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On the computation of bounds for low energy Compton scattering parameters: proof of a conjecture

by

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ABSTRACT. — A general method proposed by Raszillier [1] to obtain constraints on low energy Compton scattering parameters in terms of upper bounds on the cross sections above photoproduction threshold, is shown to give optimal results.

1. INTRODUCTION

Recently, a method has been proposed by Raszillier [1] to solve an extremum problem which arises in the derivation of dispersion inequalities for Compton scattering [1, 2].

This method should lead to an improvement of previous bounds [2] on low energy scattering parameters as functions of the fixed transfer cross section above photoproduction threshold. The results appear in the form of the so-called « inner » and « outer » approximations to the optimal bounds. The purpose of the present paper is to show that the « outer approximation » gives in fact the optimal bounds.

As the problem is of general interest, we shall not reproduce here its physical background, which can be found in ref. [1, 2]. We merely state

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it in its reduced mathematical form as presented in [1], i.e. after suitable changes of functions (= amplitudes) and conformal mapping (of the energy plane) have been made.

Let \( S \) be the class of complex vector-valued functions
\[
\tilde{w}(z) = (w_1(z), \ldots, w_n(z)),
\]
analytic in the unit disc \(| z | < 1\), such that:

\[
(1.1) \quad \sum_{i=1}^{n} |w_i(e^{i\theta})|^2 \leq 1
\]

The \( w_i(z) \)'s are assumed to be « real analytic »: \( w_i(z^*) = w_i^*(z) \). Given \( n \) real points \( x_i (i = 1, \ldots, n) \) inside the unit disc, consider the set of (real) values:

\[
\mathbf{W} = \{ W_\alpha \}_{\alpha = 1, \ldots, N} = \{ w_1(x_1), w_1'(x_1), \ldots, w_{(1)}(x_1); \ldots; w_n(x_n), w_n'(x_n), \ldots, w_{n^{(k)}}(x_n) \}
\]

where \( w_i^{(k)}(x_i) \) stands for \( \frac{d^k w_i(z)}{dz^k} \bigg|_{z = x_i} \) and \( N = \sum_{i=1}^{n} (k_i + 1) \).

The problem is to determine the range \( D \) of \( \mathbf{W} \) in \( \mathbb{R}^N \) when \( \tilde{w}(z) \) varies over the whole of \( S \).

Notice that in the special case \( n = 1 \), this is a well known problem which is easily solved by standard interpolation theory. To deal with the general case, Raszillier introduces an « outer approximation » to \( D \) in the following way.

For any positive function \( \rho(\theta) \) subjected to the normalization condition:

\[
(1.3) \quad \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \rho(\theta) = 1,
\]

one defines the class \( S_\rho \) of \( \tilde{w}(z) \)'s such that:

\[
(1.4) \quad \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \rho(\theta) \sum_{i=1}^{n} |w_i(e^{i\theta})|^2 \leq 1
\]

and the corresponding region \( D_\rho \) in \( \mathbb{R}^N \) (= range of \( \mathbf{W} \) when \( \tilde{w}(z) \) varies over \( S_\rho \)). Clearly, the class \( S \) is included in \( S_\rho \), so that \( D \) is contained in \( D_\rho \). The « outer approximation » \( D_0 \supset D \) is obtained by taking the intersection of all \( D_\rho \)'s:

\[
(1.5) \quad D_0 = \bigcap_{\rho} D_\rho
\]

where \( \rho \) ranges over the class of weight functions normalized according to eq. (1.3).
The interest of this construction lies in the fact that the regions $D_\rho$ can be determined explicitly by using interpolation theory in the Hardy space $H^2$, as explained in [1]. Moreover, in the three examples worked out in that paper, the outer approximations $D_0$ turn out to coincide with the exact regions $D$. Whether the equality $D_0 = D$ is a general property was left undecided however (although implicitly conjectured). We show here that this is true indeed, namely that:

\[(1.6) \quad D = \bigcap_\rho D_\rho \]

The key of the proof, which is given in Section 3, is the use of duality formulae in suitably defined Banach spaces of analytic functions. As a preparatory step, these spaces are defined in the next section, where the problem is also slightly generalized and reformulated in a way allowing the application of the duality argument.

2. PRELIMINARIES

For complex vector-valued function defined on the unit circle, we define the Banach spaces (on the real field):

\[(2.1) \quad \tilde{L}^p = \{ \tilde{f}(\theta) = (f_1(\theta), \ldots, f_n(\theta)) | f_i(\theta) \in L^p, f_i^*(-\theta) = f_i(-\theta) \} \]

equipped with the norms:

\[(2.2) \quad ||\tilde{f}||_\infty = \text{Ess. sup}_{-\pi < \theta < \pi} |\tilde{f}(\theta)| \quad \text{for} \quad p = \infty \]

and:

\[(2.3) \quad ||\tilde{f}||_p = \left[ \int_{-\pi}^\pi |\tilde{f}(\theta)|^p \frac{d\theta}{2\pi} \right]^{1/p} \quad \text{for} \quad 1 \leq p < \infty \]

(only the values $p = \infty$, 1 and 2 will be used).

Here, $|\tilde{f}(\theta)|$ stands for $\left[ \sum_{i=1}^n |f_i(\theta)|^2 \right]^{1/2}$.

Similarly, starting from the usual Hardy spaces $H^p$ [3], we define the vectorial Hardy spaces:

\[(2.4) \quad \tilde{H}^p = \{ \tilde{w}(z) = (w_1(z), \ldots, w_n(z)) | w_i(z) \in H^p, w_i^*(z) = w_i(z^*) \} \quad (p = \infty, 1 \text{ or } 2) \]

equipped with the norms $||\tilde{w}||_p$ above ($\tilde{w}(z)$ is identified with its boundary value $\tilde{w}(e^{i\theta}) \in L^p$), so that the Banach space $\tilde{H}^p$ is a (closed) subspace of $L^p$. 

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Given a positive function $\rho(\theta) = \rho(-\theta)$ such that $\log \rho(\theta) \in L^1$, let us introduce the outer function:

$$G(z) = \exp \left( \frac{1}{2} \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log \rho(\theta) \right).$$

Then:

$$|G(e^{i\theta})| = \sqrt{\rho(\theta)} \quad (\text{a.e.})$$

We shall say that $\tilde{w}(z) \in \tilde{H}_\rho^2$ if and only if $\tilde{h}(z) = G(z)\tilde{w}(z) \in \tilde{H}^2$. The correspondence $w \leftrightarrow \tilde{h}$ then establishes an isometric isomorphism between $H_\rho^2$ and $\tilde{H}^2$ if the norm in $H_\rho^\ast$ is taken as:

$$||\tilde{w}||_{\rho,\ast} = \left[ \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \rho(\theta) |\tilde{w}(e^{i\theta})|^2 \right]^{\frac{1}{2}}$$

The sets $S$ and $S_\rho$ now appear as the unit balls in $\tilde{H}_\rho^{\infty}$ and $H_\rho^\ast$ respectively:

$$S = \{ \tilde{w} \in \tilde{H}_\rho^{\infty} | ||\tilde{w}||_\rho^{\infty} \leq 1 \}$$

$$S_\rho = \{ \tilde{w} \in H_\rho^\ast | ||\tilde{w}||_{\rho,\ast} \leq 1 \}$$

and the sets $D$ and $D_\rho$ are the images of $S$ and $S_\rho$ in $\mathbb{R}^N$ through the linear mapping defined by eq. (1.2). Using the appropriate Cauchy representations of $w^{(k)}(x_i)$ in $H_\rho^{\infty}$ of $H_\rho^\ast$ [3], it is immediately seen that this mapping is continuous. As a consequence, the sets $D$ and $D_\rho$ are symmetric, convex and bounded. They are also closed. Let us sketch the proof of this last property (for $D$).

Since the space $\tilde{L}_\rho^{\infty}$ can be identified with the dual of $\tilde{L}^1$, the unit ball in $\tilde{L}_\rho^{\infty}$ is compact in the weak-* topology $\sigma(\tilde{L}_\rho^{\infty}, \tilde{L}^1)$ from the Alaoglu theorem [4]. Thus $S$, which is the intersection of this unit ball with the weak-* closed subspace $\tilde{H}_\rho^{\infty}$ is also weak-* compact. Now the functionals on $\tilde{H}_\rho^{\infty}$ defined by $\tilde{w} \rightarrow w^{(k)}(x_i)$ can be represented by Cauchy kernels, which belong to $\tilde{L}^1$. Therefore, the image of $S$ through the mapping defined by eq. (1.2) is itself compact in $\mathbb{R}^N$. The proof for $D_\rho$ is even simpler, because $\tilde{H}_{\rho}^{\infty}$ is a reflexive Banach space.

For the application of the duality principle we have in mind, it is necessary to replace the vector-valued functional

$$\tilde{w}(z) \rightarrow \tilde{W} = \tilde{A}(\tilde{w})$$

defined by eq. (1.2) by scalar ones. To this end, we shall consider $D$ and $D_\rho$ as intersections of symmetrical strips. To each unit vector $\tilde{a} \in \mathbb{R}^N$, we attach the functional $\Delta_\tilde{a}$ defined by the scalar product:

$$\Delta_\tilde{a} = \tilde{a} \tilde{A} \quad \text{(i.e.} \Delta_\tilde{a}(\tilde{w}) = \sum_{z=1}^{N} a_z W_z \text{)}$$
and the corresponding strips:

\( (2.11') \quad B_a = \{ \bar{U} \in \mathbb{R}^N | |\bar{a}\bar{U}| \leq \sup_{\bar{w} \in S} |\Delta_a(\bar{w})| \} \)

\( (2.11'') \quad B_{a,\rho} = \{ \bar{U} \in \mathbb{R}^N | |\bar{a}\bar{U}| \leq \sup_{\bar{w} \in S, \rho} |\Delta_a(\bar{w})| \} \)

Notice that the Sup involved in these definitions are in fact attained, for the same reasons which made \( D \) and \( D_\rho \) closed sets. Now, since \( D \) and \( D_\rho \) are convex and closed, we can write:

\( (2.12) \quad D = \bigcap_{\bar{a}} B_a, \quad D_\rho = \bigcap_{\bar{a}} B_{a,\rho} \)

where \( \bar{a} \) runs over the whole unit sphere of \( \mathbb{R}^N \). Therefore, in order to prove the conjecture (1.6), it is enough to show that, for any \( \bar{a} \):

\( (2.13) \quad B_a = \bigcap_{\rho} B_{a,\rho} \)

which in turn is equivalent to:

\( (2.14) \quad \sup_{\bar{w} \in S} |\Delta_a(\bar{w})| = \inf \sup_{\rho} \sup_{\bar{w} \in S, \rho} |\Delta_a(\bar{w})| . \)

As \( S \subset S_\rho \), the l. h. s. of this equation is certainly not greater than the r. h. s. Thus eq. (2.14) will be established if we can construct a sequence of functions \( \rho(\theta) \) such that \( \sup_{\bar{w} \in S, \rho} |\Delta_a(\bar{w})| \) converges to \( \sup_{\bar{w} \in S} |\Delta_a(\bar{w})| \).

3. PROOF

A direct proof of eq. (2.14) does not seem to be easy because one has to deal with spaces of analytic functions. We shall circumvent this difficulty by making use of duality formulae, the general form of which is:

\( (3.1) \quad \sup_{w \in T} |\phi(w)| = \min_{\psi \in T^*} \|\phi + \psi\|_{X^*} \)

where \( X \) is any Banach space, \( T \subset X \) a closed subspace, \( \phi \) a continuous functional, and \( T^\perp \subset X^* \) the annihilator of \( T \) in the dual space \( X^* \) (\( T^\perp = \{ \psi \in X^* | \psi(w) = 0 \ \forall w \in T \} \)). The notation Min specifies that the infimum is necessarily attained. To use eq. (3.1) in eq. (2.14), we want to identify the couple \( (X, T) \) successively with \( (L^2, H^2) \) and \( (L^\infty, H^\infty) \), and \( \phi \) with the corresponding functionals \( \Delta_a(\bar{w}) (w \in H^\infty) \) and

\( \Delta_{a,\rho}(\bar{h}) = \Delta_a(\bar{h}/G) \quad (\bar{h} \in H^2) \).

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One has to be careful on two points however:

i) the dual of $\mathbb{L}^\infty$ does not coincide with $\mathbb{L}^1((\mathbb{L}^\infty)^* \ni \mathbb{L}^1)$,

ii) the kernels which appear in explicit integral representations of the functional $\Delta_a$ are not necessarily the same according as $\Delta_a$ is supposed to act in $\mathbb{H}^\infty$ or in $\mathbb{H}_p^2$. When acting in $\mathbb{H}^\infty$, $\Delta_a$ will be for the moment only assumed to admit the representation:

\[
\begin{align*}
\Delta_a(\hat{w}) &= \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \hat{A}_a^*(\theta) \hat{w}(e^{i\theta}) \quad \forall \hat{w} \in \mathbb{H}^\infty, \\
\bar{\Delta}_a(\theta) &\in \mathbb{L}^1
\end{align*}
\]

(3.2)

On the other hand, a representation valid in $\mathbb{H}^2_p$ simply results from the well-known representation theorem in $\mathbb{H}^2$ [3] via the isomorphism between $\mathbb{H}^2$ and $\mathbb{H}^2_p$:

\[
\begin{align*}
\Delta_a(\hat{w}) &= \Delta_{a,\rho}(\hat{h}) = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \hat{A}_a^*\rho(\theta) G(e^{i\theta}) \hat{w}(e^{i\theta}) \quad \forall \hat{w} = \frac{\hat{h}}{G} \in \mathbb{H}^2_p \\
\bar{\Delta}_{a,\rho}(\theta) &\in \mathbb{L}^2.
\end{align*}
\]

(3.3)

The representations (3.2) and (3.3) also define natural extensions of the functionals $\Delta_a$ and $\Delta_{a,\rho}$ to the spaces $\mathbb{L}^\infty$ and $\mathbb{L}^2$ respectively. Let us first apply eq. (3.1) to:

\[
\text{Sup} \quad \Delta_a(\hat{w}) = \text{Sup} \quad |\Delta_{a,\rho}(\hat{h})| \\
||\hat{h}||_2 \leq 1
\]

(3.4)

We need a characterization of the annihilator $\mathbb{H}^{2\perp}$ of $\mathbb{H}^2$ in the dual of $\mathbb{L}^2$ (which can be identified with $\mathbb{L}^2$ itself). According to a theorem known for $\mathbb{H}^2$ [3] and readily generalized to $\mathbb{H}^2$:

\[
\psi \in \mathbb{H}^{2\perp} \iff \psi(\bar{v}) = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{i\theta}\bar{v}(e^{i\theta})\bar{f}(\theta) \quad \text{with} \quad \bar{v} \in \mathbb{H}^2
\]

(3.5)

\[
\text{clearly} \quad \psi(\hat{w}) = \int_{|z|=1} dz \bar{v}(z)\hat{w}(z) = 0 \quad \text{for} \quad \hat{w} \in \mathbb{H}^2
\]

Hence, from eqs. (3.3)-(3.5), and noticing that the canonical norm of a functional as element of the dual $(\mathbb{L}^2)^*$ coincides with the $\mathbb{L}^2$-norm (2.3) of its kernel, we obtain:

\[
\text{Sup} \quad |\Delta_a(\hat{w})| = \text{Min} \quad ||\bar{A}_a^*\rho(\theta) + e^{i\theta}\bar{v}(e^{i\theta})||_{\mathbb{L}^2}
\]

(3.6)

\[
\text{Min} \quad \left[ \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \frac{d\rho}{2\pi} \frac{|\Delta_{a,\rho}(\theta) + e^{i\theta}\bar{v}(e^{i\theta})|^2}{G^2} \right]^{\frac{1}{2}}
\]

\[\text{This is the norm of an element of} \quad \mathbb{L}_o^2 \text{of dimension}
\]

\[\text{the isomorphism between} \quad \mathbb{H}^2 \text{and} \quad \mathbb{H}^2_p.
\]
Similarly, would it be true that \((\mathcal{L}^\infty)^* = \mathcal{L}^1\), a blind application of eq. (3.1) to \(\text{Sup}_{\vec{w} \in \mathcal{S}} |\Delta_{\vec{a}}(\vec{w})|\) would give:

\[
(3.7) \quad \text{Sup}_{\vec{w} \in \mathcal{S}} |\Delta_{\vec{a}}(\vec{w})| = \text{Min}_{\vec{a} \in \mathcal{H}^1} \left\{ \int_{-\pi}^{\pi} \left| \frac{d\theta}{2\pi} |\tilde{\Delta}_a^*(\theta) + e^{i\theta} \tilde{u}(e^{i\theta})| \right| \right\},
\]

Although this « naive » formula cannot be deduced directly from eq. (3.1), it turns out to be correct as long as the functional \(\Delta_a\) admits in \(\mathcal{H}^\infty\) a representation of the form (3.2). It is actually the (easy) extension to \(\mathcal{H}^\infty\) of an analogous duality formula known for \(\mathcal{H}^\infty\). For completeness, we give the full proof in Appendix.

Now, let \(\tilde{u}_0\) be an element of \(\mathcal{H}^1\) for which the infimum is attained in the r. h. s. of eq. (3.7):

\[
(3.8) \quad \text{Sup}_{\vec{w} \in \mathcal{S}} |\Delta_{\vec{a}}(\vec{w})| = I \equiv \int_{-\pi}^{\pi} \left| \frac{d\theta}{2\pi} |\tilde{\Delta}_a^*(\theta) + e^{i\theta} \tilde{u}_0(e^{i\theta})| \right|,
\]

and suppose that the kernels of the representations (3.2) and (3.3) can be identified:

\[
(3.9) \quad \tilde{\Delta}_a(\theta)G^*\left(e^{i\theta}\right) \equiv \tilde{\Delta}_a(\theta)
\]

By comparing eqs. (3.6) and (3.8), we see that if we were allowed to choose \(\rho(\theta)\) and \(\tilde{v}(z)\) as follows:

\[
(3.10) \quad \begin{cases}
\rho(\theta) = |G(e^{i\theta})|^2 = \left| \frac{\tilde{\Delta}_a^*(\theta) + e^{i\theta} \tilde{u}_0(e^{i\theta})}{\int_{-\pi}^{\pi} \frac{d\theta}{2\pi} |\tilde{\Delta}_a^*(\theta) + e^{i\theta} \tilde{u}_0(e^{i\theta})|} \right|, \\
\tilde{v}(z) = \frac{u_0(z)}{G(z)}
\end{cases}
\]

then:

\[
(3.11) \quad \left[ \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} |\tilde{\Delta}_a^*(\theta) + e^{i\theta} \tilde{v}(e^{i\theta})|^2 \right]^{\frac{1}{2}} = \left[ \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} |\tilde{\Delta}_a^*(\theta) + e^{i\theta} \tilde{u}_0(e^{i\theta})|^2 \right]^{\frac{1}{2}} = I
\]

and the proof of eq. (2.14) would be completed.

This simple argument has to be refined however because:

i) One has to make sure that the identification (3.9) is compatible with the memberships \(\tilde{\Delta}_a, \tilde{\Delta}_a^* \in \mathcal{L}^2, \tilde{\Delta}_a \in \mathcal{L}^1\).

ii) The functions \(\rho(\theta)\) and \(\tilde{v}(z)\) given by eq. (3.10) have no reason to belong to their proper class: Log \(\rho(\theta) \in \mathcal{L}^1, \tilde{v} \in \mathcal{H}^2\).

In order to settle the point i), we first remark that in \(\mathcal{H}^\infty\), we can use a
Cauchy-type formula to represent our particular functional $\Delta_a$. Actually, the formulae:

$$w_i^{(k)}(x_i) = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \Delta_{i,\Delta}(e^{i\theta})W_k(e^{i\theta}), \quad \Delta_{i,\Delta}(z) = k! \frac{z^k}{(1 - x_iz)^{k+1}}$$

are valid when $w_i \in H^\infty$, so that the kernel $\tilde{\Delta}_a(\theta)$, which according to eqs. (1.2), (2.10) and (3.2) appears as a linear combination of the $\Delta_{i,\Delta}(e^{i\theta})$'s, obviously belongs to $L^1$. We see that $\tilde{\Delta}_a(\theta)$ is even continuous. Such a stronger property will turn out to be useful below. In fact, for our proof to be valid, we only need to assume:

$$\tilde{\Delta}_a(\theta) \in \tilde{L}^2$$

This does not imply yet that $\tilde{\Delta}_{a,\rho}(\theta) = \tilde{\Delta}_a(\theta)/G(e^{i\theta}) \in \tilde{L}^2$, because the function $G(z)$ could have zeros on the unit circle. To prevent this accident, we shall use only strictly positive functions $\rho(\theta)$:

$$m \equiv \text{Ess. inf}_{-\pi < \theta < \pi} \rho(\theta) > 0$$

Under this condition, $|\tilde{\Delta}_{a,\rho}(\theta)| \leq \frac{1}{m} |\tilde{\Delta}_a(\theta)| \text{ a. e. and } \tilde{\Delta}_a(\theta) \in \tilde{L}^2$ entails $\tilde{\Delta}_{a,\rho}(\theta) \in \tilde{L}^2$.

We now observe that the choice of $\rho(\theta)$ given in eq. (3.10) cannot be made safely. Indeed, it might happen that $|\tilde{\Delta}_a^*(\theta) + e^{i\theta} \tilde{u}_0(e^{i\theta})|$ vanishes somewhere on $-\pi \leq \theta \leq \pi$, in which case the condition (3.14) would be violated. We shall therefore abandon the choice (3.10), and exhibit instead a family of functions $\rho_\varepsilon(\theta)$ and $\tilde{v}_\varepsilon(z)$ meeting all the required conditions and such that:

$$\lim_{\varepsilon \to 0} I_\varepsilon = I$$

where:

$$I_\varepsilon \equiv \left[ \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \left| \frac{\tilde{\Delta}_a^*(\theta)}{G_a(e^{i\theta})} + e^{i\theta} \tilde{v}_\varepsilon(e^{i\theta}) \right|^2 \right]^{\frac{1}{2}}$$

and $I > 0$ is defined in eq. (3.8). Clearly, this is enough to establish the equality (2.14). To construct $\rho_\varepsilon(\theta)$, let us first introduce, for any $\varepsilon > 0$:

$$\delta_\varepsilon = \{ \theta \mid |\tilde{\Delta}_a^*(\theta) + e^{i\theta} \tilde{u}_0(e^{i\theta})| \leq \varepsilon I \}, \quad \delta_\varepsilon' = C\delta_\varepsilon$$

and:

$$N_\varepsilon = \varepsilon \int_{\delta_\varepsilon} \frac{d\theta}{2\pi} + \frac{1}{I} \int_{\delta_\varepsilon'} \frac{d\theta}{2\pi} |\tilde{\Delta}_a^*(\theta) + e^{i\theta} \tilde{u}_0(e^{i\theta})|$$

From the very definition of $I$, this can be rewritten as:

$$N_\varepsilon = 1 + \int_{\delta_\varepsilon} \frac{d\theta}{2\pi} \left[ \varepsilon - \frac{1}{I} |\tilde{\Delta}_a^*(\theta) + e^{i\theta} \tilde{u}_0(e^{i\theta})| \right]$$
so that:

\[ 1 \leq N_\varepsilon \leq 1 + \varepsilon \]

If we now define:

\[ \rho_\varepsilon(\theta) = \begin{cases} \frac{\varepsilon}{N_\varepsilon}, & \theta \in \delta_\varepsilon \\ \frac{\tilde{A}_\varepsilon^*(\theta) + e^{i\theta} \tilde{u}_0(\varepsilon e^{i\theta})}{N_\varepsilon I}, & \theta \in \delta'_\varepsilon \end{cases} \]

We immediately see that \( \rho_\varepsilon(\theta) \in L^1 \) and \( \rho_\varepsilon(\theta) \geq \varepsilon/N_\varepsilon \) a.e.

Thus condition (3.13) is verified and moreover \( \log \rho_\varepsilon(\theta) \in L^1 \), which allows us to construct the outer function \( G_\varepsilon(z) \) according to eq. (2.5). The normalization condition (1.3) is also satisfied.

We choose next:

\[ \tilde{v}_\varepsilon(z) = \frac{\tilde{u}_0(z)}{G_\varepsilon(z)} \]

One has to check that \( \tilde{v}_\varepsilon \in \tilde{H}^2 \). First of all \( \tilde{v}_\varepsilon \in \tilde{H}^1 \) because \( \tilde{u}_0 \in \tilde{H}^1 \) and \( 1/G_\varepsilon(z) \) is an outer function in \( H^\infty(\sqrt{N_\varepsilon/\varepsilon}) \). Therefore, it is sufficient [3] to show that \( \tilde{v}_\varepsilon(e^{i\theta}) \in \tilde{L}^2 \), namely that:

\[ \| \tilde{v}_\varepsilon \|_2^2 = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \left| \tilde{u}_0(e^{i\theta}) \right|^2 \]

\[ = \frac{N_\varepsilon}{\varepsilon} \int_{\delta_\varepsilon} \frac{d\theta}{2\pi} \left| \tilde{u}_0(e^{i\theta}) \right|^2 + N_\varepsilon \int_{\delta'_\varepsilon} \frac{d\theta}{2\pi} \left| \tilde{u}_0(e^{i\theta}) \right|^2 \]

is finite. Using the triangle inequality

\[ \left| \tilde{u}_0(e^{i\theta}) \right| \leq \left| \tilde{A}_\varepsilon^*(\theta) + e^{i\theta} \tilde{u}_0(e^{i\theta}) \right| + \left| \tilde{A}_\varepsilon^*(\theta) \right| \]

and the bound

\[ \left| \tilde{A}_\varepsilon^*(\theta) + e^{i\theta} \tilde{u}_0(e^{i\theta}) \right| > \varepsilon I \]

on \( \delta'_\varepsilon \), we obtain:

\[ \| \tilde{v}_\varepsilon \|_2^2 \leq \frac{N_\varepsilon}{\varepsilon} \int_{\delta_\varepsilon} \frac{d\theta}{2\pi} \left| \tilde{u}_0(e^{i\theta}) \right|^2 \]

\[ + N_\varepsilon I \int_{\delta'_\varepsilon} \frac{d\theta}{2\pi} \left[ \left| \tilde{A}_\varepsilon^*(\theta) + e^{i\theta} \tilde{u}_0(e^{i\theta}) \right| + 2 \left| \tilde{A}_\varepsilon(\theta) \right| \right] + N_\varepsilon \int_{\delta'_\varepsilon} \frac{d\theta}{2\pi} \left| \tilde{A}_\varepsilon(\theta) \right|^2 \]

All three integrals in the r.h.s. are finite: the first one because

\[ \left| \tilde{u}_0(e^{i\theta}) \right| \leq \varepsilon I + \left| \tilde{A}_\varepsilon(\theta) \right| \]

on \( \delta_\varepsilon \) and \( \tilde{A}_\varepsilon(\theta) \in \tilde{L}^2 \), the second and third ones because \( \tilde{u}_0 \in \tilde{H}^1 \) and \( \tilde{A}_\varepsilon(\theta) \in \tilde{L}^2 \subset \tilde{L}^1 \). Therefore \( \| \tilde{v}_\varepsilon \|_2 < \infty \) and \( \tilde{v}_\varepsilon \in \tilde{H}^2 \).

Finally, it remains to check eq. (3.15). According to eqs. (3.16), (3.21) and (3.22):

\[ I^2 = \frac{N_\varepsilon}{\varepsilon} \int_{\delta_\varepsilon} \frac{d\theta}{2\pi} \left| \tilde{A}_\varepsilon^*(\theta) + e^{i\theta} \tilde{u}_0(e^{i\theta}) \right|^2 + N_\varepsilon \int_{\delta'_\varepsilon} \frac{d\theta}{2\pi} \left| \tilde{A}_\varepsilon^*(\theta) + e^{i\theta} \tilde{u}_0(e^{i\theta}) \right|^2 \]

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This gives, using eq. (3.18):

\[ I_\varepsilon^2 - I^2 = \frac{N\varepsilon}{\varepsilon} \int_{\delta} \frac{d\theta}{2\pi} |\tilde{\Delta}_{\varepsilon}(\theta) e^{i\theta} \bar{u}_0(e^{i\theta})|^2 - N\varepsilon I^2 \varepsilon \int_{\delta} \frac{d\theta}{2\pi} + (N\varepsilon^2 - 1)I^2 \]

and, since \(|\tilde{\Delta}_{\varepsilon}(\theta) + e^{i\theta} \bar{u}_0(e^{i\theta})|^2 \leq \varepsilon^2 I^2\) on \(\delta\):

\[ |I_\varepsilon^2 - I^2| \leq 2N\varepsilon I^2 \varepsilon + (N\varepsilon^2 - 1)I^2 \]

Therefore \(\lim_{\varepsilon \to 0} I_\varepsilon = I\) on account of eq. (3.20), and the proof is completed.
We derive here the duality formula:

\[ \text{Sup}_{\tilde{\omega}} \left| \Delta(\tilde{\omega}) \right| = \text{Min}_{\tilde{\omega}} \left| \tilde{\omega}^*(\theta) + e^{i\theta} \tilde{u}(e^{i\theta}) \right|_{L^1}, \]

where:

\[ \Delta(\tilde{\omega}) = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \tilde{\omega}^*(\theta) \tilde{w}(e^{i\theta}), \quad \tilde{\Delta}(\theta) \in \tilde{L}^1 \]

Our proof is nothing but a straightforward extension of that given in ref. [3] (chap. 8) for $H^\infty$.

Let $\tilde{C}(\subset \tilde{L}^\infty)$ be the Banach space consisting of all continuous functions $\tilde{f}(\theta)$, equipped with the $L^\infty$-norm (2.2), and $\tilde{P} \subset H^\infty$, the (closed) subspace of $\tilde{C}$ generated by $f_i(\theta) = 1, e^{i\theta}, e^{2i\theta}, \ldots$. Then, according to the general duality formula (3.1):

\[ \text{Sup}_{\tilde{\omega} \in \tilde{P}^\perp} \left| \Delta(\tilde{\omega}) \right| = \text{Min}_{\tilde{\omega} \in \tilde{P}^\perp} \left| \Delta + \psi \right|_{\tilde{C}^*}, \]

where $\tilde{P}^\perp$, the annihilator of $\tilde{P}$ in $\tilde{C}^*$, has to be characterized. As a consequence of the well-known Riesz representation theorem in $C^*$, there exists $n$ complex measures $\nu_i(\theta) = \nu_i(-\theta)$ such that every functional $\psi \in \tilde{C}^*$ has the form:

\[ \psi(\tilde{f}) = \frac{1}{2\pi} \sum_{i=1}^{n} \int_{-\pi}^{\pi} d\mu_i(\theta) \tilde{f}(e^{i\theta}), \quad f \in \tilde{C} \]

But $\psi \in \tilde{P}^\perp$ implies $\psi(\tilde{w}) = 0$ for all $\tilde{w}$ of the form $\tilde{w}(z) = (0, \ldots, 0, z^p, 0, \ldots, 0)$ ($p = 0, 1, 2, \ldots$). Thus, for $i = 1, \ldots, n$:

\[ \int_{-\pi}^{\pi} d\mu_i(\theta) e^{ip\theta} = 0, \quad p = 0, 1, 2, \ldots \]

which in turns implies, according to the theorem of F. and M. Riesz [3]:

\[ d\mu_i(\theta) = u_i(e^{i\theta}) e^{i\theta} d\theta, \quad u_i(z) \in H^1 \]

Hence $(\Delta + \psi)$ in eq. (A.3) has the representation:

\[ (\Delta + \psi)(\tilde{f}) = \int_{-\pi}^{\pi} d\theta \left[ \tilde{\omega}^*(\theta) + e^{i\theta} \tilde{u}(e^{i\theta}) \right] \tilde{f}(e^{i\theta}) \]

with $\tilde{u} \in H^1$.

Now, for any functional $\phi \in \tilde{C}^*$ represented by a kernel $\tilde{\phi}(\theta) \in \tilde{L}^1$ as in eq. (A.7), one has:

\[ || \phi ||_{\tilde{C}^*} = || \tilde{\phi}(\theta) ||_{\tilde{L}^1} \]

as we shall prove in a moment. We conclude that:

\[ \text{Sup}_{\tilde{\omega} \in \tilde{P}^\perp} \left| \Delta(\tilde{\omega}) \right| = \text{Min}_{\tilde{\omega} \in \tilde{P}^\perp} \left| \tilde{\omega}^*(\theta) + e^{i\theta} \tilde{u}(e^{i\theta}) \right|_{\tilde{C}^*}, \]

and, since $H^\infty \supset \tilde{P}$:

\[ \text{Sup}_{\tilde{\omega} \in H^\infty} \left| \Delta(\tilde{\omega}) \right| \geq \text{Min}_{\tilde{\omega} \in \tilde{P}^\perp} \left| \tilde{\omega}^*(\theta) + e^{i\theta} \tilde{u}(e^{i\theta}) \right|_{\tilde{C}^*}. \]
On the other hand, we obviously have, for any \( w \in H^\infty \):

\[
(A.11) \quad |\Delta (w)| \leq \| w \|_\infty |\Delta^\ast (\theta) + c^\ast (\theta^\ast) + \phi^\ast (\theta)|_{L^1},
\]

Formula (A.1) then follows from eqs. (A.10) and (A.11).

Proof of eq. (A.8).

From the fact that \( C \) is a dense subset of \( L^1 \), we infer the existence of a sequence \( \{ \phi_n \} \) in \( C \) such that:

\[
(A.12) \quad \lim_{n \to \infty} \phi_n = \phi
\]

in the \( L^1 \)-norm. Then:

\[
(A.13) \quad \| \phi_n \|_{L^1} \leq \mathop{\sup}_{f \in C \atop \| f \|_{L^1} \leq 1} \left| \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \phi_n^\ast (\theta) f(\theta) \right| \leq \mathop{\sup}_{f \in C \atop \| f \|_{L^1} \leq 1} \left| \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \phi_n(\theta) f(\theta) \right|
\]

But, since \( f(\theta) = \phi_n(\theta)/\| \phi_n(\theta) \|_{L^1} + \varepsilon \in C \) and \( \| f \|_{L^1} < 1 \) for any \( \varepsilon > 0 \):

\[
(A.14) \quad \| \phi_n \|_{L^1} := \sup_{\varepsilon > 0} \left| \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \phi_n^\ast (\theta) f_\varepsilon(\theta) \right| = \sup_{\varepsilon > 0} \left| \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \phi_n(\theta) f_\varepsilon(\theta) \right| = \| \phi_n(\theta) \|_{L^1}
\]

Comparing eqs. (A.13) and (A.14), we obtain:

\[
(A.15) \quad \| \phi_n \|_{L^1} = \| \phi_n(\theta) \|_{L^1}, \quad \forall n
\]

On the other hand, eq. (A.12) obviously implies:

\[
(A.16) \quad \lim_{n \to \infty} \| \phi_n(\theta) \|_{L^1} = \| \phi(\theta) \|_{L^1}
\]

and also:

\[
(A.17) \quad \lim_{n \to \infty} \| \phi_n \|_{L^1} = \| \phi \|_{L^1}
\]

because:

\[
(A.18) \quad \left| \frac{\| \phi_n(\theta) \|_{L^1} - \| \phi(\theta) \|_{L^1}}{\| \phi(\theta) \|_{L^1}} \right| \leq \frac{\| \phi_n(\theta) - \phi(\theta) \|_{\infty}}{\| \phi(\theta) \|_{L^1}} \leq \| \phi_n(\theta) - \phi(\theta) \|_{L^1}.
\]

Eqs. (A.15)-(A.17) yield the announced formula (A.8).

REFERENCES

[5] See e. g. ref. [3], Theorem 7.1. (Manuscrit reçu le 9 mars 1979)