ and $p_{a(v)}^{(1,1)\alpha}$ are given by (80.*b*); $q_{a(v)}^{(1,2)\alpha}$ and $p_{a(v)}^{(1,2)\alpha}$ by (83.*b*); $q_{a(v)}^{(2,1)\alpha}$ and $p_{a(v)}^{(2,1)\alpha}$ by (83.*c*); and $q_{a(v)}^{(3)\alpha}$ and $p_{a(v)}^{(3)\alpha}$ by

$$q_{a(v)}^{(3)\alpha} = -\gamma\pi_a^{-2}\pi_{a\rho}\int_{v\infty}^{0} dy F^{\alpha\rho}(x_a^\sigma + y\pi_a^\sigma), \qquad p_a^{(3)\alpha} = -\gamma\pi_a^{-2}\pi_{a\rho}F^{\alpha\rho}(x_a^\sigma). \tag{95}$$

Then, there exist canonical coordinates $(q_{a(v)}^\alpha, p_{a(v)}^\alpha)$ of $\Omega_{(v)}$

$$(\text{i. e.,} \qquad \Omega_{(v)} = dq_{a(v)}^\alpha \wedge dp_{\alpha(v)}^a)$$

having the form

$$\left\{\begin{aligned} q_{a(v)}^\alpha &= x_a^\alpha + e_a e_{a'} q_{a(v)}^{(1,1)\alpha} + e_a^3 q_{a(v)}^{(3)\alpha} + e_a e_{a'}^2 q_{a(v)}^{(1,2)\alpha} + e_a^2 e_{a'} q_{a(v)}^{(2,1)\alpha} + \dots \\ p_{a(v)}^\alpha &= \pi_a^\alpha + e_a A^\alpha(x_a^\rho) + e_a e_{a'} p_{a(v)}^{(1,1)\alpha} + e_a^3 p_a^{(3)\alpha} + e_a e_{a'}^2 p_{a(v)}^{(1,2)\alpha} + e_a^2 e_{a'} p_{a(v)}^{(2,1)\alpha} + \dots, \end{aligned}\right. \tag{96}$$

defined, except for the canonical transformation that has the form

$$
\begin{cases}
A_\alpha(x_a^\rho) \to A_\alpha(x_a^\rho) + \dfrac{\partial S}{\partial x_a^\alpha}(x_a^\sigma), \\
q_{a(v)}^{(1,1)\alpha} \to q_{a(v)}^{(1,1)\alpha} + \dfrac{\partial R}{\partial \pi_\alpha^a}(x_b^\beta, \pi_c^\gamma), \qquad p_{a(v)}^{(1,1)\alpha} \to p^{(1,1)\alpha}_{a(v)} - \dfrac{\partial R}{\partial x_\alpha^a}, \\
q_{a(v)}^{(3)v} \to q_{a(v)}^{(3)v} + \dfrac{\partial T}{\partial \pi_\alpha^a}(x_b^\beta, \pi_c^\gamma), \qquad p_a^{(3)\alpha} \to p_a^{(3)\alpha} - \dfrac{\partial T}{\partial x_\alpha^a}, \\
q_{a(v)}^{(1,2)\alpha} \to q_{a(v)}^{(1,2)\alpha} + Q_a^\alpha(x_b^\beta, \pi_c^\gamma), \qquad p_{a(v)}^{(1,2)\alpha} \to p_{a(v)}^{(1,2)\alpha} + P_a^\alpha(x_b^\beta, \pi_c^\gamma), \\
q_{a(v)}^{(2,1)\alpha} \to q_{a(v)}^{(2,1)\alpha} + \bar{Q}_a^\alpha(x_b^\beta, \pi_c^\gamma), \qquad p_{a(v)}^{(2,1)\alpha} \to p_{a(v)}^{(2,1)\alpha} + \bar{P}_a^\alpha(x_b^\beta, \pi_c^\gamma),
\end{cases}
\tag{97}
$$

where S, R, T, are arbitrary functions of their arguments and $(Q_a^\alpha, P_b^\beta, \bar{Q}_a^\alpha, \bar{P}_b^\beta)$ is the general solution of the following exterior system

$$
\begin{cases}
dx_a^\rho \wedge d(P_\rho^a + \bar{P}_\rho^a) + d(Q_a^\rho + \bar{Q}_a^\rho) \wedge d\pi_\rho^a \\
\qquad + dq_{a(v)}^{(1,1)\rho} \wedge d\left(\dfrac{\partial S}{\partial x_\rho^a}\right) + d\left(\dfrac{\partial R}{\partial \pi_a^\rho}\right) \wedge dA_a^\rho + d\left(\dfrac{\partial R}{\partial \pi_a^\rho}\right) \wedge d\left(\dfrac{\partial S}{\partial x_\rho^a}\right) = 0 \\
dx_a^\rho \wedge dP_{a\rho} + dQ_a^\rho \wedge d\pi_{a\rho} + dx_{a'}^\rho \wedge d\bar{P}_{a'\rho} + d\bar{Q}_{a'}^\rho \wedge d\pi_{a'\rho} \\
\quad + dq_{a'(v)}^{(1,1)\rho} \wedge d\left(\dfrac{\partial S}{\partial x^{a'\rho}}\right) + d\left(\dfrac{\partial R}{\partial \pi^{a'\rho}}\right) \wedge dA_{a'}^\rho + d\left(\dfrac{\partial R}{\partial \pi^{a'\rho}}\right) \wedge d\left(\dfrac{\partial S}{\partial x_\rho^{a'}}\right) = 0 .
\end{cases}
\tag{98}
$$

By choosing the canonical coordinates (96), we can express the Hamiltonians H_a as functions of these coordinates. This can be carried out by reversing developments (96), and the result is

$$
[H_a(q_b^\beta, p_c^\gamma) = -\frac{1}{2}[p_a^\rho - e_a A^\rho(q_a^\sigma)][p_{a\rho} - e_a A_\rho] - e_a^2 e_{a'} \underset{\sim}{q}_{a(v)}^{(1,1)\rho} p_a^\sigma \frac{\partial A_\sigma}{\partial q^{a\rho}} + \dots \tag{99}
$$

where the functions $\underset{\sim}{q}_{a(v)}^{(1,1)\alpha}(q_b^\beta, p_c^\sigma) \equiv q_{a(v)}^{(1,1)\alpha}(x_b^\rho \to q_b^\rho, \pi_c^\sigma \to p_c^\sigma)$ and

$$A^\alpha(q_a^\rho) \equiv A^\alpha(x^\rho \to q_a^\rho).$$

Finally, we remark two different difficulties in the construction of a Quantum Theory of said non-isolated systems: *i*) the position variables are not canonical coordinates, and *ii*) the arbitrariness for the canonical coordinates in (97) is clear. Therefore, it is necessary to establish a selection principle in order to restrict the coordinates, unless the « quantification » makes all the canonical coordinates equivalent.

APPENDIX

This Appendix is a simple generalization of the Appendix developed in ref. [*24*].

A. THE LAGRANGIAN FORMULATION IN THE TWO FORMALISMS

A.1. The (usual) Lagrangian formulation in the predictive formalism.

Given the dynamic system on $\mathbb{R}^{3N}$

$$\frac{dx_a^i}{dt} = v_a^i, \qquad \frac{dv_a^i}{dt} = \mu_a^i(t, x_b^j, v_c^k), \tag{100}$$

it can be considered equivalent to the vector field on $\mathbb{R} \times (T\mathbb{R}^3)^N$

$$\vec{F} \equiv \frac{\partial}{\partial t} + v_b^j \frac{\partial}{\partial x_b^j} + \mu_b^j \frac{\partial}{\partial v_b^j}. \tag{101}$$

$\vec{F}$ is said to be a (regular) Lagrangian dynamic system if a function $\mathscr{L}(t, x_a^i, v_b^j)$ exists on $\mathbb{R} \times (T\mathbb{R}^3)^N$, satisfying $\det\left(\frac{\partial^2 \mathscr{L}}{\partial v_a^i \partial v_b^j}\right) \neq 0$ and such that (100) is equivalent to the following system

$$\pounds(\vec{F}) \frac{\partial \mathscr{L}}{\partial v_a^i} - \frac{\partial \mathscr{L}}{\partial x_a^i} = 0 \qquad \text{(Lagrange equations)} \tag{102}$$

On the other hand, as is well-known in mathematics and physics literature (cf., for example, R. Abraham [*19*], chapt. IV; C. Godbillon [*20*], chapt. VII; Y. Choquet-Bruhat [*32*], p. 294; J. L. Sanz [*33*]) the following equivalence theorem can be proven:

i) If we are dealing with a (regular) Lagrangian dynamic system on $\mathbb{R}^{3N}$ of type (100), we can construct a presymplectic [*34*] form ω on $\mathbb{R} \times (T\mathbb{R}^3)^N$, possessing the following properties

$$\begin{cases} \omega \wedge dt \wedge dx_1^1 \wedge \ldots \wedge dx_N^3 = 0, & (103.a) \\ i(\vec{F})\omega = 0. & (103.b) \end{cases}$$

Clearly, ω has the structure $\omega = -dt \wedge d\mathscr{H} + dx_a^i \wedge d\left(\frac{\partial \mathscr{L}}{\partial v_a^i}\right)$, $\mathscr{H}$ being the Hamiltonian

$$\mathscr{H}(t, x_a^i, p_j^b) = \underset{*}{v}{}_a^i p_i^e - \mathscr{L}(t, x_a^i, \underset{*}{v}{}_b^j)$$

where $\underset{*}{v}{}_a^i(t, x_b^j, p_k^c)$ are the functions obtained by inverting $p_i^a = \dfrac{\partial \mathscr{L}}{\partial v_a^i}$.

ii) Given a dynamic system on $\mathbb{R}^{3N}$ of type (100) and a presymplectic form on $\mathbb{R} \times (T\mathbb{R}^3)^N$, satisfying (103), $\vec{F}$ is then a Lagrangian dynamic system.

Thus, the following: « $\vec{F}$ is said to be a Lagrangian dynamic system if a presymplectic form on $\mathbb{R} \times (T\mathbb{R}^3)^N$ satisfying (103) exists » can be considered an alternative definition to the classical and usual definition of a Lagrangian dynamic system in the predictive formalism.

A.2. The Lagrangian formulation in the covariant formalism.

We have mentioned in Section II that an alternative to the classical definition of a Lagrangian dynamic system on $(M_4)^N$ is the following: « $\vec{H}$ is said to be a (regular) Lagrangian dynamic system if a symplectic form on $(TM_4)^N$ satisfying (22) and (23) exists ».

A.3. The predictive Lagrangian formulation induced by a given covariant Lagrangian one.

We shall assume that a given $\vec{H}$ on $(TM_4)^N$ is a predictive Lagrangian dynamic system, i. e., that conditions (22) and (24) are verified. Moreover, we shall suppose that the H_a functions associated to $\vec{H}_a$ generators by $i(\vec{H}_a)\Omega = -dH_a$, are precisely arbitrary functions of π_b

$$H_a = H_a(\pi_b). \tag{104}$$

On the other hand, let us consider a function $f(x_a^\alpha, \pi_b^\beta)$ on $(TM_4)^N$ and define $\bar{f}$

$$\bar{f}(t, x_a^i, v_b^j; m_c) = f[x_a^0 \equiv t, x_b^i, \pi_c^0 \equiv m_c(1 - v_c^2)^{-1/2}, \pi_d^j \equiv m_d(1 - v_d^2)^{-1/2} v_d^j] \tag{105}$$

that is, the restriction of f over the hyper-surfaces $\pi_b = m_b$. For each m_b given, the corresponding hyper-surface can be identified with $\mathbb{R} \times (T\mathbb{R}^3)^N$. The following properties can be verified:

i) $\overline{df} = d\bar{f}$, d being the exterior differential operator.

ii) $\varepsilon^a m_a^{-1}(1 - v_a^2)^{1/2}\overline{\pounds(\vec{H}_a)f} = \pounds(\vec{F})\bar{f}$, $\vec{F}$ being the vector field on $\mathbb{R} \times (T\mathbb{R}^3)^N$ induced by $\vec{H}$ through (13), i. e.,

$$\mu_a^i = (1 - v_a^2)(\delta_j^i - v_a^i v_{aj})\bar{\xi}_a^j.$$

Then, it is very easy to prove that the restriction $\bar{\Omega}$ of the two-form Ω on the hyper-surface $\pi_b = m_b$ possesses the following properties:

i) $\bar{\Omega}$ is a presymplectic form on $\mathbb{R} \times (T\mathbb{R}^3)^N$,

ii) $i(\vec{F})\bar{\Omega} = 0$,

iii) $\bar{\Omega} \wedge dt \wedge dx_1^1 \wedge \ldots \wedge dx_N^3 = 0$.

Thus, if we are dealing with a Lagrangian dynamic system $\vec{H}$ on $(TM_4)^N$, verifying (104), the induced dynamic system $\vec{F}$ on $\mathbb{R} \times (T\mathbb{R}^3)^N$ is a Lagrangian dynamic system in the usual sense.

Moreover, as Ω can be written

$$\Omega = dx_a^\alpha \wedge dp_\alpha^a, \tag{106.a}$$

where p_α^a are functions on $(TM_4)^N$, the restriction is

$$\bar{\Omega} = -dt \wedge d(\varepsilon^a \bar{p}_a^0) + dx_a^i \wedge d\bar{p}_i^a \tag{106.b}$$

and the function $\mathscr{H} \equiv \varepsilon^a \bar{p}_a^0$ can be identified as the Hamiltonian associated to $\vec{F}$.

The Lagrangian formulation can be used to describe the system constituted by N-non-interacting charges affected by an external electromagnetic field $F_{\alpha\beta}(x^\rho)$.

B. A GENERALIZED HAMILTONIAN FORMULATION IN THE TWO FORMALISMS

Let us consider $\underline{N}$ vector fields $\vec{H}_a$ on $(TM_4)^N$ and a symplectic form Ω on $(TM_4)^N$, such that (51) are verified. Moreover, we shall assume that the generating functions H_a associated to $\vec{H}_a$ satisfy (104). By using the restriction operation over the hyper-surface $\pi_b = m_b$

(as defined in Appendix A.3), we can conclude that the restriction $\bar{\Omega}$ of the 2- form Ω on this hyper-surface possesses the property

$$i(\vec{F})\bar{\Omega} = 0, \tag{107}$$

F being the dynamic system on $\mathbb{R} \times (T\mathbb{R}^3)^N$ induced by $\vec{H}$ by means of (13). Moreover, as Ω can be written (according to the Darboux theorem):

$$\Omega = dq^\alpha_a \wedge dp^a_\alpha, \tag{108.a}$$

where q^α_a, $p^b_\beta(x^\rho_c, \pi^\sigma_d)$ are functions on $(TM_4)^N$, the restriction is

$$\bar{\Omega} = d\bar{q}^\alpha_a \wedge d\bar{p}^a_\alpha. \tag{108.b}$$

It is clear that in this case, there is no rule to identify a non-covariant Hamiltonian $\mathscr{H}$ in the predictive formalism because of the general form (108.*b*). We remark that in the Lagrangian case this rule is given by the canonical character of the position variables x^i_a.

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[3] $\alpha, \beta, \gamma, \rho, \sigma, \ldots = 0, 1, 2, 3$; $i, j, k, \ldots = 1, 2, 3$; $a, b, c, a', \ldots = 1, \ldots, N$; a' is different from a, $a' \neq a$; all indices follow the summation convention; M_4 denotes the Minkowski space and $\eta_{\alpha\beta}$: $\eta_{00} = -1$, $\eta_{ij} = \delta_{ij}$, $\eta_{0i} = 0$, its metric tensor; $(ab) \equiv a^\rho b_\rho$, $(\vec{a}\,\vec{b}) \equiv a^i b_i$; the velocity of light in vacuum will be $c = 1$.
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[15] L. BEL, *Ann. Inst. H. Poincaré*, t. **14**, 1971, p. 189.
[16] L. BEL, *Doctoral course, Departamento de Fisica Teórica*, Universidad de Barcelona, 1976.
[17] TM_4 is the M_4 tangent vector bundle (since M_4 is flat, T_*M_4 can be identified with TM_4).
[18] $\pounds(\vec{A})$ is the Lie derivative associated to $\vec{A}$ and $[\vec{A}, \vec{B}] \equiv \pounds(\vec{A})\vec{B}$ denote the Lie bracket of the two vector fields $\vec{A}$ and $\vec{B}$.
[19] R. ABRAHAM, *Foundations of Mechanics*, W. A. Benjamin, 1967.
[20] C. GODBILLON, *Géométrie diferentielle et Mécanique Analitique*, Hermann, 1969.
[21] L. BEL, *Doctoral Course*, Universidad Autónoma de Madrid, 1972, unpublished.
[22] J. MARTIN and J. L. SANZ, to appear in *J. Math. Phys.*, Section 2.

[23] Let us consider a symplectic form on $(TM_4)^N$ with local expression

$$\Omega = \frac{1}{2}\Omega_{AB}dy^A \wedge dy^B \qquad (A, B = 0, \ldots 8N - 1 ; y^\alpha = x_1^\alpha, \ldots, y^{4(N-1)+\alpha} = x_N^\alpha, y^{4N+\alpha} = \pi_1^\alpha, \ldots, y^{4(2N-1)+\alpha} = \pi_N^\alpha)$$

where Ω_{AB} are skewsymmetric functions on $(TM_4)^N$. The Poisson bracket of two functions f and g on $(TM_4)^N$ is defined by

$$[f, g] = - \Omega^{-1AB}\frac{\partial f}{\partial y^A}\frac{\partial g}{\partial y^B}$$

where Ω^{-1AB} is the inverse matrix of Ω_{AB} (i. e., $\Omega^{-1AB}\Omega_{BC} = \delta_C^A$). As is well-known in the literature (see, for example, L. BEL, *Ann. Inst. H. Poincaré*, t. **18 A**, 1973, p. 57; H. P. KUNZLE, *Symposia Mathematica*, t. **14**, 1974, p. 53; *J. Math. Phys.*, t. **15**, 1974, p. 1033) condition (22) can be equivalently written in the form $[x_a^\alpha, x_b^\beta] = 0$ ([,] being the Poisson bracket relative to Ω), which is the classical form of expressing the canonical character of the position variables x_a^α.

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[26] R. N. HILL, *J. Math. Phys.*, t. **8**, 1967, p. 1756.

[27] J. MARTIN and J. L. SANZ, *J. Math. Phys.*, t. **19**, 1978, p. 780.

[28] D. HIRONDEL, *J. Math. Phys.*, t. **15**, 1974, p. 1689.

[29] F. J. KENNEDY, *J. Math. Phys.*, t. **16**, 1975, p. 1844.

[30] L. BEL and J. MARTIN, *Ann. Inst. H. Poincaré*, *22 A*, 1957, p. 173.

[31] We shall assume the necessary conditions on the field $F_{\alpha\beta}(x^\rho)$ in order to warrant the existence of the integral $\Phi_{\alpha\beta(\nu)}(x^\rho) \equiv \int_{\nu\infty}^0 dy F_{\alpha\beta}(x^\sigma + y\pi^\sigma)$. Also, we shall assume the weak conditions $\lim_{x\to\nu\infty} \partial_\rho F_{\alpha\beta} = \lim_{x\to\nu\infty} \Phi_{\alpha\beta(\nu)} = \lim_{x\to\nu\infty} \partial_\rho \Phi_{\alpha\beta(\nu)} = 0$.

[32] Y. CHOQUET-BRUHAT, *Géométrie différentielle et systèmes extérieures*, Dunod, 1968.

[33] J. L. SANZ, *Tesis Doctoral*, Universidad Autónoma de Madrid, 1976.

[34] A presymplectic form is a two-form on a variety of odd dimension that is closed and has maximum rank.

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