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Two-channel Hamiltonians and the optical model of nuclear scattering

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ABSTRACT. — We study some qualitative properties of the scattering operator for a two-channel Hamiltonian which describes elastic neutron scattering from a heavy nucleus. Conditions are given under which the directly scattered wave-function may be separated from that subject to compound scattering, and under which the former may be calculated using an optical potential. We also adapt the Kato-Birman theory to prove the existence and completeness of the scattering operator for a non-unitary time evolution with an arbitrarily large coupling constant.

§ 1. INTRODUCTION

The extremely complicated energy dependence of the cross-sections for the various reactions occurring when fast neutrons are scattered off nuclei indicate that their detailed explanation may require a full treatment of the associated many-body problem. Much of the theory therefore concerns phenomenological models which explain the general features of some particular reaction over a specified range of neutron energies and nuclear masses. These models make use of many interesting ideas, some of which are capable of rigorous analysis, the purpose of which is to clarify the nature of the approximations involved in their use.

In this paper we study a model for shape and compound elastic neutron scattering from a nucleus which is heavy enough to have a very large number of internal excited states. The physical literature [*10*, *14*] tells us that the wave-function of an elastically scattered neutron can be separated as a sum of two parts, one of which is scattered directly, the other being absorbed to form a compound nucleus which subsequently decays re-emitting the neutron in a direction independent of its initial velocity. The directly scattered part of the wave-function can be computed approximately by the use of an optical potential, which is a non-self-adjoint energy-dependent operator accounting for the effect of the compound nucleus channel on the free neutron channel.

It turns out that the features described above can be described by a two-channel Hamiltonian with three natural time scales. The separation between the direct and compound scattering processes corresponds to the division between the standard time scale and a much longer one, determined by the half-life for the decay for the compound nucleus. The optical model for the directly scattered wave-function is then justified by the existence of a very short time scale for the entry of the neutron into the compound nucleus. Our treatment has the virtues of clarifying the distinct nature of these two approximations and of being sufficiently general that it can be applied in other two-channel scattering problems [*15*].

We let $\mathcal{H}_0$ be the Hilbert space of the free neutron channel and $\mathcal{H}_1$ the Hilbert space of the compound nucleus, and put $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$. The free Hamiltonians of the two channels denoted by H_0 and H_1 respectively. The channels are coupled by a self-adjoint operator A which, for the sake of simplicity, we assume to be bounded. If P_i are the projections of $\mathcal{H}$ onto $\mathcal{H}_i$ we write $A_{ij} = P_i A P_j$ and assume that $A_{00} = A_{11} = 0$. The total Hamiltonian is

$$H = H_0 + H_1 + \mu A. \tag{1.1}$$

We are primarily interested in the free neutron part of the time evolution in the interaction picture, that is in $P_0 S_t P_0$ where

$$S_t = e^{i(H_0+H_1)t} e^{-2iHt} e^{i(H_0+H_1)t}.$$

We define the associated scattering operator S to be the limit of S_t as $t \to \infty$, when this limit exists. In Section 2 we study the separation between direct and compound scattering as the density of energy levels in the compound nucleus increases, and in Section 3 we follow this up by analysing some situations where the use of optical potentials can be theoretically justified. In both cases we aim not for greatest generality but consider the simplest models which exhibit clearly the underlying mechanisms. Sections 5 and 6 are of a rather more sophisticated nature, showing how the Kato-Birman trace class conditions for scattering theory may be adapted to one-parameter contraction semigroups.

§ 2. COMPOUND AND DIRECT SCATTERING

Although the energy spectrum of the total elastic cross-section is very irregular, these irregularities are not observable if the incoming wave function has a significant energy spread. We suppose that such wave functions lie in a dense subspace $\mathscr{L}$ of $\mathscr{H}_0$ and that $\mathscr{L}$ has a norm $|\cdot|$ such that

$$\int_{-\infty}^{\infty} \| A_{10}e^{-iH_0t}\phi \| (1 + |t|^{\alpha})dt \leqslant |\phi|$$

for all $\phi \in \mathscr{L}$ and some $0 < \alpha \leqslant 1$. The fairly rapid decrease of $A_{10}e^{-iH_0t}\phi$ in time corresponds to a slow variation of the energy density of ϕ, by the uncertainty principle.

An important condition is that the discrete spectrum of the compound nucleus Hamiltonian H_1 has on the average a smooth variation with energy. In our model this corresponds to the assumption that there exists a Hamiltonian H_1' on $\mathscr{H}_1$ with absolutely continuous spectrum such that $\| H_1 - H_1' \| < \varepsilon$ for some small $\varepsilon > 0$. The following theorem describes the sense in which the time evolution with respect to H and H′ (defined by replacing H_1 by H_1' in (1.1)) are close uniformly in time.

THEOREM 2.1. — If $\phi, \psi \in \mathscr{L}$ and $0 \leqslant t < \infty$ then

$$|\langle S_t\phi, \psi \rangle - \langle S_t'\phi, \psi \rangle| \leqslant 2\varepsilon^{\alpha} |\phi| |\psi|.$$

Proof. — Starting from the expansion

$$e^{-iHt} = e^{-i(H_0+H_1)t} - i\mu\int_{s=0}^{t} e^{-i(H_0+H_1)(t-s)}Ae^{-i(H_0+H_1)s}ds$$
$$- \mu^2\int_{s=0}^{t}\int_{u=0}^{s} e^{-i(H_0+H_1)(t-s)}Ae^{-iH(s-u)}Ae^{-i(H_0+H_1)u}duds$$

we obtain

$$\langle S_t\phi, \psi \rangle = \langle \phi, \psi \rangle - \mu^2\int_{s=-t}^{t}\int_{u=-t}^{s} \langle e^{iH_0s}Ae^{-iH(s-u)}Ae^{-iH_0u}\phi, \psi \rangle\, duds$$

so

$$|\langle S_t\phi, \psi \rangle - \langle S_t'\phi, \psi \rangle|$$
$$\leqslant \mu^2\int_{s=-t}^{t}\int_{u=-t}^{s} \| e^{-iH(s-u)} - e^{-iH'(s-u)} \| \| Ae^{-iH_0u}\phi \| \| Ae^{-iH_0s}\psi \|\, duds.$$

Now interpolating between

$$\| e^{-iHs} - e^{iH's} \| \leqslant \begin{cases} 2 \\ \| H - H' \| |s| \end{cases}$$

we obtain

$$\| e^{-iH(s-u)} - e^{-iH'(s-u)} \| \leqslant 2^{1-\alpha}\varepsilon^{\alpha} |s - u|^{\alpha}$$
$$\leqslant 2\varepsilon^{\alpha}(|s|^{\alpha} + |u|^{\alpha})$$

so

$$|\langle S_t\phi, \psi \rangle - \langle S_t'\phi, \psi \rangle|$$
$$\leqslant \int_{s=-t}^{t}\int_{u=-t}^{s} \varepsilon^{\alpha}(|s|^{\alpha} + |u|^{\alpha}) \| Ae^{-iH_0u}\phi \| \| Ae^{-iH_0s}\psi \|\, duds$$
$$\leqslant \varepsilon^{\alpha} |\phi| |\psi|.$$

As we shall see, it is not possible to prove that S_t and S'_t are close in the strong operator topology uniformly in time under the same hypotheses. One has only the following weaker result.

THEOREM 2.2. — If $\phi \in \mathscr{L}$ and $0 \leqslant s, t < \infty$ then

$$\| e^{-iH(t+s)}e^{iH_0 t}\phi - e^{-iH'(t+s)}e^{iH_0 t}\phi \| \leqslant 2\mu\varepsilon^{\alpha}(1 + s^{\alpha}) \mid \phi \mid. \tag{2.1}$$

Proof. — Starting from

$$e^{-iH(t+s)}\phi = e^{-iH_0(t+s)}\phi - i\mu\int_{u=0}^{t+s} e^{-iH(t+s-u)}Ae^{-iH_0 u}\phi du$$

we deduce that

$$e^{-iH(t+s)}e^{iH_0 t}\phi = e^{-iH_0 s}\phi - i\mu\int_{u=-t}^{s} e^{-iH(s-u)}Ae^{-iH_0 u}\phi du.$$

If I denotes the left-hand side of (2.1) then

$$\begin{aligned} I &\leqslant \mu\int_{u=-t}^{s} \| e^{-iH(s-u)} - e^{-iH'(s-u)} \| \, \| Ae^{-iH_0 u}\phi \| du \\ &\leqslant 2\mu\varepsilon^{\alpha}\int_{u=-t}^{s} (\mid s \mid^{\alpha} + \mid u \mid^{\alpha}) \| Ae^{-iH_0 u}\phi \| du \\ &\leqslant 2\mu\varepsilon^{\alpha}(1 + s^{\alpha}) \mid \phi \mid. \end{aligned}$$

The fact that the two evolution operators are close in the weak operator topology uniformly in time but close in the strong operator topology only up to finite positive times has a direct physical interpretation. An ingoing wave function has one part which is directly scattered and another part which is absorbed to form a compound nucleus. This second part remains in the compound nucleus for a long time but eventually must be re-emitted into the free neutron channel if H_1 has pure point spectrum. Correspondingly if $\phi \in \mathscr{L} \subseteq \mathscr{H}_0$ then $S\phi$ lies entirely in $\mathscr{H}_0$ but is the sum of a term approximately equal to $P_0S'\phi$ and a term which has a long time delay and therefore is weakly negligible although not small in norm. For the approximate Hamiltonian H′ the second part of the wave-function remains in $\mathscr{H}_1$ for all times $t > 0$.

§ 3. THE OPTICAL MODEL

We have seen that the average cross-section for *direct* elastic scattering may be described using a model where the compound nucleus Hamiltonian has absolutely continuous spectrum. With this assumption, which is discussed further in Section 4, we now turn to the theoretical justification for the use of the optical potential.

The optical potential B is required to be an operator on $\mathscr{H}_0$ such that

$e^{(-iH_0-B)t}$ and $P_0e^{-iHt}P_0$ are close to each other uniformly in time. If the coupling constant μ is very small then the choice

$$B_+ = \mu^2\int_0^\infty e^{iH_0t}A_{01}e^{-iH_1t}A_{10}dt \tag{3.1}$$

is suggested both by the rigorous theory of master equations (2, 3) and by the following perturbation theoretic argument.

If one expands the scattering operator formally in powers of μ one obtains

$$\lim_{t\to\infty} P_0e^{iH_0t}e^{-2iHt}e^{iH_0t}P_0 = P_0 - \mu^2\int_{t=-\infty}^{\infty}\int_{s=-\infty}^{t} e^{iH_0t}A_{01}e^{-iH_0(t-s)}A_{10}e^{-iH_0s}dsdt + O(\mu^4).$$

A similar calculation for the optical evolution in the subspace $\mathscr{H}_0$ leads to

$$\lim_{t\to\infty} e^{iH_0t}e^{(iH_0-B)2t}e^{iH_0t} = 1 - \int_{s=-\infty}^{\infty} e^{iH_0s}Be^{-iH_0s}ds + O(B^2).$$

The first non-trivial terms of these expansions are identical if one defines B by (3.1).

In spite of the above, two contrary arguments raise doubts about (3.1) being the correct choice for B, except in the weak coupling limit. The first is that the identity mentioned would be satisfied equally well by the choice

$$B_- = \mu^2\int_0^\infty A_{01}e^{-iH_1t}A_{10}e^{iH_0t}dt.$$

The second is that simple calculations in the case where A_{01} is of rank one make it clear that neither of $(-iH_0 - B_\pm)$ is generally dissipative so that the associated semigroups are not contraction semigroups. This conflicts with the fact that $P_0e^{-iHt}P_0$ are contractions for all $t \in \mathbb{R}$.

It eventually emerges that the correct choice for B is that made by Feshbach [*7*, *8*] namely

$$B = \mu^2\int_0^\infty A_{01}e^{-iH_1t}A_{10}dt = \lim_{\varepsilon\to 0} \mu^2A_{01}(\varepsilon + iH_1)^{-1}A_{10}. \tag{3.2}$$

The operator $-B$ is always dissipative by a trivial calculation and it turns out that in circumstances where the optical approximation is a good one, B, B_+ and B_- are all approximately equal. As well as being better behaved theoretically the simple resolvent expression (3.2) for B encourages belief that the practical use of such operators is not inconceivable.

Having chosen B, we have the problem of establishing in what circumstances the optical approximation is a good one. Among the various methods available we choose the smooth perturbation technique of Kato [*11*] because it enables one to handle the scattering theory of the non-unitary group $e^{(-iH_0-B)t}$ with least trouble. We need to assume that μ is small enough for certain power series to converge, but do not consider the weak coupling limit $\mu \to 0$. We emphasise this point, and shall return to it, because many people have strong feelings that weak coupling limit calculations are not of much physical significance.

We assume that the operator A_{10} may be factorised in the form $A_{10} = C_1^* C_0$ where the operator-valued functions

$$X_j(t) = \begin{cases} C_j e^{-iH_j t} C_j^* & \text{if} \quad t > 0 \\ 0 & \text{if} \quad t \leqslant 0 \end{cases}$$

are both L^1 norm integrable. We shall in fact assume that both operators are bounded and norm continuous for $t \neq 0$, but allow $\| X_j(t) \|$ to become unbounded as $t \to 0$; this type of condition is frequently satisfied in non-relativistic potential scattering theory. By changing μ if necessary we normalise $X_j(t)$ by

$$\int_{-\infty}^{\infty} \| X_j(t) \| \, dt = 1$$

for $i = 1, 2$. The operator

$$\overline{X}_1 = \int_0^{\infty} C_1 e^{-iH_1 t} C_1^* dt$$

is then bounded and

$$B = \mu^2 C_0^* \overline{X}_1 C_0.$$

We borrow from Kato [*11*] the crucial estimate

$$\int_{-\infty}^{\infty} \| C_j e^{-iH_j t} \phi \|^2 \, dt \leqslant \| \phi \|^2$$

valid for all $\phi \in \mathscr{H}_j$. This enables us to deduce that the Dyson expansion for e^{iHt} in the interaction picture is norm uniformly convergent in time provided the coupling constant μ is smaller than unity, which we henceforth assume.

If $\phi_1, \phi_2 \in \mathscr{H}_0$ and

$$S^t = P_0 e^{iH_0 t} e^{-2iHt} e^{iH_0 t} P_0$$

then the Dyson expansion gives

$$\begin{aligned} \langle S^t \phi_1, \phi_2 \rangle &= \langle \phi_1, \phi_2 \rangle \\ &- \mu^2 \int_{s=-t}^{t} \int_{u=-t}^{s} \langle e^{iH_0 s} A_{01} e^{-iH_1(s-u)} A_{10} e^{-iH_0 u} \phi_1, \phi_2 \rangle \, du ds \\ &+ \dots \end{aligned} \tag{3.3}$$

Writing

$$\psi_i^t(s) = \begin{cases} C_0 e^{-iH_0 s} \phi_i & \text{if} \quad | s | \leqslant t \\ 0 & \text{otherwise} \end{cases}$$

so that ψ_i^t lies in the Hilbert space $L^2(\mathbb{R}, \mathscr{H}_0)$, we may rewrite (3.3) in the form

$$\begin{aligned} \langle S^t \phi_1, \phi_2 \rangle = \langle \phi_1, \phi_2 \rangle &- \mu^2 \langle X_1 \circ \psi_1^t, \psi_2^t \rangle \\ &+ \mu^4 \langle X_1 \circ X_0 \circ X_1 \circ \psi_1^t, \psi_2^t \rangle + \dots \end{aligned} \tag{3.4}$$

where $\circ$ denotes convolution.

If one defines the operator S^t_{opt} on $\mathscr{H}_0$ by

$$S^t_{opt} = e^{iH_0t}e^{(-iH_0-B)2t}e^{iH_0t}$$

then one obtains the corresponding expansion

$$\langle S^t_{opt}\phi_1, \phi_2 \rangle = \langle \phi_1, \phi_2 \rangle - \mu^2 \langle \overline{X}_1 . \psi^t_1, \psi^t_2 \rangle + \mu^4 \langle \overline{X}_1 . X_0 \circ \overline{X}_1 . \psi^t_1, \psi^t_2 \rangle + \dots \quad (3.5)$$

where $\circ$ denotes convolution and $\cdot$ denotes pointwise multiplication.

THEOREM 3.1. — If

$$\alpha = \| X_1 \circ \psi^t_1 - \overline{X}_1 . \psi^t_1 \|_2 \quad (3.6)$$

and

$$\beta = \| X_1 \circ X_0 - \overline{X}_1 . X_0 \|_1 \quad (3.7)$$

then

$$\| S^t\phi_1 - S^t_{opt}\phi_1 \| \leqslant \mu^2\alpha(1-\mu^2)^{-1} + \mu^4\beta \| \phi_1 \| (1-\mu^2)^{-2}.$$

Proof. — Collecting together terms of (3.4) and (3.5) involving the same powers of μ, we obtain

$$\begin{aligned} I \equiv | \langle S^t\phi_1, \phi_2 \rangle - \langle S^t_{opt}\phi_1, \phi_2 \rangle | & \\ \leqslant \mu^2 | \langle X_1 \circ \psi^t_1, \psi^t_2 \rangle - \langle \overline{X}_1 . \psi^t_1, \psi^t_2 \rangle | & \\ + \mu^4 | \langle X_1 \circ X_0 \circ X_1 \circ \psi^t_1, \psi^t_2 \rangle - \langle \overline{X}_1 . X_0 \circ \overline{X}_1 . \psi^t_1, \psi^t_2 \rangle | & \\ + \dots & \end{aligned}$$

Using (3.6) and (3.7) together with the estimates

$$\| X_0 \|_1 = \| X_1 \|_1 = 1 \quad , \quad \| \psi^t_1 \|_2 \leqslant \| \phi_1 \| \quad , \quad \| \psi^t_2 \|_2 \leqslant \| \phi_2 \|$$

and $\| \overline{X}_1 \| \leqslant 1$, leads to

$$\begin{aligned} I \leqslant \mu^2\alpha \| \phi_2 \| + \mu^4(\alpha \| \phi_2 \| + \beta \| \phi_1 \| \| \phi_2 \|) & \\ + \mu^6(\alpha \| \phi_2 \| + 2\beta \| \phi_1 \| \| \phi_2 \|) + \dots & \\ = \mu^2\alpha \| \phi_2 \| (1-\mu^2)^{-1} + \mu^4\beta \| \phi_1 \| \| \phi_2 \| (1-\mu^2)^{-2} & \end{aligned}$$

from which the theorem follows.

While it is the case that the scattering operators are approximately equal if μ is small enough, this is not interesting because each of them is then approximately equal to the identity operator. The significant point is that the scattering operators are also approximately equal if α and β are both small. For given ϕ_1 and X_1 it is intuitively obvious that α and β get smaller as $X_1(t)$ becomes more closely concentrated around the point $t = 0$. This corresponds to a difference of natural time scale between the two channels, and seems to be characteristic of all situations where Markovian or optical approximations can be justified. The shortness of the natural time scale for the compound nucleus channel might be explained in terms of its multiparticle nature since it is commonplace that the rate of decay of wave functions increases with the number of degrees of freedom.

If one wishes to have explicit error estimates the above theorem concludes the section, but to obtain a more mathematically clean result one can proceed as follows. We assume that the ratio of the natural time scales of the two channels is described by a parameter λ and that C_1 or H_1 or both depend on λ in such a way that

$$X_1^\lambda(t) = \lambda^{-2} X_1(\lambda^{-2} t).$$

The Hamiltonian H then depends on λ, as does the interaction operator S_λ^t. However since

$$\int_0^\infty \lambda^{-2} X_1(\lambda^{-2} t) dt = \overline{X}_1$$

the operator S_{opt}^t is independent of λ.

THEOREM 3.2. — If $\phi_1 \in \mathscr{H}_0$ then

$$\lim_{\lambda \to 0} \sup_{0 \leqslant t < \infty} \| S_\lambda^t \phi_1 - S_{\text{opt}}^t \phi_1 \| = 0$$

Proof. — It is sufficient to prove that

$$\lim_{\lambda \to 0} \| X_1^\lambda \circ \psi_1^t - \overline{X}_1 . \psi_1^t \|_2 = 0$$

$$\lim_{\lambda \to 0} \| X_1^\lambda \circ X_0 - \overline{X}_1 . X_0 \|_1 = 0$$

which may be done by standard approximation arguments using only the facts that $\psi_1^t \in L^2(\mathbb{R}, \mathscr{H}_0)$ and $X_0 \in L^1(\mathbb{R}, \mathscr{L}(\mathscr{H}_0))$.

Although we have emphasised that the limit $\lambda \to 0$ is very different from the weak coupling limit $\mu \to 0$, there is a sense in which the two are related. The easiest way of obtaining the λ-dependence of $X_1(t)$ is to replace C_1 by $\lambda^{-1} C_1$ and H_1 by $\lambda^{-2} H_1$.

The Hamiltonian then becomes

$$H = \lambda^{-2} \begin{bmatrix} \lambda^2 H_0 & \mu\lambda A_{01} \\ \mu\lambda A_{10} & H_1 \end{bmatrix}$$

The overall factor λ^{-2} can be interpreted as a time rescaling and one then has a weak coupling limit problem as $\lambda \to 0$, in which the « system » Hamiltonian $\lambda^2 H_0$ is λ-dependent. This is exactly the situation in which the master equation theory of [*2*, *3*] applies, although our problem is new in that context because it involves uniform limits in time.

We mention that other studies show that the weak coupling limit theory may also be used to solve singular coupling problems [*5*, *16*] and even problems concerning the interaction of a system with a reservoir in the low density limit [*17*]. All of the problems seem to have a common mathematical core, even if their physical interpretations are very different.

§ 4. EFFECTS OF OTHER REACTIONS

It was necessary in the above theory to modify the compound nucleus Hamiltonian before considering the optical potential approximation, because the requirement that

$$\int_{-\infty}^{\infty} \| X_1(t) \| \, dt < \infty \tag{4.1}$$

cannot be satisfied if H_1 has pure point spectrum.

The condition that the spectrum of H_1 is continuous may also be defended on the grounds that gamma decay involves a coupling of the compound nucleus to the electromagnetic field, and leads to the energy levels of the nucleus being diffuse.

There is another way of allowing for the effect of gamma decay, or other reactions. Namely we can give up the requirement that H_1 is self-adjoint and suppose instead that it has complex eigenvalues determined by the energies and decay rates of the bound states. Then e^{-iH_1t} is a one-parameter contraction semigroup on $\mathscr{H}_1$ and, even if this space is finite-dimensional, it is still possible for (4.1) to be satisfied. With this change the calculations of Section 3 may still be applied provided the eigenvalues of H_1 have large imaginary parts.

We thus see that the optical approximation may be justified either by the eigenvalues of H_1 being real but very numerous and close together, or by their having large imaginary parts. The non-unitary nature of the scattering matrix is explained in the first case by compound nuclear scattering and in the second case by gamma decay. Of course it is easy to combine these effects by letting H_1 have continuous and complex spectrum.

§ 5. NON-UNITARY SCATTERING THEORY

The work of the earlier sections indicates the desirability of developing a general scattering theory for one-parameter contraction semigroups without any restriction on the size of the coupling constant. We start from first principles, since the literature is rather restricted in scope [*9*, *11*, *13*].

We assume that e^{At} is a one-parameter unitary group on the Hilbert space $\mathscr{H}$ and that e^{Bt} is a strongly continuous one-parameter contraction semigroup. If J is a contraction on $\mathscr{H}$ we investigate the existence of the limits

$$W_1\phi = \lim_{t\to\infty} e^{Bt}Je^{-At}\phi \tag{5.1}$$

$$W_2^*\phi = \lim_{t\to\infty} e^{-At}Je^{Bt}\phi \tag{5.2}$$

the corresponding expressions for $t \to -\infty$ not being meaningful. We shall not mention W_2, but write W_2^* in (5.2) as a concession to established notation in the unitary case. The scattering operator S is defined as $W_2^* W_1$, and is a contraction on its domain.

We define the subspace M(B) of $\mathscr{H}$ by

$$M(B) = \left\{ \phi \in \mathscr{H} : \int_0^\infty |\langle e^{Bt}\phi, \psi \rangle|^2 dt \leqslant c_\phi \| \psi \|^2 \quad \text{for all} \quad \psi \in \mathscr{H} \right\}$$

and write $\mathscr{H}_B^{ac}$ for the closure of M(B). It is an easy exercise to prove that if e^{Bt} is a unitary group then M(B) may be characterised as the set of all $\phi \in \mathscr{H}$ such that

$$\int_{-\infty}^\infty |\langle e^{Bt}\phi, \psi \rangle|^2 dt \leqslant c_\phi \| \psi \|^2$$

for all $\psi \in \mathscr{H}$, and that $\mathscr{H}_B^{ac}$ is then the usual absolutely continuous subspace. In spite of this some remarks below indicate that our definition of $\mathscr{H}_B^{ac}$ may not be the right one for some purposes.

LEMMA 5.1. — The subspaces M(B) and $\mathscr{H}_B^{ac}$ are invariant under e^{Bt}. If $\phi \in \mathscr{H}_B^{ac}$ then

$$\lim_{t\to\infty} \langle e^{Bt}\phi, \psi \rangle = \lim_{t\to\infty} \| Ce^{Bt}\phi \| = 0$$

for all $\psi \in \mathscr{H}$ and all compact operators C on $\mathscr{H}$.

Proof. — The invariance is obvious. If $\phi \in M(B)$ and $\psi \in \mathscr{H}$ and

$$f(t) = \langle e^{Bt}\phi, \psi \rangle$$

then as $\varepsilon \to 0$

$$|f(t+\varepsilon) - f(t)| = |\langle e^{Bt}(e^{B\varepsilon}\phi - \phi), \psi \rangle| \leqslant \| \psi \| \; \| e^{B\varepsilon}\phi - \phi \| \to 0$$

so f is uniformly continuous. Since also $f \in L^2$ it follows that $f(t) \to 0$ as $t \to \infty$. The result for general $\phi \in \mathscr{H}_B^{ac}$ follows by approximation, and that for compact C by approximation from the case where C has finite rank.

The first part of the following theorem is an adaptation of a famous result of Cook [*12, 19*].

THEOREM 5.2. — If there is a dense subspace $\mathscr{L}$ of $\mathscr{H}_A^{ac}$ such that

$$\int_0^\infty \| Ce^{-At}\phi \| \, dt < \infty$$

for all $\phi \in \mathscr{L}$, where $C = BJ - JA$, then $W_1\phi$ exists for all $\phi \in \mathscr{H}_A^{ac}$ and is a contraction from $\mathscr{H}_A^{ac}$ into $\mathscr{H}_B^{ac}$ such that

$$e^{Bt}W_1 = W_1 e^{At}$$

for all $t \geqslant 0$.

Proof. — We prove only the last statement. If $\phi \in \mathscr{H}_A^{ac}$ then

$$\begin{aligned} W_1\phi &= \lim_{t\to\infty} e^{B(t+a)}Je^{-A(t-a)}\phi \\ &= e^{Ba}W_1e^{-Aa}\phi \end{aligned}$$

so $W_1e^{Aa}\phi = e^{Ba}W_1\phi$ as stated. If also $\phi \in M(A)$ then

$$\begin{aligned} \int_0^\infty |\langle e^{Bt}W_1\phi, \psi\rangle|^2dt &= \int_0^\infty |\langle e^{At}\phi, W_1^*\psi\rangle|^2dt \\ &\leqslant \| W_1^*\psi \|^2 \leqslant \| \psi \|^2 \end{aligned}$$

so $W_1\phi \in M(B)$. By approximation we deduce that $W_1(\mathscr{H}_A^{ac}) \subseteq \mathscr{H}_B^{ac}$.

While the above theorem can also be used to prove the existence of W_2^* in principle, the conditions are not easy to verify in the cases of interest. We therefore use instead a modification of the Kato-Birman theory as presented in [*18*, *19*]. It is rather strange that the following analysis seems *not* to be applicable to the apparently simpler problem of the existence of W_1.

THEOREM 5.3. — If $C = JB - AJ$ lies in the trace ideal $\mathscr{I}_1$ then $W_2^*\phi$ exists for all $\phi \in \mathscr{H}_B^{ac}$.

Proof. — We rewrite the proofs of [*18*, *19*] in a form which does not require e^{Bt} to be a unitary group. Writing $W(t) = e^{-At}Je^{Bt}$ we see that

$$W(t)' = e^{-At}Ce^{Bt} \in \mathscr{I}_1$$

so

$$W(t) - W(s) = \int_s^t W'(x)dx \in \mathscr{I}_1 \subseteq \mathscr{I}_\infty$$

for all $s, t \geqslant 0$. Hence if $\phi \in M(B)$

$$\begin{aligned} &\lim_{x\to\infty} \| W(t+x)\phi - W(s+x)\phi \| \\ &\leqslant \lim_{x\to\infty} \| (W(t) - W(s))e^{Bx}\phi \| = 0. \end{aligned}$$

It follows that

$$\begin{aligned} \| W(t)\phi - W(s)\phi \|^2 &= - \int_0^\infty \frac{\partial}{\partial x} \| W(t+x)\phi - W(s+x)\phi \|^2 dx \\ &\leqslant \sum_{r=1}^{8} \int_0^\infty f_r(s, t, x)dx \end{aligned}$$

where

$$\begin{aligned} f_1(s, t, x) &= |\langle W(t+x)'\phi, W(s+x)\phi\rangle| \\ &= |\langle e^{-A(t+x)}Ce^{B(t+x)}\phi, e^{-A(s+x)}Je^{B(s+x)}\phi\rangle| \\ &= |\langle Ce^{B(t+x)}\phi, e^{A(t-s)}Je^{B(s+x)}\phi\rangle| \end{aligned}$$

and the other f_r have similar forms.

Since $C \in \mathcal{I}_1$ it has an expansion

$$C = \sum_{r=1}^{\infty} \lambda_r | \phi_r \rangle \langle \psi_r |$$

where $\| \phi_r \| = \| \psi_r \| = 1$, $\lambda_r \geqslant 0$ and $\Sigma \lambda_r < \infty$. Thus

$$\left\{ \int_0^\infty f_1(s, t, x) dx \right\}^2$$

$$\leqslant \left\{ \int_0^\infty \sum_r \lambda_r | \langle e^{B(t+x)}\phi, \psi_r \rangle \langle e^{A(t-s)} J e^{B(s+x)} \phi, \phi_r \rangle | dx \right\}^2$$

$$\leqslant \| J \|^2 \sum_r \lambda_r . \int_t^\infty \sum_r \lambda_r | \langle e^{Bx}\phi, \psi_r \rangle |^2 dx$$

which converges to zero as $t \to \infty$. Carrying out similar calculations for the other f_r we obtain

$$\lim_{s,t \to \infty} \| W(t)\phi - W(s)\phi \| = 0$$

from which the theorem follows.

For applications we need the following routine extension of the above theorem.

THEOREM 5.4. — If there exist non-negative integers m, n such that

$$(1 - A)^{-m}(JB - AJ)(1 - B)^{-n} \varepsilon \mathcal{I}_1$$

and if

$$\| \, | A |^{\frac{1}{2}} J\phi \| \leqslant C(\| B\phi \| + \| \phi \|)$$

for all $\phi \in \mathcal{H}_B^{ac} \cap$ *Dom* B then $W_2^* \phi$ exists for all $\phi \in \mathcal{H}_B^{ac}$.

We omit the proof, which is virtually the same as in the unitary case.

Example 1. — Suppose that $J = 1$ and that A has absolutely continuous spectrum. Then the above theorems enable one to prove the existence of

$$S = W_2^* W_1 = s\text{-}\lim_{t \to \infty} e^{-At} e^{2Bt} e^{-At}$$

as a contraction on $\mathcal{H}$. However there remain unsolved many problems which cannot arise for unitary groups. For example the equality of the ranges of W_1 and W_2 is not obvious because

i) $\mathcal{H}_B^{ac}$ and $\mathcal{H}_{B^*}^{ac}$ are not known to be equal;

ii) if B has an isolated eigenvalue with finite multiplicity and negative real part then by [*13*] the range of W_1 does not equal $\mathcal{H}_B^{ac}$.

Example 2. — We consider a two-channel system with $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$

$$A = \begin{bmatrix} -iH & 0 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} -iH & -X^* \\ X & Z \end{bmatrix} \qquad J = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

and where H is a self-adjoint operator on $\mathscr{H}_1$ with absolutely continuous spectrum, Z is the generator of a contraction semigroup on $\mathscr{H}_2$ and X is a bounded operator, for simplicity.

THEOREM 5.5. — The limit

$$S = s\text{-}\lim_{t\to\infty} e^{-At}Je^{2Bt}Je^{-At}$$

exists and defines a contraction on $\mathscr{H}_1$ commuting with H provided:

i) there is a dense subspace $\mathscr{L}$ of $\mathscr{H}_1$ such that

$$\int_0^\infty \| Xe^{-iHt}\phi \| \, dt < \infty$$

for all $\phi \in \mathscr{L}$;

ii) $(1 + iH)^{-m}X^* \in \mathscr{I}_1$ for some $m \geqslant 0$.

Proof. — Since B is a bounded perturbation of $\begin{bmatrix} -iH & 0 \\ 0 & Z \end{bmatrix} \equiv B_0$ it is the generator of a strongly continuous one-parameter contraction semigroup on $\mathscr{H}$. By a direct computation

$$BJ - JA = \begin{bmatrix} 0 & 0 \\ X & 0 \end{bmatrix} \quad , \quad JB - AJ = \begin{bmatrix} 0 & -X^* \\ 0 & 0 \end{bmatrix}.$$

Conditions *i*) and *ii*) now allow the application of Theorems 5.2 and 5.4. Note particularly that since A commutes with J and B is a bounded perturbation of B_0

$$\begin{aligned} \| \, |A|^{\frac{1}{2}}J\phi \| &= \| J \, |A|^{\frac{1}{2}}\phi \| \leqslant \| \phi \| + \| A\phi \| \\ &\leqslant \| \phi \| + \| B_0\phi \| \leqslant C(\| \phi \| + \| B\phi \|) \end{aligned}$$

for all $\phi \in$ *Dom* B.

Notes i) The conditions of the theorem involve no knowledge about the operator Z describing the time evolution in the unobserved channel. In particular it is irrelevant whether Z has discrete or continuous spectrum;

ii) One may allow X to be relatively bounded without difficulty;

iii) In a multi-channel system one may similarly prove the existence of

$$S_{ij} = s\text{-}\lim_{t\to\infty} e^{-At}J_ie^{2Bt}J_je^{-At}$$

where the projections J_i onto the subspaces for different reaction products all commute with the skew-adjoint operator A.

Example 3. — We consider a simple model for the partial reflection and partial absorption of an electron by a « half-line » of solid. This model can be greatly generalised by the methods of [6], whose results we use freely.

We let e^{Bt} be a strongly continuous one-parameter contraction semigroup on $\mathscr{H} = L^2(\mathbb{R})$ such that B is a local (differential) operator. If $A = i\Delta$ we assume that *Dom* B $\subseteq$ *Dom* $(|A|^{\frac{1}{2}})$ and that

$$\| |A|^{\frac{1}{2}} f \| \leqslant C(\| Bf \| + \| f \|)$$

for all $f \varepsilon$ *Dom* B. We assume that $A = B$ on $(-\infty, 0)$ and in particular that

$$L^2(-\infty, 0) \cap Dom\, A = L^2(-\infty, 0) \cap Dom\, B.$$

We let J be a C^∞ function on $\mathbb{R}$ such that $0 \leqslant J \leqslant 1$, $J(x) = 1$ if $x \leqslant a$, $J(x) = 0$ if $x \geqslant b$, where $-\infty < a < b < 0$. The asymptotic subspaces $\mathscr{H}_1^\pm$ are defined by

$$\begin{aligned} \mathscr{H}_1^\pm &= \{ f : \lim_{t \to \pm\infty} e^{-At} J e^{At} f = f \} \\ &= \{ f : \lim_{t \to \pm\infty} \| J e^{At} f \| = \| f \| \}. \end{aligned} \tag{5.3}$$

These subspaces do not depend on the particular choice of J and satisfy $\mathscr{H} = \mathscr{H}_1^+ \oplus \mathscr{H}_1^-$.

THEOREM 5.6. — The operator

$$S = s\text{-}\lim_{t \to \infty} e^{-At} J e^{2Bt} J e^{-At}$$

exists, is independent of the particular choice of J and is a contraction from $\mathscr{H}_1^-$ into $\mathscr{H}_1^+$ which commutes with A.

Proof. — Since B is local, $(BJ - JA)$ and $(JB - AJ)$ are first order differential operators with C^∞ coefficients of compact support. The existence of

$$W_1 \phi = \lim_{t \to \infty} e^{Bt} J e^{-At} \phi$$

for all $\phi \in \mathscr{H}$ follows from Theorem 5.2 since

$$\int_0^\infty \| (BJ - JA) e^{-At} \phi \| \, dt$$

holds for all ϕ in a dense subspace of $\mathscr{H}$. Theorem 5.4 implies that

$$W_2^* \phi = \lim_{t \to \infty} e^{-At} J e^{Bt} \phi$$

exists for all $\phi \in \mathscr{H}_B^{ac}$ since

$$(1 - A)^{-m} (JB - AJ) \in \mathscr{I}_1$$

for all $m \geqslant 2$. Moreover since J is a bounded operator on the quadratic form domain of A

$$\| |A|^{\frac{1}{2}} J \phi \| \leqslant a \| |A|^{\frac{1}{2}} \phi \| \leqslant ac(\| B\phi \| + \| \phi \|).$$

The fact that the range of S lies in $\mathscr{H}_1^+$ is proved as follows. If J' is a function of the same type as J and satisfying J'J = J we define the asymptotic projection P_1^+ as in [6] by

$$P_1^+ = s\text{-}\lim_{t\to\infty} e^{-At}J'e^{At}.$$

If $\phi \in \mathscr{H}_B^{ac}$ then

$$\begin{aligned} P_1^+W_2^*\phi &= \lim_{t\to\infty} e^{-At}J'e^{At}W_2^*\phi \\ &= \lim_{t\to\infty} [e^{-At}J'e^{At}\{W_2^*\phi - e^{-At}Je^{Bt}\phi\} + e^{-At}Je^{Bt}\phi] \end{aligned}$$

so

$$\| P_1^+W_2^*\phi - W_2^*\phi \| \leqslant \lim_{t\to\infty} \| e^{-At}Je^{At}\{W_2^*\phi - e^{-At}Je^{Bt}\phi\} \| = 0.$$

Finally to prove that S is independent of J, we note that if J' is another function of the same type then (J − J') is a C^∞ function of compact support. Since $(J - J')(1 - A)^{-1/2}$ is compact

$$\lim_{t\to\infty} e^{Bt}(J - J')e^{-At}\phi = 0$$

for all $\phi \in (1 - A)^{-1/2}\mathscr{H}$ and hence all $\phi \in \mathscr{H}$. Since also

$$(J - J')(1 - B)^{-1} = (J - J')(1 - A)^{-1/2} \cdot (1 - A)^{1/2}(1 - B)^{-1}$$

is also compact

$$\lim_{t\to\infty} e^{-At}(J - J')e^{Bt}\phi = 0$$

for all $\phi \in (1 - B)^{-1}\mathscr{H}_B^{ac}$ and hence all $\phi \in \mathscr{H}_B^{ac}$.

Finally to prove that S commutes with A we note that if $t \geqslant 0$

$$Se^{At} = W_2^*W_1e^{At} = W_2^*e^{Bt}W_1 = e^{At}W_2^*W_1 = e^{At}S.$$

From this we deduce that

$$e^{-At}S = e^{-At}(Se^{At})e^{-At} = e^{-At}(e^{At}S)e^{-At} = Se^{-At}$$

so S commutes with the unitary group and hence with A.

§ 6. COMPLETENESS OF THE SCATTERING OPERATOR

For unitary time evolution with J = 1 the scattering is said to be complete if $S = W_2^*W_1$ is unitary, or equivalently if

$$\text{Range } W_1 = \text{Range } W_2.$$

The condition that this range equals the absolutely continuous subspace of B is often added, but is not always appropriate. Martin [13] has proposed a non-unitary analogue of the above.

We shall adopt a different definition for the non-unitary case with $J \neq 1$, motivated by the idea that S captures the whole of the wave-function still existing at large positive time. Specifically we demand that

$$\| S\phi \| = \lim_{t\to\infty} \| e^{-At}e^{2Bt}e^{-At}\phi \|$$

for all ϕ in the subspace $\mathscr{H}_1^-$ defined in (5.3).

LEMMA 6.1. — The scattering operator S is complete provided

$$\lim_{t\to\infty} \| (1 - J)e^{Bt}\psi \| = 0 \tag{6.1}$$

for all $\psi \in W_1(\mathscr{H}_1^-)$.

Proof. — Since

$$S\phi = \lim_{t\to\infty} e^{-At}Je^{Bt}W_1\phi$$

$$\begin{aligned} \Big| \| S\phi \| - \lim_{t\to\infty} \| e^{-At}e^{2Bt}e^{-At}\phi \| \Big| & \\ &\leqslant \lim_{t\to\infty} \| e^{-At}Je^{Bt}W_1\phi - e^{-At}e^{2Bt}e^{-At}\phi \| \\ &\leqslant \lim_{t\to\infty} \| e^{-At}Je^{Bt}W_1\phi - e^{-At}e^{Bt}W_1\phi \| \\ &+ \lim_{t\to\infty} \| e^{-At}e^{Bt}(W_1\phi - e^{Bt}e^{-At}\phi) \| \\ &\leqslant \lim_{t\to\infty} \| (1 - J)e^{Bt}W_1\phi \| + \lim_{t\to\infty} \| W_1\phi - e^{Bt}e^{-At}\phi \|. \end{aligned}$$

The first limit vanishes by hypothesis and the second is dominated by

$$\begin{aligned} &\lim_{t\to\infty} \| e^{Bt}Je^{-At}\phi - e^{Bt}e^{-At}\phi \| \\ &= \lim_{t\to\infty} \| e^{Bt}Je^{-At}\phi - e^{Bt}e^{At}P_1^-\phi \| \\ &\leqslant \lim_{t\to\infty} \| e^{At}Je^{-At}\phi - P_1^-\phi \| = 0. \end{aligned}$$

We shall in fact prove that (6.1) holds for all ψ in the subspace

$$\mathscr{H}_B^0 = \{ \phi : \lim_{t\to\infty} \langle e^{Bt}\phi, \psi \rangle = 0 \quad \text{for all } \psi \in \mathscr{H} \}.$$

This subspace contains $\mathscr{H}_B^{ac}$, is invariant under e^{Bt}, and has the following additional properties.

LEMMA 6.2. — If $B\phi = \lambda\phi$ then $\phi \in \mathscr{H}_B^0$ if $\mathrm{Re}\lambda < 0$ and $\phi \perp \mathscr{H}_B^0$ if $\mathrm{Re}\lambda = 0$. Moreover $\mathscr{H}_B^0 = \mathscr{H}$ under either of the following conditions:

i) e^{Bt} is completely non-unitary in the sense of [20].

ii) $B = -iH - V$ where $H = H^*$ has absolutely continuous spectrum and $V \geqslant 0$ is bounded.

Proof. — The result for $\mathrm{Re}\lambda < 0$ is trivial. If $\mathrm{Re}\lambda = 0$ then

$$\begin{aligned}\langle e^{\lambda t}e^{B^*t}\phi, \phi \rangle &= \langle e^{\lambda t}\phi, e^{Bt}\phi \rangle \\ &= e^{\lambda t}e^{\bar{\lambda} t}\langle \phi, \phi \rangle = \| \phi \|^2\end{aligned}$$

and $\| e^{\lambda t}e^{B^*t}\phi \| \leqslant \| \phi \|$ so $e^{\lambda t}e^{B^*t}\phi = \phi$. Now given $\psi \in \mathscr{H}^0_B$

$$\begin{aligned}0 &= \lim_{t\to\infty} \langle e^{Bt}\psi, \phi \rangle = \lim_{t\to\infty} \langle \psi, e^{B^*t}\phi \rangle \\ &= \lim_{t\to\infty} e^{\lambda t}\langle \psi, \phi \rangle\end{aligned}$$

so $\langle \psi, \phi \rangle = 0$.

The proofs that *i)* and *ii)* imply $\mathscr{H}^0_B = \mathscr{H}$ depend heavily upon the theory of unitary dilations of one-parameter contraction semigroups [*20*]. There exists a unique decomposition $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$ into invariant subspaces, there $\mathscr{H}_1$ is the largest subspace on which e^{Bt} are unitary operators, by ([*20*], p. 145). If $\phi \in \mathscr{H}_2$ then $\phi \in \mathscr{H}^0_B$ since the minimal unitary dilation of e^{Bt} on this subspace always has absolutely continuous spectrum by ([*20*], § 3.8 and 3.9). This completes *i)* since for e^{Bt} to be completely non-unitary means that $\mathscr{H}_1 = 0$.

Now suppose that $\mathscr{H}_1 \neq 0$ and that *ii)* is satisfied. If $\phi \in \mathscr{H}_1 \cap \mathit{Dom}\, B$ then

$$0 = \frac{d}{dt}\| e^{Bt}\phi \|^2 = -2\langle Ve^{Bt}\phi, e^{Bt}\phi \rangle.$$

Since $V \geqslant 0$ it follows that $Ve^{Bt}\phi = 0$ so

$$\frac{d}{dt}(e^{Bt}\phi) = -iH(e^{Bt}\phi)$$

and

$$e^{Bt}\phi = e^{-iHt}\phi$$

for all $t \geqslant 0$. By the absolute continuity of H we deduce that $\phi \in \mathscr{H}^0_B$. The result follows for all $\phi \in \mathscr{H}_1$ by approximation.

THEOREM 6.3. — Let e^{Bt} be a one-parameter contraction semigroup on $\mathscr{H}$ and J a bounded self-adjoint operator such that

$$B^*J^2 + J^2B = C + D$$

where $D \leqslant -\alpha J^2$ for some $\alpha > 0$ and $(1 - B^*)^{-n}C(1 - B)^{-n}$ is compact for some integer $n \geqslant 0$. Then

$$\lim_{t\to\infty} \| Je^{Bt}\phi \| = 0$$

for all $\phi \in \mathscr{H}^0_B$.

Proof. — If $\phi \in \mathscr{H}^0_B \cap \mathit{Dom}\, B$ and

$$f(t) = \| Je^{Bt}\phi \|^2$$

then

$$\begin{aligned} f'(t) &= \langle (B^*J^2 + J^2B)e^{Bt}\phi, e^{Bt}\phi \rangle \\ &= \langle (C + D)e^{Bt}\phi, e^{Bt}\phi \rangle \\ &\leqslant g(t) - \alpha f(t) \end{aligned} \tag{6.2}$$

where

$$g(t) = \langle Ce^{Bt}\phi, e^{Bt}\phi \rangle.$$

Now $\lim_{t\to\infty} g(t) = 0$ for all $\phi \in \mathscr{H}_B^0$ by the compactness of $(1 - B^*)^{-n}C(1 - B)^{-n}$. Elementary calculus applied to the differential inequality (6.2), together with the fact that $f(t) \geqslant 0$ for all $t \geqslant 0$, implies that $f(t) \to 0$ as $t \to \infty$.

Example 1. — Let $\mathscr{H} = L^2(\mathbb{R})$ and let $B = i\dfrac{d^2}{dx^2} - V$ where $V \geqslant 0$ is bounded and $V(x) \geqslant \alpha > 0$ for all $x \geqslant 0$. Let J be a C^∞ function with $0 \leqslant J \leqslant 1$, $J(x) = 1$ for $x \geqslant 0$ and $J(x) = 0$ for $x \leqslant -a < 0$.

THEOREM 6.4. — $\lim_{t\to\infty} \| Je^{Bt}\phi \| = 0$ for all $\phi \in \mathscr{H}$.

Proof. — We see by Lemma 6.2 *ii*) that $\mathscr{H}_B^0 = \mathscr{H}$. Now

$$\begin{aligned} &B^*J^2 + J^2B \\ &= (iP^2 - V)J^2 + J^2(-iP^2 - V) \\ &= A_0 + A_1P + D \end{aligned}$$

where A_0, A_1 are C^∞ functions of compact support and $D \leqslant -\alpha J^2$. Since V is bounded B and P^2 have the same domain and

$$\| (1 + P^2)(1 - B)^{-1} \| < \infty$$

Therefore

$$\begin{aligned} &(1 - B^*)^{-1}(A_0 + A_1P)(1 - B)^{-1} \\ &\quad = (1-B^*)^{-1}(1+P^2)\cdot(1+P^2)^{-1}(A_0+A_1P)(1+P^2)^{-1}\cdot(1+P^2)(1-B)^{-1} \end{aligned}$$

which is compact.

Example 2. — If $J = 1$ and $B = i\Delta - V$ on $\mathscr{H} = L^2(\mathbb{R}^n)$ where $V \geqslant 0$ is bounded and $V(x) \geqslant \alpha > 0$ for all x outside some compact set then

$$\lim_{t\to\infty} \| e^{Bt}\phi \| = 0$$

for all $\phi \in \mathscr{H}$. Norm convergence of e^{Bt} to zero has been proved in [*4*] under slightly different hypotheses and we conjecture that it is also valid in the present circumstances.

It is obvious that most of our applications to Schrödinger operators can be extended to unbounded potentials in higher space dimensions by the methods of [*1*, *6*], but we leave this question to a future publication.

REFERENCES

[1] M. Combescure and J. Ginibre, Scattering and local absorption for the Schrödinger operator. *J. Functional Anal.* t. **29**, 1978, p. 54-73.
[2] E. B. Davies, Markovian master equations. *Commun. Math. Phys.*, t. **39**, 1974, p. 91-110.
[3] E. B. Davies, Markovian master equations. II. *Math. Ann*, t. **219**, 1976, p. 147-158.
[4] E. B. Davies, A model for absorption or decay. *Helv. Phys. Acta*, t. **48**, 1975, p. 365-382.
[5] E. B. Davies, Asymptotic analysis for some abstract evolution equations. *J. Functional Anal.*, t. **25**, 1977, p. 81-101.
[6] E. B. Davies and B. Simon, Scattering theory for systems with different spatial asymptotics on the left and right. *Commun. Math. Phys.*, t. **63**, 1978, p. 277-301.
[7] H. Feshbach, Unified theory of nuclear reactions. I. *Ann. Phys.*, t. **5**, 1958, p. 357-390.
[8] H. Feshbach, Unified theory of nuclear reactions. II. *Ann. Phys.*, t. **19**, 1962, p. 287-313.
[9] C. Goldstein, Perturbation of non-self-adjoint operators. II. *Arch. Rat. Mech. Anal.*, t. **42**, 1971, p. 380-402.
[10] P. E. Hodgson, *Nuclear reactions and nuclear structure.* Clarendon Press, Oxford, 1971.
[11] T. Kato, Wave operators and similarity for some non-self-adjoint operators. *Math. Ann.*, t. **162**, 1966, p. 258-279.
[12] T. Kato, *Perturbation theory for linear operators.* Springer, 1966.
[13] Ph. A. Martin, Scattering theory with dissipative interactions and time delay *Nuovo Cim.*, t. **30** B, 1975, p. 217-238.
[14] W. E. Meyerhof, *Elements of Nuclear Physics.* McGraw-Hill, 1967.
[15] R. G. Newton, *Scattering theory of waves and particles.* McGraw-Hill, 1966.
[16] P. F. Palmer, The singular coupling and weak coupling limits. *J. Math. Phys.*, t. **18**, 1977, p. 527-529.
[17] P. F. Palmer, *D. Phil. thesis*, Oxford, 1976.
[18] D. Pearson, A generalisation of the Birman trace theorem. *J. Functional Anal.* t. **28**, 1978, p. 182-186.
[19] M. Reed and B. Simon, *Methods of modern mathematical physics. III. Scattering theory.* Academic Press, 1979.
[20] B. Sz.-Nagy and C. Foias, *Harmonic analysis of operators on Hilbert space.* North-Holland, 1970.

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