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Variational principle for quasi-local algebras over the lattice


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by

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ABSTRACT. — It is shown that a variational principle holds for certain quasi-local algebras over the lattice.

1. INTRODUCTION AND DEFINITION

In general, as in Ruelle’s book [3, 6.2.4], to describe the infinite systems of statistical mechanics over the lattice $Z^v$, we associate finite-dimensional algebras $\mathcal{A}_A$ with finite subsets $A$ of $Z^v$ and we assume that:

a) If $\Lambda \subseteq \Lambda'$, an (identity-preserving) isomorphism $\alpha_{\Lambda'\Lambda}$ of $\mathcal{A}_\Lambda$ into $\mathcal{A}_{\Lambda'}$ is given such that if $\Lambda \subseteq \Lambda' \subseteq \Lambda''$, then $\alpha_{\Lambda''\Lambda} = \alpha_{\Lambda'\Lambda} \circ \alpha_{\Lambda''\Lambda}$. 

b) An isomorphism $\tau^\Lambda_n$ of $\mathcal{A}_\Lambda$ onto $\mathcal{A}_{\Lambda+n}$ is given for each translation $n \in Z^v$ and each $\Lambda$ such that $\tau^\Lambda_{n+m} = \tau^\Lambda_n \circ \tau^\Lambda_m$; and if $\Lambda \subseteq \Lambda'$, then

$$\tau^\Lambda_{n+m} \circ \alpha_{\Lambda'\Lambda} = \alpha_{\Lambda'+n,\Lambda'+n} \circ \tau^\Lambda_n.$$

By using $a$) we define the C*-inductive limit $\mathcal{A}$ of the family $\{ \mathcal{A}_\Lambda, \alpha_{\Lambda'\Lambda} \}$, i.e. we have a unique C*-algebra $\mathcal{A}$ and isomorphisms $\alpha_{\Lambda}$ of $\mathcal{A}_\Lambda$ into $\mathcal{A}$ such that if $\Lambda \subseteq \Lambda'$, $\alpha_{\Lambda'} \circ \alpha_{\Lambda'\Lambda} = \alpha_{\Lambda}$; and the union of $\alpha_{\Lambda}(\mathcal{A}_\Lambda)$ is dense in $\mathcal{A}$. By using $b$) we define a homomorphism $\tau$ of the group $Z^v$ into the automorphism group of $\mathcal{A}$ such that $\tau_n \circ \alpha_{\Lambda} = \alpha_{\Lambda+n} \circ \tau^\Lambda_n$. The triple $(\mathcal{A}, Z^v, \tau)$ is called the « quasi-local » algebra constructed from the « local » algebras $\mathcal{A}_\Lambda$.

We further assume the following properties, the first part of which is commonly assumed:

$c)$ If $\Lambda_1 \cap \Lambda_2 = \phi$, then $\alpha_{\Lambda_1\Lambda_2}(\mathcal{A}_{\Lambda_1})$ is in the commutant of $\alpha_{\Lambda_1\Lambda_2}$. (\mathcal{A}_{\Lambda_1}),$
and $\mathcal{A}(\mathcal{U}_\Lambda)$ and $\mathcal{A}_\Lambda(\mathcal{U}_\Lambda)$ generate a subalgebra of $\mathcal{U}_\Lambda$ whose relative commutant is its center, where $\Lambda = \Lambda_1 \cup \Lambda_2$.

The second part of c) also seems quite natural. It requires that observables in $\mathcal{U}_\Lambda$ which commute with all strictly local ones, i.e. elements in $\mathcal{A}_\Lambda(\mathcal{U}_\Lambda)$, $n \in \Lambda$, must be generated by strictly local observables.

Now, if $\Lambda_1 \cap \Lambda_2 = \phi$, we define a homomorphism $\Phi_{\Lambda_1,\Lambda_2}$ of $\mathcal{U}_{\Lambda_1} \otimes \mathcal{U}_{\Lambda_2}$ into $\mathcal{U}_\Lambda$ by:

$$\Phi_{\Lambda_1,\Lambda_2}(\sum a_1 \otimes b_1) = \sum \mathcal{A}_{\Lambda_1 \cup \Lambda_2, \Lambda_1}(a_1) \mathcal{A}_{\Lambda_1 \cup \Lambda_2, \Lambda_2}(b_1).$$

It is easily shown that well-defined and satisfies:

c'1) the restriction of $\Phi_{\Lambda_1,\Lambda_2}$ to $\mathcal{U}_{\Lambda_1}$ (identified with $\mathcal{U}_{\Lambda_1} \otimes 1$) is $\mathcal{A}_{\Lambda_1 \cup \Lambda_2, \Lambda_1}$;

c'2) if $\Lambda_1$, $\Lambda_2$ and $\Lambda_3$ are mutually disjoint,

$$\Phi_{\Lambda_1 \cup \Lambda_2, \Lambda_3} \circ \Phi_{\Lambda_1,\Lambda_2} \otimes I = \Phi_{\Lambda_1,\Lambda_2 \cup \Lambda_3} \circ I \otimes \Phi_{\Lambda_2,\Lambda_3}$$

where $I$ is the identity isomorphism;

c'3) the quotient $\mathcal{U}_{\Lambda_1} \otimes \mathcal{U}_{\Lambda_2}/\ker \Phi_{\Lambda_1,\Lambda_2}$ is mapped into $\mathcal{U}_{\Lambda_1 \cup \Lambda_2}$ with multiplicity 1 under the induced isomorphism; and

c'4) $\tau^\Lambda_{\Lambda_1 \cup \Lambda_2} \circ \Phi_{\Lambda_1,\Lambda_2} = \Phi_{\Lambda_1 + n \Lambda_2 + n} \circ \tau^\Lambda_{\Lambda_1} \otimes \tau^\Lambda_{\Lambda_2}$ for $n \in \mathbb{Z}^\Lambda$.

In the rest of this note we show that the variational principle holds for the systems satisfying a), b) and c). In particular we show that if $\mathcal{U}_\Lambda$ is generated by $\mathcal{A}_\Lambda(\mathcal{U}_\Lambda)$, $n \in \Lambda$ for any $\Lambda$, then the systems are classical (or rather semi-quantum) and the ones considered by Ruelle, i.e. the systems, restricted to closed invariant subsets of the whole configuration space (see the announcement in [4]).

In section 2, we derive some results on the algebra $\mathcal{U}$ from the condition c) and in section 3 we show the existence of thermodynamic quantities. The main result is shown in section 4 and examples are given in section 5.

2. STRUCTURE OF $\mathcal{U}$

Suppose a), b) and c). If $\Lambda_1$, ..., $\Lambda_k$ are mutually disjoint, we inductively define a homomorphism $\Phi_{\Lambda_1,\ldots,\Lambda_k}$ of $\mathcal{U}_{\Lambda_1} \otimes \ldots \otimes \mathcal{U}_{\Lambda_k}$ into $\mathcal{U}_\Lambda$ with $\Lambda = \Lambda_1 \cup \ldots \cup \Lambda_k$ by $\Phi_{\Lambda_1 \cup \ldots \cup \Lambda_k} \circ \Phi_{\Lambda_1,\ldots,\Lambda_k} \otimes I$, where if $k = 2$, $\Phi_{\Lambda_1} = I$ and $\Phi_{\Lambda_1,\Lambda_2}$ is already defined. By using c'2) we have the identity:

$$\Phi_{\Lambda' \cup \Lambda_{k-1}} \circ \Phi_{\Lambda_1,\ldots,\Lambda_{k-1}} \otimes I = \Phi_{\Lambda' \cup \Lambda_{k-1}} \circ \Phi_{\Lambda',\Lambda_{k-1}} \otimes I \circ \Phi_{\Lambda_1,\ldots,\Lambda_{k-2}} \otimes I \otimes I = \Phi_{\Lambda' \cup \Lambda_{k-1}} \circ \Phi_{\Lambda_1,\ldots,\Lambda_{k-2}} \otimes \Phi_{\Lambda_{k-1},\Lambda_k}$$

where $\Lambda' = \Lambda_1 \cup \ldots \cup \Lambda_{k-2}$. Thus the homomorphism $\Phi_{\Lambda_1, \ldots, \Lambda_k}$ does not depend on the order of $\{ \Lambda_{k-1}, \Lambda_k \}$, and so, inductively, on the order of $\{ \Lambda_1, \ldots, \Lambda_k \}$. Note that the properties as in c'1), c'3) and c'4) still hold for $\Phi_{\Lambda_1,\ldots,\Lambda_k}$.
For a finite \( \Lambda = \{n_1, \ldots, n_k\} \) let \( \tilde{\mathcal{A}}_\Lambda \) be the tensor product of \( \mathcal{A}_{(n_i)} \), \( i = 1, \ldots, k \), and let \( \Phi_\Lambda = \Phi_{(n_1), \ldots, (n_k)} \). If \( \Lambda \subset \Lambda' \), we have that \( \alpha_{\Lambda' \Lambda} \circ \tilde{\Phi}_\Lambda = \tilde{\Phi}_\Lambda \circ l \) where \( l \) is the natural embedding of \( \tilde{\mathcal{A}}_\Lambda \) into \( \tilde{\mathcal{A}}_{\Lambda'} \). In more general we have the following commutative diagram: if \( \Lambda_1 \cap \Lambda_2 = \phi \),

\[
\begin{array}{ccc}
\tilde{\mathcal{A}}_{\Lambda_1} \otimes \tilde{\mathcal{A}}_{\Lambda_2} & = & \tilde{\mathcal{A}}_{\Lambda_1 \cup \Lambda_2} \\
\downarrow \tilde{\Phi}_{\Lambda_1} \otimes \tilde{\Phi}_{\Lambda_2} & & \downarrow \tilde{\Phi}_{\Lambda_1 \cup \Lambda_2} \\
\mathcal{A}_{\Lambda_1} \otimes \mathcal{A}_{\Lambda_2} & \xrightarrow{\Phi_{\Lambda_1 \cup \Lambda_2}} & \mathcal{A}_{\Lambda_1 \cup \Lambda_2}
\end{array}
\]

This is shown by the induction on the cardinality of \( \Lambda_1 \cup \Lambda_2 \), and by the identities:

\[
\Phi_{\Lambda_1 \cup \Lambda_2} \circ (\tilde{\Phi}_{\Lambda_1} \otimes \tilde{\Phi}_{\Lambda_2}) = \Phi_{\Lambda_1 \cup \Lambda_2} \circ (\tilde{\Phi}_{\Lambda_1} \otimes (\Phi_{\Lambda_2(n)} \circ \tilde{\Phi}_{\Lambda_2} \otimes l)) = \Phi_{\Lambda_1 \cup \Lambda_2} \circ (l \otimes \Phi_{\Lambda_2(n)}) \circ (\tilde{\Phi}_{\Lambda_1} \otimes \tilde{\Phi}_{\Lambda_2} \otimes l) = \Phi_{\Lambda_1 \cup \Lambda_2(n)} \circ ((\Phi_{\Lambda_1 \Lambda_2} \circ \tilde{\Phi}_{\Lambda_1} \otimes \tilde{\Phi}_{\Lambda_2}) \otimes l)
\]

where \( n \in \Lambda_2 \) and \( \Lambda_2' = \Lambda_2 \setminus \{n\} \).

Let \( \tilde{\mathcal{A}} \) be the \( \mathcal{C}^* \)-inductive limit of \( \tilde{\mathcal{A}}_\Lambda \). We have a homomorphism \( \tilde{\Phi} \) of \( \tilde{\mathcal{A}} \) into \( \mathcal{A} \) such that \( \alpha_\Lambda \circ \tilde{\Phi}_\Lambda = \tilde{\Phi} \) on \( \tilde{\mathcal{A}}_\Lambda \). If \( \tau_{\mathcal{A}} \) denotes the action of \( Z' \) on \( \tilde{\mathcal{A}} \) extending \( \otimes \tau_{\mathcal{A}}(n) \) of \( \tilde{\mathcal{A}}_\Lambda \) into \( \tilde{\mathcal{A}}_{\Lambda+n} \), we have that \( \tau_n \circ \tilde{\Phi} = \tilde{\Phi} \circ \tau_{\mathcal{A}} \).

**Theorem 1.** — Let \( \mathcal{A}_\Lambda \) satisfy a), b) and c). Further suppose that \( \mathcal{A}_\Lambda \) is generated by \( \alpha_{\Lambda(n)}(\mathcal{A}_{(n)}) \), \( n \in \Lambda \), for any \( \Lambda \). Then the quasi-local algebra \( (\mathcal{A}, Z', \tau) \) is isomorphic to \( (\tilde{\mathcal{A}}/I, Z', \tau') \), where \( I \) is the kernel of \( \tilde{\Phi} \) and \( \tau' \) is the induced action on the quotient algebra \( \tilde{\mathcal{A}}/I \) from \( \tau_{\mathcal{A}} \).

If \( \mathcal{A}_{(n)} \) is commutative, say the algebra \( \mathcal{C}(F) \) of (continuous) functions on a finite set \( F \), then \( \tilde{\mathcal{A}} \simeq \mathcal{C}(F^{Z'}) \) and \( \tilde{\mathcal{A}}/I \simeq \mathcal{C}(\Omega) \) where

\[
\Omega = \{ x \in F^{Z'} : f(x) = 0, f \in I \}
\]

is a translation invariant closed set. Hence, this system has a good thermodynamic property (cf. [4]). This is easily extended to the « semi-quantum » case, i.e. the case that \( \mathcal{A}_{(n)} \) is not commutative. In particular, we have a theorem similar to [2, 8.3].

In general there is a unique projection \( p \) of norm 1 of \( \mathcal{A} \) onto \( \tilde{\Phi}(\tilde{\mathcal{A}}) \), such that \( p(\mathcal{A}_\Lambda) = \tilde{\Phi}(\tilde{\mathcal{A}}_\Lambda) \). This is shown by using the fact

\[
\mathcal{A}_\Lambda \cap \tilde{\Phi}(\tilde{\mathcal{A}}_\Lambda)' = \tilde{\Phi}(\tilde{\mathcal{A}}_\Lambda) \cap \tilde{\Phi}(\tilde{\mathcal{A}}_\Lambda)'.
\]

Now we associate central projections $e_\Lambda$ of $\mathfrak{U}_\Lambda$ with finite $\Lambda$ such that \( \ker \Phi_\Lambda = (1 - e_\Lambda)\mathfrak{U}_\Lambda \). If $\Lambda_1 \cap \Lambda_2 = \phi$, we have that $e_{\Lambda \setminus \Lambda_1 \cap \Lambda_2} \geq e_{\Lambda_1 \cup \Lambda_2}$ since the kernel of $\Phi_{\Lambda_1 \cup \Lambda_2}$ is larger than that of $\Phi_{\Lambda_1} \otimes \Phi_{\Lambda_2}$. In particular, if $\Lambda \subset \Lambda'$, then $e_{\Lambda'} \leq e_\Lambda$ in $\mathfrak{U}$. And the kernel of $\Phi$ is generated by $1 - e_\Lambda$, with finite $\Lambda$.

Further we have that if $\Lambda_1 \cap \Lambda_2 = \phi$, $\Phi_{\Lambda_1} \otimes \Phi_{\Lambda_2}(e_{\Lambda_1 \cup \Lambda_2})$ is in the center of $\mathfrak{U}_{\Lambda_1} \otimes \mathfrak{U}_{\Lambda_2}$ and that

$$ \ker \Phi_{\Lambda_1 \cup \Lambda_2} = \Phi_{\Lambda_1} \otimes \Phi_{\Lambda_2}(e_{\Lambda_1} \otimes e_{\Lambda_2} - e_{\Lambda_1 \cup \Lambda_2}) \cdot \mathfrak{U}_{\Lambda_1} \otimes \mathfrak{U}_{\Lambda_2} $$

Let $\mathcal{B}_\Lambda = \bigotimes_{\Lambda \in \Lambda} \mathcal{B}_\Lambda$ and let $\mathfrak{U}_\Lambda = e_\Lambda \mathfrak{U}_\Lambda e_\Lambda$. There is a unique subalgebra $\mathfrak{U}_\Lambda$ of $\mathfrak{U}_\Lambda$ such that $\mathfrak{U}_\Lambda$ is isomorphic to $\mathfrak{U}_\Lambda$ by an isomorphism extending $\Phi_\Lambda$ of $\mathfrak{U}_\Lambda e_\Lambda(\subset \mathfrak{U}_\Lambda)$ into $\mathfrak{U}_\Lambda$. If $\Lambda \subset \Lambda'$, there is a natural embedding of $\mathfrak{U}_\Lambda$ into $\mathfrak{U}_{\Lambda'}$ given by the multiplication of $e_{\Lambda'}$. At this point we do not know if there are isomorphisms $\beta_\Lambda$ of $\mathfrak{U}_\Lambda$ onto $\mathfrak{U}_{\Lambda'}$ satisfying the obvious consistency relations. But examples given in section 5 have the structure of $(\mathfrak{U}_\Lambda)$. So we give a remark: let $\mathcal{B}$ be the $\mathrm{C}^*$-inductive limit of $\mathcal{B}_\Lambda$ and let $\mathcal{D}$ be the $\mathrm{C}^*$-subalgebra generated by $\mathfrak{U}_\Lambda$ with all $\Lambda$, in $\mathcal{B}$. Let $I$ be the ideal of $\mathcal{D}$ generated by $1 - e_\Lambda$ with all $\Lambda$. Then the $\mathrm{C}^*$-inductive limit $\mathfrak{U}$ of $\mathfrak{U}_\Lambda$ is isomorphic to the quotient $\mathcal{D}/I$ (the action of $Z'$ on $\mathfrak{U}$ should be the induced one from the natural translations on $\mathcal{B}$). Hence, $\mathfrak{U}$ has a good thermodynamic property, too, as maybe shown in the same way as in the classical case.

If $\Lambda_1, \ldots, \Lambda_k$ are mutually disjoint and if $\beta_{\Lambda_i}$ are isomorphisms of $\mathfrak{U}_{\Lambda_i}$ onto $\mathfrak{U}_{\Lambda_i}$ with $\Phi_{\Lambda_i} \circ \beta_{\Lambda_i} = l$ on $\Phi(\mathfrak{U}_{\Lambda_i})$, it is shown that there is an isomorphism $\beta_\Lambda$ of $\mathfrak{U}_\Lambda$ onto $\mathfrak{U}_\Lambda$ with $\Lambda = \Lambda_1 \cup \ldots \cup \Lambda_k$ such that $\beta \circ \Phi_{\Lambda_1} \cdots \Lambda_k = e_{\Lambda} \otimes \beta_{\Lambda_1} \otimes \ldots \otimes \beta_{\Lambda_k}$. For instance if $k = 2$, and if $x \in \mathfrak{U}_{\Lambda_1} \otimes \mathfrak{U}_{\Lambda_2}$, $e_{\Lambda_1} \beta_{\Lambda_1}(x) = \beta_{\Lambda_1}(e_{\Lambda_1}x)$. So if $\Phi_{\Lambda_1 \cup \Lambda_2} e_{\Lambda}$ denotes the isomorphism of $e_{\Lambda} \mathfrak{U}_{\Lambda_1} \otimes \mathfrak{U}_{\Lambda_2}$ into $\mathfrak{U}_{\Lambda}$, $\beta_{\Lambda}$ is an extension of $\beta_{\Lambda_1} \otimes \beta_{\Lambda_2} \circ (\Phi_{\Lambda_1 \cup \Lambda_2} e_{\Lambda})^{-1}$.

Let $t$ be the unique tracial state of $\mathcal{B}$. We define a trace $t_\Lambda$ on $\mathfrak{U}_\Lambda$ by $t \circ \beta_{\Lambda}$, which does not depend on $\beta_{\Lambda}$, and takes the same value on each minimal projection of $\mathfrak{U}_{\Lambda}$. Note that $t_{\Lambda}(1)^{-1} t_{\Lambda'} \circ s_{\Lambda'_\Lambda} = t_{\Lambda}(1)^{-1} t_{\Lambda}$ does not hold in general ($\Lambda \subset \Lambda'$).

### 3. THERMODYNAMIC QUANTITIES

Let $\omega$ be a translation invariant state of $\mathfrak{U}$. For each $\Lambda$ let $\rho_\Lambda = \rho_\Lambda(\omega)$ be an element of $\mathfrak{U}_\Lambda$ satisfying that $t_\Lambda(\rho_\Lambda A) = \omega(A)$ for all $A \in \mathfrak{U}_\Lambda$ and set

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\[ S(\Lambda) = - t_\Lambda(\rho_\Lambda \log \rho_\Lambda). \] If \( \Lambda_1 \cap \Lambda_2 = \phi \), we have the subadditivity
\[ S(\Lambda_1 \cup \Lambda_2) \leq S(\Lambda_1) + S(\Lambda_2), \] by the inequality [3, 2.5.3] :
\[ t(\beta_\Lambda(\rho_\Lambda) \log \beta_\Lambda(\rho_\Lambda)) - t(\beta_{\Lambda_1}(\rho_{\Lambda_1}) \log \beta_{\Lambda_1}(\rho_{\Lambda_1}) \otimes \beta_{\Lambda_2}(\rho_{\Lambda_2})) \geq 0 \]
where \( \Lambda = \Lambda_1 \cup \Lambda_2 \), and \( \beta_\Lambda \), \( \beta_{\Lambda_1} \) and \( \beta_{\Lambda_2} \) satisfy that
\[ \beta_\Lambda \circ \Phi_{\Lambda_1,\Lambda_2} = e_\Lambda \cdot \beta_{\Lambda_1} \otimes \beta_{\Lambda_2}. \]
Hence, we can define the mean entropy :
\[ s(\omega) = \lim_{N} |\Lambda(N)|^{-1} S(\Lambda(N)) \]
where \( \Lambda(N) = \{ n \in \mathbb{Z}^r; 0 \leq n_i < N_i \} \) for \( N \in \mathbb{Z}^r \) with \( N_i > 0 \) and \(|\Lambda(N)| \)
is the cardinality of \( \Lambda(N) \) (cf. [3, 7.2.11]).

Let \( \Phi \) be a (translation-invariant) potential \( \mathfrak{A} \), i.e. \( \Phi \) is a family of \( \Phi(\Lambda) \in \mathbb{A}_\Lambda \) with all non-empty finite subsets \( \Lambda \) of \( \mathbb{Z}^r \) satisfying that
\( \Phi(\Lambda)^* = \Phi(\Lambda) \), \( \tau_\Lambda \Phi(\Lambda) = \Phi(\Lambda + n) \) and \( \| \Phi \| \equiv \sum_{\Lambda \neq 0} \| \Lambda \|^{-1} \| \Phi(\Lambda) \| < \infty. \)
We set
\[ U_\Lambda = U_\Lambda^\Phi = \sum_{\Gamma \subset \Lambda} \alpha_{\Lambda\Gamma}(\Phi(\Gamma)), \]
\[ p_\Lambda = p_\Lambda(\Phi) = |\Lambda|^{-1} t_\Lambda(e^{-U_\Lambda}) \]
If \( \Lambda_1, \ldots, \Lambda_k \) are mutually disjoint, we have
\[ t_\Lambda \left( \exp \left( - \sum_{i=1}^{k} \alpha_{\Lambda_{\Lambda_i}}(U_{\Lambda_i}) \right) \right) = t(\epsilon_\Lambda \exp \left( - \sum \beta_{\Lambda_i}(U_{\Lambda_i}) \right)) \]
\[ \leq t \left( \exp \left( - \sum \beta_{\Lambda_i}(U_{\Lambda_i}) \right) \right) = \prod_{i=1}^{k} t_{\Lambda_i}(e^{-U_{\Lambda_i}}) \]
where \( \Lambda = \Lambda_1 \cup \ldots \cup \Lambda_k \), and \( \beta_\Lambda \circ \Phi_{\Lambda_1,\ldots,\Lambda_k} = e_\Lambda \cdot \beta_{\Lambda_1} \otimes \ldots \otimes \beta_{\Lambda_k}. \)

If \( \Phi \) is of finite range, we can show, as in the proof of [3, 2.3.1], that
\[ p_{\Lambda(M)} \leq p_{\Lambda(N)} + \varepsilon_N + \delta_M(N) \]
where \( \varepsilon_N \) tends to zero (independently of \( M \)) as \( N \to \infty \) and \( \delta_M(N) \) tends to zero for each \( N \) as \( M \to \infty \). Thus we have
\[ \sup lim p_{\Lambda(M)} \leq \inf lim p_{\Lambda(N)}. \]

By the same reasoning as in [3, 2.3.3] we have the pressure
\[ p(\Phi) = \lim p_{\Lambda(N)}(\Phi) \] for any \( \Phi \) (with \( \| \Phi \| < \infty \)).

From the special case \( \Phi = 0 \), we have that
\[ p(0) = \lim |\Lambda|^{-1} \log t_\Lambda(1). \]
Hence, replacing \( t_\Lambda \) by the normalized \( t_\Lambda(1)^{-1} t_\Lambda \) in the definition of entropy and pressure implies replacing \( s(\omega) \) by \( s(\omega) - p(0) \) and \( p(\Phi) \) by \( p(\Phi) - p(0) \).
For any invariant state \( \omega \) of \( \mathcal{U} \) and any potential \( \Phi \), as easily shown, we have the mean energy

\[
\omega(A_\Phi) = \lim \left| \Lambda(N) \right|^{-1} \omega(U_{\Lambda(N)}^\Phi)
\]

where

\[
A_\Phi = \sum_{\Lambda \neq 0} \left| \Lambda \right|^{-1} \Phi(\Lambda).
\]

4. VARIATIONAL PRINCIPLE

Let \( \omega \) be a translation invariant state of \( \mathcal{U} \) and let \( \Phi \) be a potential. For each \( \Lambda \) we have

\[
\log t_\Lambda \left( e^{-U_\lambda} \right) \geq -t_\Lambda (\rho_\Lambda(\omega) \log \rho_\Lambda(\omega)) - \omega(U_\lambda).
\]

Thus, we obtain the variational inequality: \( p(\Phi) \geq s(\omega) - \omega(A_\Phi) \).

Let \( N \in \mathbb{Z}^v \) with \( N_i > 0 \) and let \( \mathcal{U}(N) = \bigotimes_{n \in \mathbb{Z}^v} \mathcal{U}_{\Lambda(N) + nN} \) and so especially \( \mathcal{U}(1, \ldots, 1) = \tilde{\mathcal{H}} \). In the same way as to construct \( \Phi \) in section 2, we have a homomorphism \( \Phi_N \) of \( \mathcal{U}(N) \) into \( \mathcal{U} \), extending

\[
\Phi_{\Lambda(N)+n_1N, \ldots, \Lambda(N)+n_kN} \left( \{n_1, \ldots, n_k\} \right) \in \mathcal{Z}^v.
\]

Furthermore, we have the natural action of \( \mathbb{Z}^v \) on \( \mathcal{U}(N) \) such that \( \tau_{Nn} \circ \Phi_N = \Phi_N \circ \tau_{Nn} \).

Let \( M \) be also in \( \mathbb{Z}^v \) with \( M_i > 0 \). We have a homomorphism \( \Phi_{N,M} \) of \( \mathcal{U}(N) \) into \( \mathcal{U}(NM) \) given by

\[
\bigotimes_{n} \Phi_{\Lambda(N)+(aM+a)N, \ldots, \Lambda(N)+(aM+b)N}
\]

with \( \{a, \ldots, b\} = \Lambda(M) \). We have that \( \Phi_{NM} \circ \Phi_{N,M} = \Phi_N \); and

\[
\tau_{NMn} \circ \Phi_{N,M} = \Phi_{N,M} \circ \tau_{NMn}.
\]

Let \( \Phi \) be a potential of finite range and let \( \varphi_N \) be a product state of \( \mathcal{U}(N) = \bigotimes_{n} \mathcal{U}_{\Lambda(N)+nN} \) such that

\[
\rho_{\Lambda(N)+nN}(\varphi_N) = e^{-U_{\Lambda(N)+nN}/t_{\Lambda(N)+nN}}(e^{-U_{\Lambda(N)+nN}}).
\]

Then \( \varphi_N \) is \( \tau_N \)-invariant. Let \( \varphi_{N,M} = \varphi_{NM} \circ \Phi_{N,M} \), which is a \( \tau_{NM} \)-invariant state of \( \mathcal{U}(N) \), and let \( \varphi_{N,M} \) be the \( \tau_N \)-invariant state obtained by averaging \( \varphi_{N,M} \) over the translations \( \mathbb{N} \mathbb{Z}^v \). Then, by using the product trace of \( t_{\Lambda(N)+nN} \) in the definition of entropy, we have,

\[
s(\varphi_{NM}) = |\Lambda|^{-1} S_{\Lambda}(\varphi_{NM}) \leq |\Lambda|^{-1} S_{\Lambda}(\varphi_{NM} \circ \Phi_{N,M})
\]

\[
= s(\varphi_{N,M}) = s(\varphi_{N,M}).
\]
and

\[ s(\varphi_{NM}) = |\Lambda|^{-1} t_\Lambda(e^{-U}\Lambda') + |\Lambda|^{-1} t_\Lambda(U_\Lambda e^{-U}\Lambda')/t_\Lambda(e^{-U}\Lambda') \]

where \( \Lambda = \Lambda(NM) \). A simple argument shows that there is a constant \( \varepsilon_N \) which tends to zero as \( N \to \infty \) such that

\[ s(\varphi_{N,M}) \geq |\Lambda(NM)|^{-1} t_{\Lambda(NM)}(e^{-U\Lambda(NM)}) + |\Lambda(N)|^{-1} \varphi_{N,M}(U_{\Lambda(N)}) - \varepsilon_N. \]

Let \( \omega_N \) be a weak limit point of \( \varphi_{N,M} \) as \( M \to \infty \). By the upper semi-continuity of \( s(.) \) we have

\[ s(\omega_N) \geq p(\Phi) - |\Lambda(N)|^{-1} \omega_N(U_{\Lambda(N)}) - \varepsilon_N. \]

For any \( m \in \mathbb{Z}^* \) with \( m_i > 0 \), \( \varphi_{N,M}(\epsilon_{\Lambda+n\Lambda'}) = 1 \) with \( \Lambda = \Lambda(Nm) \) if \( \Lambda + n\Lambda \subset \Lambda(NM) \). Thus, we have that \( \omega_N(\epsilon_{\Lambda+n\Lambda'}) = 1 \) for any \( n \in \mathbb{N} \). Since the kernel of \( \Phi_N \) is generated by \( 1 - e_{\Lambda(NM)+n\Lambda} \), we have a unique \((\tau_{n\Lambda}-\text{invariant})\) state \( \hat{\omega}_N \) of \( \Phi_N(\mathfrak{H}(N)) \) such that \( \omega_N = \hat{\omega}_N \circ \Phi_N \). We extend \( \hat{\omega}_N \) to a state of \( \mathfrak{H} \), denoted by \( \bar{\omega}_N \), also, by using a unique projection of norm 1 of \( \mathfrak{H} \) onto \( \Phi_N(\mathfrak{H}(N)) \) mapping \( \mathfrak{H}_{\Lambda(NM)} \) onto \( \Phi_N(\bigotimes_{m \in \Lambda(NM)} \mathfrak{H}_{\Lambda(N)+m\Lambda}). \)

Let \( \bar{\omega}_N \) be the \( \tau \)-invariant state of \( \mathfrak{H} \) obtained by averaging \( \omega_N \) over \( \mathbb{Z}^* \).

Then, we have \( s(\bar{\omega}_N) = s(\hat{\omega}_N) = s(\omega_N) \). Thus,

\[ s(\bar{\omega}_N) \geq p(\Phi) - |\Lambda(N)|^{-1} \bar{\omega}_N(U_{\Lambda(N)}) - \varepsilon_N \]

where \( U_{\Lambda(N)} \) is identified with \( \pi_{\Lambda(N)}(U_{\Lambda(N)}) \). Again a simple argument shows that \( \hat{\omega}_N \) can be replaced by \( \bar{\omega}_N \) in the above inequality with \( \varepsilon_N \) replaced by a different constant \( \varepsilon'_N \) tending to zero as \( N \to \infty \). If \( \omega \) is a weak limit point of \( \bar{\omega}_N \) as \( N \to \infty \), we have

\[ s(\omega) \geq p(\Phi) - \omega(\Lambda_N) \]

Hence, the equality holds and further this equality holds for any \( \Phi \) (not only of finite range) (cf. [3, 7.4.1]).

**Theorem 2.** — Let \((\mathfrak{H}, \Lambda)\) satisfy a) b) and c). Then, the thermodynamic qualities can be defined and the variational principle holds.

### 5. Examples

First we give a known example in classical case, i.e. lattice gas with hard core of radius 1. Let \( F = \{0, 1\} \). For each finite \( \Lambda \) let \( \Omega_\Lambda \) be a subset of \( F^\Lambda \) such that \( \Omega_\Lambda = \{ \xi \in F^\Lambda; \xi_n \xi_m = 0 \text{ if } |n - m| = 1 \} \) where \( |n| = \Sigma_1^n |n_i| \).

Let $\mathcal{A}_\Lambda = C(\Omega_\Lambda)$. If $\Lambda \subset \Lambda'$, there is a natural injection of $\mathcal{A}_\Lambda$ into $\mathcal{A}_{\Lambda'}$, since the projection of $\Omega_{\Lambda'}$ into $\Omega_{\Lambda}$ generates $\Lambda$. If $\Lambda_1 \cap \Lambda_2 = \phi$, it follows from $\Omega_{\Lambda_1} \times \Omega_{\Lambda_2} \rightarrow \Omega_{\Lambda_1 \cup \Lambda_2}$ that there is a homomorphism of $\mathcal{A}_{\Lambda_1} \otimes \mathcal{A}_{\Lambda_2}$ into $\mathcal{A}_{\Lambda_1 \cup \Lambda_2}$, given by restriction. Further we have all properties given in $a)$, $b)$ and $c)$.

The corresponding quantum model is constructed as follows: we associate a $2 \times 2$ matrix algebra $\mathcal{B}_{(n)}$ with each $n \in \mathbb{Z}$, such that $\mathcal{B}_{(n)} \cong C(F_n)$ with $F_n = F$. Let $\partial \Lambda = \{ n \in \Lambda; \exists m \in \Lambda$ s. t. $|n - m| = 1 \}$. With each $\Lambda$ and $\xi \in \Omega_{\partial \Lambda}$ we associate a subfactor $\mathcal{A}_{\Lambda}^\xi$ of $\mathcal{B} = \otimes \mathcal{B}_{(n)}$, by $\mathcal{A}_{\Lambda}^\xi = \chi_{\Omega_{\Lambda}}^\xi \mathcal{B}_{(n)} \chi_{\Omega_{\Lambda}}^\xi$, where $\chi_{\Omega_{\Lambda}}^\xi$ is the characteristic function of $\chi_{\Omega_{\Lambda}}^\xi = \{ \eta \in \Omega_{\Lambda} : \eta \mid \partial \Lambda = \xi \}$. Let $\mathcal{A}_\Lambda$ be the algebra generated by $\mathcal{A}_{\Lambda}^\xi$, $\xi \in \Omega_{\partial \Lambda}$; $\mathcal{A}_\Lambda \simeq \oplus \mathcal{A}_{\Lambda}^\xi$.

If $\Lambda \subset \Lambda'$ and $\xi \in \Omega_{\partial \Lambda}$ and $\eta \in \Omega_{\partial \Lambda'}$, the map $\mathcal{A}_{\Lambda}^\xi$ of $\mathcal{A}_{\Lambda}^\xi$ into $\mathcal{A}_{\Lambda}^\eta$, is given by: $\Lambda \mapsto A_{\xi E_{\eta}}$, where $E_{\eta} = \{ \xi \in \Omega_{\Lambda' \setminus \Omega_{\partial \Lambda}} : \xi \mid \partial \Lambda = \xi, \xi \mid \partial \Lambda' = \eta \}$, which may be empty. Since $\bigcup_{\eta} E_{\eta} \neq \phi$ for each $\xi \in \Omega_{\partial \Lambda}$, this map is injective.

Let $\Lambda_1 \cap \Lambda_2 = \phi$ and let $\xi_1 \in \Omega_{\partial \Lambda_1}$, and $\xi_2 \in \Omega_{\partial \Lambda_2}$. The map $\Phi_{\Lambda_1 \Lambda_2}$ of $\mathcal{A}_{\Lambda_1}^\xi_1 \otimes \mathcal{A}_{\Lambda_2}^\xi_2$ into $\mathcal{A}_{\Lambda_1 \cup \Lambda_2}$ is given by: $\Lambda \mapsto A_{\xi_1 \times \xi_2}$ if $\xi_1 \times \xi_2 \in \Omega_{\partial \Lambda_1 \cup \Lambda_2}$ and $\Lambda \mapsto 0$ otherwise. It is easily shown that all properties in $a)$, $b)$ and $c)$ hold.

This is maximal in the sense that if there are a family $(\mathcal{A}_\Lambda)$ of local algebras satisfying $a)$, $b)$ and $c)$ and a family $(\phi_{\Lambda})$ of isomorphisms of $\mathcal{A}_\Lambda$ into $\mathcal{A}_\Lambda$ with multiplicity 1 satisfying the obvious consistency relations, then all $\phi_{\Lambda}$ are surjective.

Hence, we can take as $\mathcal{D}$ in section 2 the C*-subalgebra of $\mathcal{B} = \otimes \mathcal{B}_{(n)}$ of elements which commute with all $\chi_{\Omega}^\xi$, and as $I$ the ideal of $\mathcal{D}$ generated by all $1 - \chi_{\Omega}^\xi$. Then the family $(\mathcal{A}_\Lambda)$ constructed above is isomorphic to $(q(\mathcal{D} \cap \mathcal{B}_{(n)}))$ where $q$ is the quotient map of $\mathcal{D}$ onto $\mathcal{D}/I$.

We notice that if the distance between $\partial \Lambda$ and $\partial \Lambda'$ is larger than 1 in case $\Lambda \subset \Lambda'$, then $E_{\eta} \neq \phi$ for any $\xi \in \Omega_{\partial \Lambda}$ and $\eta \in \Omega_{\partial \Lambda'}$. Thus each subfactor $\mathcal{A}_{\Lambda}^\xi$ of $\mathcal{A}_\Lambda$ is mapped into each subfactor $\mathcal{A}_{\Lambda'}^\eta$ of $\mathcal{A}_{\Lambda'}$. So the C*-inductive limit $\mathcal{A}$ of $\mathcal{A}_\Lambda$ is simple [7].

Both the classical and quantum models above satisfy: if $\Lambda_1 \cap \Lambda_2 = \phi$ and the distance between $\Lambda_1$ and $\Lambda_2$ is larger than 1, $\Phi_{\Lambda_1 \Lambda_2}$ is an isomorphism of $\mathcal{A}_{\Lambda_1} \otimes \mathcal{A}_{\Lambda_2}$ onto $\mathcal{A}_{\Lambda_1 \cup \Lambda_2}$.

Any finite-dimensional abelian algebra $C$ can be a quasi-local algebra by setting $\mathcal{A}_\Lambda = C$ for all $\Lambda$. This is maximal in the sense above but not simple.

There is an example of local algebras where $(\mathcal{A}_\Lambda)$ satisfies $a)$, $b)$ and $c)$ except the second part of $c)$. Let $\mathcal{B}_\Lambda$ be a usual quantum lattice system and set $\mathcal{A}_\Lambda = \mathcal{B}_\Lambda^0$ (or $\mathcal{B}_\Lambda \otimes \mathcal{B}_\Lambda^0$) with $\mathcal{B}_\phi = C.1$, where $\Lambda^0$ is the interior of $\Lambda$, i. e. $\Lambda = \Lambda \setminus \partial \Lambda$. 

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