ANNALES DE L'I. H. P., SECTION A

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Annales de l'I. H. P., section A, tome 29, nº 2 (1978), p. 179-205
<http://www.numdam.org/item?id=AIHPA_1978__29_2_179_0>

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Ann. Inst. Henri Poincaré,
Vol. XXIX, n° 2, 1978, p. 179-205.

Section A :
Physique théorique.

The inverse problem at fixed energy for finite range complex potentials

by

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ABSTRACT. — In the inverse problem of scattering theory, a simple transformation may be used to map the results for a fixed $l = 0$ value of the angular momentum onto similar results at fixed energy concerning a finite-range potential. This property is applied to non self-adjoint operators corresponding to complex potentials. Following a study of such operators at fixed l value, one gives the condition on the inverse problem data to be coherent, the fundamental equation is derived and the uniqueness of the reconstructed potential is proved. A last paragraph shows that the inverse problem data of this final potential are the initial ones.

RÉSUMÉ. — Dans l'étude du problème inverse de la diffusion, une transformation simple permet de passer des résultats obtenus à valeur fixée nulle du moment angulaire à des résultats semblables à énergie fixe concernant un potentiel de portée finie. Cette propriété est appliquée ici aux opérateurs non-self adjoints associés aux potentiels complexes. L'étude du problème inverse correspondant à de tels opérateurs ayant été faite à l fixé, celui du problème inverse à énergie fixée s'en déduit aisément : condition pour que les données soient cohérentes, dérivation de l'équation fondamentale, et unicité du potentiel reconstruit. Le dernier paragraphe assure la cohérence de l'ensemble en prouvant que les données du problème inverse correspondant au potentiel final sont bien celles qu'on a choisies au départ.

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The inverse problem has given rise to many publications. A glance at the concerned litterature shows that there are principaly two different ways of studying this problem:

— the first attitude is to start from the potential itself. The aim of the study will be to find the functions or the groups of functions which are necessary to reproduce the given potential. The question of uniqueness of the reproduction of the potential must be solved;

— the second attitude is somewhat different. We want to derive some potential from experimental data. Is this problem solvable and with what method? If the problem is solvable, is it uniquely? Let us remark that this potential may reproduce fairly well some experimental data (in particular the phase-shifts for one given value of l or one given value of the energy), but will not necessarily reproduce the other ones.

In this paper we shall adopt the first point of view, and we shall show that the inverse problem data are not the experimental scattering data.

I. POSITION OF THE PROBLEM AND NOTATIONS

The results of Agranovich and Marchenko [*1*] concerning the inverse problem are well known. These authors considered the radial Schrödinger equation with fixed $l = 0$ value of the angular momentum and real potentials

$$\left\{ \frac{d^2}{dr^2} - \mathrm{U}(r) + k^2 \right\} u(r, k) = 0 \qquad 0 \leqslant r < \infty \tag{1}$$

Ljance [*2*, *3*] has extended their method for potentials $\mathrm{U}(r)$ which are no longer real functions of r. We are also interested in non self-adjoint operators, but our aim is to solve the corresponding inverse problem at fixed energy.

Loeffel [*4*] has shown that for finite range potentials a relatively simple transformation permits in the real case to map the results of the inverse problem at fixed $l = 0$ angular momentum onto analogous results for the inverse problem at fixed energy. In this paper we shall consider complex finite range potentials. And we will show that the same transformation as in the real case may be used for complex potentials. Using this transformation and the results of Ljance [*3*] we shall be able to derive explicitely the fundamental equation from the inverse problem data of the problem.

So, following Loeffel [*4*], we suppose a potential of finite range equal to 1 (the extension to any other value being straightforward). And we write:

$$x = e^{-r} \qquad r = -\log x \qquad x \in]0, 1] \tag{2.a}$$

$$\nu = -ik \tag{2.b}$$

The first transformation is a bijection which maps the open half-line $[0, \infty[$ of the variable r onto the open interval $[1, 0[$ of the corresponding variable x. The second transformation performs a rotation of $-\pi/2$ in the whole plane of the variable k to obtain the new variable ν. To these changes of variable one associates the following transformation from $f \in \mathscr{L}^2(0, \infty)$ onto $g \in \mathscr{L}^2(0, 1)$:

$$f(r, k) = e^{-r/2}g(e^{-r}, -ik) = x^{1/2}g(x, \nu) \tag{3}$$

such a transformation is a unitary isomorphism, and $-\frac{d}{dr}$ is transformed into $D = x\frac{d}{dx} + \frac{1}{2} = \frac{1}{2}\left\{x, \frac{d}{dx}\right\}$.

Let us introduce the solution of Eq. (1) satisfying the integral equation:

$$u(r, k) = e^{ikr} - \int_r^\infty \frac{\sin k(r-t)}{k} U(t)u(t, k)dt \tag{4}$$

Obviously we have:

$$\lim_{r \to +\infty} e^{-ikr}u(r, k) = 1 .$$

Transformations (2) and (3) map $u(r, k)$ onto $\varphi(x, \nu)$ such as:

$$u(r, k) = e^{-r/2}\varphi(e^{-r}, -ik) = x^{1/2}\varphi(x, \nu) \tag{5}$$

$$\lim_{x \to 0} \varphi(x, \nu)x^{-\nu+1/2} = 1 \tag{6}$$

Besides we introduce $q(x)$ as:

$$U(r) = q(e^{-r}) = q(x) \tag{7}$$

With our new notations (Eq. 1) writes:

$$\left\{-\frac{d}{dx}\ x^2\frac{d}{dx}\ + q(x) + \nu^2 - \frac{1}{4}\right\}\varphi(x, \nu) = 0 \qquad 0 < x \leqslant 1 \tag{8}$$

which is the equation of the inverse problem at fixed energy studied in the real case by Loeffel. We shall call $\mathscr{L}$ the non self-adjoint operator generated in the Hilbert space $\mathscr{L}^2(0, 1)$ by this equation, the eigenvalues of $\mathscr{L}$ being $-\nu^2$. We recall that $\nu^2 - \frac{1}{4} = l(l+1)$. $V(x)$ is defined by:

$$q(x) = x^2[V(x) - 1] \tag{9}$$

and condition (6) shows that $\varphi(x, \nu)$ is the regular solution of (8). This last solution exists in the complex case and has been considered by various authors [*5*, *3*]. The wronskian associated to $\mathscr{L}$ is:

$$\begin{aligned} Wr(f, g)(x) &= x^2[f'(x)g(x) - f(x)g'(x)] \\ &= \frac{d}{dx}[xf(x)]xg(x) - xf(x)\frac{d}{dx}[xg(x)] \end{aligned}$$

We shall in the following use the double transformation (2) to map the results of Ljance [3] onto analoguous results of the inverse problem at fixed energy; so we shall suppose $q(x)$ continuous and assume everywhere the condition on $U(r)$ imposed by Ljance:

$$\int_0^\infty e^{\varepsilon r} \, | U(r) | \, dr < \infty \qquad \text{for some} \qquad \varepsilon > 0 \tag{10}$$

or after transformation (2.a):

$$\int_0^1 x^{-1-\varepsilon} \, | q(x) | \, dx < \infty \qquad \text{for some} \qquad \varepsilon > 0 \tag{11}$$

which is the condition imposed by Loeffel, after identifying ε and 2η.

II. THE DIRECT PROBLEM

1. A fundamental system of solutions and the kernel K(x, u)

We have already introduced the two corresponding functions $u(r, k)$ and $\varphi(x, v)$. The domain of holomorphy of φ deduces itself from that of u: for $0 < x \leqslant 1$, it is the half-plane delimited by the condition: Re $v > -\dfrac{\varepsilon}{2}$.

Another useful solution of Eq. (10) is defined by the equation:

$$u_1(r, k) = e^{-ikr} + \frac{1}{2ik} \int_a^r e^{ik(r-t)} U(t) u_1(t, k) dt + \frac{1}{2ik} \int_r^\infty e^{ik(t-r)} U(t) u_1(t, k) dt \qquad 0 < a \leqslant r < \infty \tag{12}$$

This solution is holomorphic in k when Im $k \geqslant 0$, $| k | > k_\delta$ and satisfies the limit relation

$$\lim_{r \to \infty} e^{ikr} u_1(r, k) = 1 .$$

So we introduce:

$$u_1(r, k) = e^{-r/2} \psi(e^{-r}, -ik) = x^{1/2} \psi(x, v) \tag{13}$$

The corresponding limit relation will be:

$$\lim_{r \to 0} \psi(x, v) x^{v+1/2} = 1 \tag{14}$$

and ψ is another solution of Eq. 8. Its domain of holomorphy is related to that of u_1 by a rotation of $-\pi/2$ in the k-plane for $0 < x \leqslant 1$, and for each $v_0 > 0$, ψ is holomorphic in v when $| v | > v_0$, Re $v > 0$.

In the case of a real potential V(x), $\varphi^*(x, -v^*)$ and $\psi(x, v)$ are solutions of the same differential equation with the same limit condition. They are identical. However, in the case we are interested in, $\varphi^*(x, -\nu^*)$ is no more a solution of Eq. 8, for it corresponds to the potential $V^*(x) \neq V(x)$, and to $q^*(x) \neq q(x)$. The value of $Wr(\nu)$, wronskian of $x\varphi(x, \nu)$ and of $x\varphi(x, -\nu)$ in their common domain of holomorphy, is given immediately from the corresponding wronskian of $u(r, k)$ and $u(r, -k)$:

$$Wr(\nu) = 2\nu \qquad |\,\mathrm{Re}\ \nu\,| < \frac{\varepsilon}{2} \tag{15}$$

So $x\varphi(x, \nu)$ and $x\varphi(x, -\nu)$ form a fundamental system of functions. Except perhaps at the origin, it is the same for $\varphi(x, \nu)$ and $\varphi(x, -\nu)$.

Similarly, for $|\,\nu\,| > \nu_0$, $\mathrm{Re}\ \nu > 0$, the wronskian $Wr_1(\nu)$ of $x\varphi(x, \nu)$ and $x\psi(x, \nu)$ writes:

$$Wr_1(\nu) = 2\nu \tag{16}$$

and the same conclusion holds.

Under very general assumptions, it has been proved [6] that $u(r, k)$ admits an integral representation:

$$u(r, k) = e^{ikr} + \int_r^\infty k(r, t)e^{ikt}dt \tag{17}$$

and that $k(r, t)$ has continuous derivatives with respect to r and to t. If we set:

$$k(r, t) = e^{-r/2}e^{-t/2}K(e^{-r}, e^{-t}) = (xu)^{1/2}K(x, u) \tag{18}$$

Eq. 17 becomes

$$\varphi(x, \nu) = x^{\nu-1/2} + \int_0^x K(x, u)u^{\nu-1/2}du \tag{19}$$

and a Mellin transform takes place for the Fourier transform. Estimates for K and its derivatives are easily obtained from the corresponding ones [3]:

$$|\,K(x, u)\,| < C(xu)^{\frac{\varepsilon-1}{2}} \tag{20}$$

$$|\,K'_x(x, u)\,|,\ |\,K'_u(x, u)\,| \leqslant x^{-3/2}u^{-3/2}\left\{C(xu)^{\frac{\varepsilon-1}{2}} + C'(x^{3/2}u)^{\varepsilon} + \frac{1}{4}|\,q[(xu)^{1/2}]\,|\right\} \tag{21}$$

where C and C′ are certain numbers.

As last if we define $\tilde{K}(x, -\nu)$ by:

$$\tilde{K}(x, -\nu) = \int_0^x K(x, u)u^{\nu-1/2}du \tag{22}$$

it is easily seen that $\tilde{K}(x, -\nu)$ is holomorphic in ν for $\mathrm{Re}\ \nu > -\frac{\varepsilon}{2}$ for each value of x such as $0 < x \leqslant 1$. Furthermore, for each $\delta < \frac{\varepsilon}{2}$ when $\mathrm{Re}\, v \geqslant -\delta$:

$$|\tilde{\mathrm{K}}(x, -v) \leqslant \frac{\mathrm{C}_\delta}{1 + |v|} x^{\varepsilon + \mathrm{Re}\, v - \frac{1}{2}} \tag{23}$$

where C_δ is a certain positive number.

2. The spectrum of the non self-adjoint operator $\mathscr{L}$

DEFINITION 1. — A function $\mathscr{E}_1(v)$ will be said a function of type (E_1) in the half plane $\mathrm{Re}\, v > -\varepsilon_0$ ($\varepsilon_0 > 0$) if:

a) $\mathscr{E}_1(v)$ is holomorphic when $\mathrm{Re}\, v > -\varepsilon_0$ and for each $\eta < \varepsilon_0$ one has:

$$\mathscr{E}_1(v) = 1 + \mathcal{O}\left(\frac{1}{v}\right) \tag{24}$$

uniformly in the half-plane $\mathrm{Re}\, v \geqslant -\eta$.

b) $\mathscr{E}_1(v) \neq 0$ for $0 < |\mathrm{Re}\, v| < \varepsilon_0$.

c) If $\mathrm{Re}\, v = 0$, $v \neq 0$ and $\mathscr{E}_1(v) = 0$ then $\mathscr{E}_1(-v) \neq 0$.

d) If $\mathscr{E}_1(0) = 0$ then $\mathscr{E}'_1(0) \neq 0$.

Let $w(v)$ be defined by the relation:

$$w(v) = \varphi(1, v) \tag{25}$$

As $\varphi(1, v) = u(0, -ik)$ the properties of $w(v)$ deduce easily from these of $u(0, -ik)$, and we can state Theorem 1 (cf. Ljance [3], Lemma 3.1):

THEOREM 1. — The singular numbers of the operator $\mathscr{L}$ are the roots of $w(v)$ in the domain $v \neq 0$, $\mathrm{Re}\, v > 0$. Their number is finite. If ε_1 is the distance from the imaginary axis to the non purely imaginary roots of the function $w(v)$, and if ε_0 is the minimum of ε and ε_1 then $w(v)$ is a function of type (E_1) in the half-plane $\mathrm{Re}\, v > -\varepsilon_0$.

We shall call β the number of the roots v_k of $w(v)$:

$$v_1, v_2, \ldots, v_\alpha \text{ are such that } \mathrm{Re}\, v_i > 0 \qquad i = 1, 2, \ldots, \alpha.$$

$$v_{\alpha+1}, v_{\alpha+2}, \ldots, v_\beta \text{ are such that } \mathrm{Re}\, v_i = 0 \qquad i = \alpha + 1, \ldots, \beta.$$

The multiplicity of the root will be in any case m_k.

For a real potential the roots of $w(v)$ are all real and simple ($m_k = 1$).

Let us now introduce the solution $\Phi(x, -v^2)$ of Eq. 8 defined by the initial conditions:

$$\Phi(1, -v^2) = 0 \tag{26.a}$$

$$\frac{d}{dx}[\Phi(x, -v^2)]_{x=1} = -1 \tag{26.b}$$

In their common domain of holomorphy, i. e. for $|\mathrm{Re}\, v| < \varepsilon/2$, the wronskian of $x\varphi(x, v)$ and $x\varphi(x, -v)$ is not zero (except for $v = 0$ but the functions are then confounded), so these functions are linearly independent. For $x \neq 0$, it will be the same for $\varphi(x, v)$ and $\varphi(x, -v)$ so every solution of Eq. 8 may be written as one of their linear combinations. One obtains readily:

$$\Phi(x, -v^2) = \frac{w(-v)\varphi(x, v) - w(v)\varphi(x, -v)}{2v} \tag{27}$$

Besides for $|v| > v_0$, $\text{Re } v > 0$ the same considerations are valid for $\varphi(x, v)$ and $\psi(x, v)$:

$$\Phi(x, -v^2) = \frac{w_1(v)\varphi(x, v) - w(v)\psi(x, v)}{2v} \tag{28}$$

where:

$$w_1(v) = \psi(1, v) \tag{29}$$

(we remark that for $V(x)$ real $w_1(v) = w^*(-v^*)$).

Let us define, for f finite:

$$\varpi(f, -v^2) = \int_0^1 f(x)\Phi(x, -v^2)dx \tag{30}$$

As said before the transformation (3) is unitary. So it is easy to transform Parseval's equality given by Ljance [3] and to obtain:

$$\int_0^1 f_1(x)f_2(x)dx = -\frac{i}{\pi}\int_0^\infty \varpi(f_1, -v^2)\varpi(f_2, -v^2)\frac{vdv^2}{w(v)w(-v)}$$
$$-\sum_{k=1}^{\alpha}\left\{\left(\frac{d}{dv^2}\right)^{m_k - 1} \mathrm{M}_k(-v^2)\varpi(f_1, -v^2)\varpi(f_2, -v^2)\right\}_{v^2 = v_k^2} \tag{31}$$

where the summation upon k is reduced to the non purely imaginary roots of $w(v)$, the multiplicity of each one being m_k. $\mathrm{M}_k(-v^2)$ may be deduced from the equivalent quantity in the k-plane

$$\mathrm{M}_k(-v^2) = -\frac{(v^2 - v_k^2)}{(m_k - 1)!}\,\frac{\dfrac{d}{dr}[e^{-r/2}\varphi(e^{-r}, v)]_{r=0}}{w(v)}$$
$$= -\frac{(v^2 - v_k^2)}{(m_k - 1)!}\,\frac{\left\{-\dfrac{1}{2}\varphi(1, v) + \left[\dfrac{d\varphi}{dx}(x, v)\right]_{x=1}\right\}}{w(v)}$$

Let us introduce the function $\xi(v)$, logarithmic derivative of $\varphi(x, v)$ at the point $x = 1$. We obtain:

$$\mathrm{M}_k(-v^2) = -\frac{(v^2 - v_k^2)}{(m_k - 1)!}\left[\xi(v) - \frac{1}{2}\right] \qquad \begin{cases} \text{Re } v > 0 \\ k = 1, 2, \ldots, \alpha \end{cases} \tag{32}$$

Parseval's equality is valid for any functions f_1 and f_2 finite and subject to the condition that one of f_1 and f_2 is such that:

$$\left(\frac{d}{d\nu^2}\right)^{j_k} \varpi(f, -\nu_k^2) = 0 \tag{33}$$

the ν_k being now the purely imaginary roots of $w(\nu)$ and j_k varying from 0 to $m_k - 1$. The set of functions satisfying Eq. 33 is dense.

DEFINITION 2. — A function $\mathscr{S}_1(\nu)$ will be said a function of type (S_1) in the strip $|\operatorname{Re} \nu| < \varepsilon_0$ if:

a) $\mathscr{S}_1(\nu)$ is meromorphic in this strip and for each $\eta < \varepsilon_0$ one has: $\mathscr{S}_1(\nu) = 1 + \mathcal{O}(1/\nu)$, $|\nu| \to \infty$ uniformly in the strip $|\operatorname{Re} \nu| \leqslant \eta$;

b) $\mathscr{S}_1(\nu)$ does not have non purely imaginary poles in the strip $|\operatorname{Re} \nu| < \varepsilon_0$;

c) $\mathscr{S}_1(\nu)\mathscr{S}_1(-\nu) = +1$. In particular $\mathscr{S}_1(0) = \pm 1$.

Let $S_1(\nu)$ be defined by:

$$S_1(\nu) = \frac{w(-\nu)}{w(\nu)} \tag{34}$$

The properties of $S_1(\nu)$ are easily deduced from the properties of the scattering function corresponding to Eq. 1. In fact:

$$S_1(\nu) = \frac{\varphi(1, -\nu)}{\varphi(1, \nu)} = \frac{u(0, ik)}{u(0, -ik)} = S(-ik) \tag{35}$$

THEOREM 2. — The function $S_1(\nu)$ is a function of type (S_1) in the strip $|\operatorname{Re}| < \varepsilon_0$ where ε_0 is defined in theorem 1. Furthermore $w(0) = 0$ if and only if $S_1(0) = -1$.

This theorem is deduced immediately from the corresponding Lemma 5.1 of Ljance [3]. So the singularities of $S_1(\nu)$ in the strip $|\operatorname{Re} \nu| < \varepsilon_0$ are only located on the imaginary axis. They are the roots of $w(\nu)$: $\rho_{\alpha+1}, \ldots, \rho_\beta$ each one with multiplicity m_k.

In the case of real potentials there are no such poles in the strip $|\operatorname{Re} \nu| < \varepsilon_0$: all the roots of $w(\nu)$ are real positive, the first of them being at the distance $\varepsilon_1 \geqslant \varepsilon_0$ of the origin. Let us now introduce the eigenfunction of the operator $\mathscr{L}$:

$$\Theta(x, \nu) = \Phi(x, -\nu^2)\frac{2\nu}{w(\nu)} \qquad \operatorname{Re} \nu > -\frac{\varepsilon}{2} \tag{36}$$

The function ϖ which appears in Parseval's equality (31) may be written, with the help of this new function:

$$\varpi(f, -\nu^2) = \int_0^1 f(x)\Theta(x, -\nu^2)\frac{w(\nu)}{2\nu}dx \qquad \text{for} \quad \operatorname{Re} \nu > -\frac{\varepsilon}{2} \tag{37.a}$$

$$= -\int_0^1 f(x)\Theta(x, -\nu^2)\frac{w(-\nu)}{2\nu}dx \quad \text{for} \quad \operatorname{Re}(-\nu) > -\frac{\varepsilon}{2} \tag{37.b}$$

So let us introduce:

$$\Omega(f, \nu) = \int_0^1 f(x)\Theta(x, -\nu^2)dx \quad \text{for} \quad \text{Re}\, \nu > -\frac{\varepsilon}{2} \tag{38}$$

Then

$$\begin{aligned}\int_0^\infty \varpi(f_1, -\nu^2)\varpi(f_2, -\nu^2)\frac{\nu d\nu^2}{w(\nu)w(-\nu)} &= \int_0^\infty \varpi(f_1, -\nu^2)\varpi(f_2, -\nu^2)\frac{2\nu^2 d\nu}{w(\nu)w(-\nu)} \\ &= \int_{-\infty}^{+\infty} \varpi(f_1, -\nu^2)\varpi(f_2, -\nu^2)\frac{\nu^2 d\nu}{w(\nu)w(-\nu)} \\ &= \frac{1}{4}\int_{-\infty}^{+\infty} \Omega(f_1, \nu)\Omega(f_2, -\nu)d\nu\end{aligned} \tag{39}$$

III. THE INVERSE PROBLEM

1. The symmetric factorization of a function of type (S_1) (cf. Riemann problem)

Following Ljance [3], we shall call « problem of symmetric factorization » the following problem. Given:

— a function $\mathscr{S}_1(\nu)$ of type (S_1) in the strip $|\text{Re}\, \nu| < \varepsilon_0$,

— a set of complex numbers $R_1, \ldots, R_\gamma$ belonging to the half-plane $\text{Re}\, \nu \geqslant \varepsilon_0$,

— a corresponding set of natural numbers $M_1, \ldots, M_\gamma$, we require the determination of the function $\mathscr{E}_1(\nu)$ of type (E_1) of the half-plane $\text{Re}\, \nu > -\varepsilon_0$ such as:

— $$\frac{\mathscr{E}_1(-\nu)}{\mathscr{E}_1(\nu)} = \mathscr{S}_1(\nu) \qquad |\text{Re}\, \nu| < \varepsilon_0$$

— $\mathscr{E}_1(\nu)$ possesses a root of multiplicity M_k at the point R_k $(k=1, \ldots, \gamma)$,

— $\mathscr{E}_1(0) \neq 0$ if $\mathscr{S}_1(0) = 1$.

— $\mathscr{E}_1(0) = 0$ if $\mathscr{S}_1(0) = -1$.

$\mathscr{S}_1(\nu)$, and the set of numbers (R_i, M_i) will be said inverse problem data Ljance [3] has solved the same problem in the plane of the variable $k = i\nu$, i. e. after rotation of the total plane of $+\pi/2$. He has given the condition which secures the unicity of the solution. We shall not here give his demonstration, but only transpose his result by effecting a rotation of $-\pi/2$.

We shall define the ν-index of a function $\mathscr{S}_1(\nu)$ of type (S_1) in the strip $|\text{Re}\, \nu| < \varepsilon_0$. Let $\mathscr{C}$ be a curve in this strip running from $+i\infty$ to $-i\infty$ and leaving all the roots (poles) of $\mathscr{S}_1(\nu)$ at its right (left) side. We shall call ν-index of $\mathscr{S}_1(\nu)$ and denote it by $\text{ind}_\nu \mathscr{S}_1$ the increment, divided by 2π, of the argument of a continuous branch arg $\mathscr{S}_1(\nu)$ when ν varies on $\mathscr{C}$ from $+i\infty$ to $-i\infty$.

THEOREM 3. — The problem of symmetric factorization is solvable if and only if:

$$\text{ind}_\nu \mathscr{S}_1 + 2(\text{M}_1 + \ldots + \text{M}_\gamma) + \frac{1}{2}[1 - \mathscr{S}_1(0)] = 0 \qquad (40)$$

Then the solution of the problem is unique.

We deduce immediately from theorem 3 that the function $w(\nu)$ is uniquely determined by the knowledge of the function $\text{S}_1(\nu)$, of the non purely imaginary singular numbers $\nu_1, \ldots, \nu_\alpha$ and of their multiplicities $m_1, \ldots, m_\alpha$. The condition of this solution is the following:

$$\text{ind}_\nu \text{S}_1 + 2(m_1 + \ldots + m_\alpha) + \frac{1}{2}[1 - \text{S}(0)] = 0 \qquad (41)$$

2. The function $\text{F}_s(x)$

To write the fundamental equations when the angular momentum is fixed, a generalized Fourier transform of the scattering function $\text{S}(k)$ is needed. This gives the function:

$$f_s(r) = \frac{1}{2\pi}\int_{-\infty+i\eta}^{+\infty+i\eta} [\text{S}(k) - 1] \exp irkdk \qquad 0 < \eta < \varepsilon_0 \qquad (42)$$

As preceedingly when we put: $f_s(r) = e^{-r/2}\text{F}_s(e^{-r}) = x^{1/2}\text{F}_s(x)$ this Fourier transform is replaced by a Mellin transform, and we introduce:

$$\text{F}_s(x) = \frac{i}{2\pi}\mathscr{L}_2 \lim_{a\to\infty}\int_{ia+\eta}^{-ia+\eta} [\text{S}_1(\nu) - 1]x^{\nu-1/2}d\nu \qquad 0 < \eta < \varepsilon_0 \qquad (43)$$

$\text{F}_s(x)$ does not depend on the value of η : $\text{S}_1(\nu)$ is meromorphic in the strip $|\,\text{Re}\,\nu\,|\,\varepsilon_0$ and its poles are all situated on the imaginary axis. $\text{S}_1(\nu)$ is a function of type (S_1) so there exists a number $\text{D}_1(\delta)$ such as:

$$|\,\text{S}_1(\nu) - 1\,| < \frac{\text{D}_1(\delta)}{|\,\nu\,|} \qquad \delta \leqslant |\,\text{Re}\,\nu\,| \leqslant \varepsilon_0 - \delta \qquad (44)$$

For Mellin transforms, Parseval's equality writes:

$$\int_0^\infty |\,\tilde{f}(x)\,|^2 dx = \int_{-\infty}^{+\infty} |\,f(y)\,|^2 dy \qquad (45)$$

with

$$\tilde{f}(x) = \frac{1}{\sqrt{2\pi}}\mathscr{L}_2 \lim_{a\to\infty}\int_{-a}^{+a} x^{-iy-1/2} f(y)dy \qquad (46)$$

Equation 41 may be written:

$$x^{-\eta}F_s(x) = \frac{1}{2\pi}\mathscr{L}_2 \lim_{a\to\infty}\int_{-a}^{+a}[S_1(\eta - iy) - 1]x^{-iy-1/2}dy \tag{47}$$

so:

$$\int_0^\infty |x^{-\eta}F_s(x)|^2 dx = \frac{1}{2\pi}\int_{-\infty}^{+\infty}|S_1(\eta - iy) - 1|^2 dy$$

$$\int_0^\infty |x^{-\eta}F_s(x)|^2 dx \leqslant \frac{D_1(\delta)}{2\pi}\int_{-\infty}^{+\infty}\frac{dy}{|\eta - iy|^2}$$

In conclusion, for any δ, $0 < \delta < \varepsilon_0/2$, there exists a number $D(\delta)$ such that:

$$\int_0^\infty |x^{-\eta}F_s(x)|^2 dx < D(\delta) \qquad \delta \leqslant \eta \leqslant \varepsilon_0 - \delta \tag{48}$$

3. The normalization polynomials

To solve the inverse problem, we need informations on the continuous spectrum of $\mathscr{L}$, and it is the role of the function $F_s(x)$, but also on its point spectrum, so we must introduce it. Let us define the analog of the function $f_k(r)$ of Ljance [3]:

$$F_k(x) = \frac{i}{2\pi}\int_{C_k}\left[\frac{w_1(v)}{w(v)} - 1\right]x^{v-1/2}dv \tag{49}$$

where C_k is constituted of a parallel to the imaginary axis situated in the interval $]0, \varepsilon_0[$ and of a circle of infinite radius in the right half plane, this curve being described in the direct sense. It would be straightforward to verify that $f_k(r) = e^{-r/2}F_k(e^{-r}) = x^{1/2}F(x)$. In formula 49, the integrant is meromorphic inside the curve C_k and its poles are the roots v_k of $w(v)$ (*). So:

$$F_k(x) = -\operatorname{Re} s_{v=v_k}\frac{w_1(v)}{w(v)}x^{v-1/2} \qquad (k = 1, 2, \ldots, \alpha) \tag{50}$$

We shall write:

$$F_k(x) = \bar{p}_k(x)x^{v_k-1/2} = p_k(\operatorname{Log} x)x^{v_k-1/2} = p_k\left(\frac{d}{dv}\right)[x^{v-1/2}]_{v=v_k} \tag{51}$$

where $p_k(x)$ are polynomials of degree $m_k - 1$ (m_k is the multiplicity of the root v_k). We call $\bar{p}_k(x)$ the corresponding logarithmic polynomials. The functions $\bar{p}_k(x)$ will be called the normalization polynomials of the operator $\mathscr{L}$. They reduce to constants in the case of a real potential, for all zeros of $w(v)$ are simple. They are in this last case the analog of the norma-

(*) These roots are not roots of $w_1(v)$ (cf. equations 16 and 29).

lization multipliers [1]: in the Appendix, following a method given by R. G. Newton [7], we calculate their value and we find Loeffel's result [4]: the polynomials $\bar{p}_k(x)$ reduce to constants C_k such as:

$$C_k = \left[\int_0^1 \varphi^2(x, \nu_k) dx \right]^{-1} \tag{52}$$

In every case we remark that the normalization polynomials determine the asymptotic behaviour for $x \to \infty$ of the eigenfunctions and associated functions of the operator $\mathscr{L}$.

Indeed if we combine equations 28 and 36 we obtain:

$$\Theta(x, \nu) = \frac{w_1(\nu)}{w(\nu)} \varphi(x, \nu) - \psi(x, \nu) \qquad \text{for} \quad |\nu| > \nu_0 \ \mathrm{Re}\, \nu > 0 \tag{53}$$

For every value of k ($k = 1, 2, \ldots, \alpha$), $\mathrm{Re}\, \nu_k > \varepsilon_0$. And ν_0 may be chosen $< \varepsilon_0$. So $\psi(x, \nu)$ is holomorphic in a neighbourhood of every ν_k. And:

$$\mathrm{Re}\, s_{\nu = \nu_k} \Theta(x, \nu) = \mathrm{Re}\, s_{\nu = \nu_k} \frac{w_1(\nu)}{w(\nu)} \varphi(x, \nu) \tag{54}$$

Taking in account equation 19 gives:

$$\mathrm{Re}\, s_{\nu = \nu_k} \Theta(x, \nu) = - \bar{p}_k(x) x^{\nu_k - 1/2} - \int_0^x \mathrm{K}(x, u) u^{\nu_k - 1/2} \bar{p}_k(u) du \tag{55}$$

This formula describes the asymptotics for $x \to \infty$ of the eigenfunctions and associated functions of the operator $\mathscr{L}$, corresponding to the point spectrum. Another remark concerns $F_k(x)$

$$\begin{aligned} F_k(x) &= \bar{p}_k(x) x^{\nu k - 1/2} \\ &= p_k (\mathrm{Log}\, x) x^{\nu k_1 - 1/2} x^i \nu^{k_2} \qquad \text{where} \quad \nu_k = \nu_{k_1} + i\nu_{k_2} \, | F_k(x) | < C x^{\varepsilon - 1/2} \end{aligned} \tag{56}$$

for one has: $0 < x \leqslant 1$.

4. The fundamental equation

In order to determine the operator $\mathscr{L}$ from the corresponding inverse problem data (i. e. the function $S_1(\nu)$, the singular numbers $\nu_1, \ldots, \nu_\alpha$ and their normalization polynomials), we introduce the following function which includes information at once on the continuous spectrum and on its point spectrum:

$$F(x) = F_s(x) - \sum_{k=1}^{\alpha} F_k(x) = F_s(x) - \sum_{k=1}^{\alpha} \bar{p}_k(x) x^{\nu_k - 1/2} \qquad 0 < x \leqslant 1 \tag{57}$$

Estimates 47 and 55 show that for any δ, $0 < \delta < \frac{\varepsilon_0}{2}$, there exists a number $D_1(\delta)$ such that for all η, $0 < \eta < \varepsilon_0 - \delta$:

$$\int_0^1 | x^{-\eta} F(x) |^2 dx < D_1(\delta) \tag{58}$$

Ljance [3] has shown the existence of the fundamental equation:

$$k(x, t) - \int_x^\infty k(x, u) f(u + t) du = f(x + t) \tag{59}$$

where

$$f(x) = f_s(x) + \sum_{k=1}^{\alpha} f_k(x) \tag{60}$$

and $k(x, t)$ is defined by means of Eq. 17.

Transformation (18) and its analog for f:

$$f(r) = e^{-r/2} F(e^{-r}) = x^{1/2} F(x) \tag{61}$$

give immediately the fundamental equation in the case of fixed energy:

$$K(x, u) - \int_0^x K(x, z) F(zu) dz = F(xu) \qquad 0 < u \leqslant x \leqslant 1 \tag{62}$$

for the corresponding functions $F(x)$ and $K(x, u)$.

5. Uniqueness of the solution of the inverse problem

DEFINITION 3. — A function $\varphi(x)$ defined on the interval (0, 1) will be said a function of type (F_1, ε) ($\varepsilon > 0$) if the two following condition hold:

a) the function $x^{1/2}\varphi(x)$ possesses a continuous derivative such as:

$$\int_0^1 x^{-\varepsilon/2} \left| \frac{d}{dx} [x^{1/2}\varphi(x)] \right| dx < \infty \tag{63}$$

b) if the function $Y_x(t) t^{\frac{\varepsilon - 1}{2}}$ is summable on $0 < t < x \leqslant 1$ and if

$$Y_x(t) = \int_0^x Y_x(u) \varphi(ut) du \qquad \text{for} \quad t < x \tag{64}$$

then $Y_x(t) = 0$ for $t < x$.

THEOREM 3. — The function $F(x)$ defined by equation 57 is a function of type (F_1, ε), where ε is the number from condition 11. Let us begin by showing condition *a*.

Let

$$\sigma(x) = \int_0^x |q(t)| \frac{dt}{t} \qquad \text{for} \quad x \leqslant 1 \tag{65}$$

$$\begin{aligned} x^{-\varepsilon}\sigma(x) &\leqslant \int_0^x t^{-1-\varepsilon} |q(t)| dt \qquad\qquad \text{for} \quad \varepsilon > 0 \\ &\leqslant \int_0^x t^{-1-\varepsilon} |q(t)| dt + \int_x^1 t^{-1-\varepsilon} |q(t)| dt \\ &\leqslant \int_0^1 t^{-1-\varepsilon} |q(t)| dt . \end{aligned}$$

So:

$$\sigma(x) \leqslant C_1 x^{\varepsilon} \qquad \text{with} \quad C_1 > 0 \tag{66}$$

Besides, Ljance [3] deduces from the fundamental equation 59 the existence of a continuous derivative $f'(r)$ and the estimate:

$$\left| f'(2r) - \frac{1}{4} U(r) \right| < C_2[\sigma_1(r)]^2 \qquad \text{with} \qquad \sigma_1(r) = \int_r^\infty | U(t) | dt \tag{67}$$

for the function $f(r)$ defined by equation 60. Applying our transformation $2.a$ gives readily:

$$\begin{aligned} \sigma_1(r) &= \sigma(x) \\ f(2r) &= xF(x^2) \\ f'(2r) &= -x^2 \frac{d}{d(x^2)} [xF(x^2)] \end{aligned}$$

and this last equality proves the existence of the derivative of $x^{1/2}F(x)$.
So equality 67 becomes

$$\left| -x^2 \frac{d}{d(x^2)} [xF(x^2)] - \frac{1}{4} q(x) \right| < C_2[\sigma(x)]^2 \tag{67.a}$$

Or, applying estimate 66:

$$\left| x \frac{d}{dx} [x^{1/2}F(x)] + \frac{1}{4} q(x^{1/2}) \right| < C_2 C_1^2 x^{\varepsilon}$$

Multiplication of this relation by $x^{-1-\varepsilon/2}$ then integration from 0 to 1 provides, after application of the inequality: $|a| - |b| \leqslant |a + b|$

$$\int_0^1 \left| x \frac{d}{dx} [x^{1/2}F(x)] \right| x^{-1-\varepsilon/2} dx - \frac{1}{4} \int_0^1 | q(x^{1/2}) | x^{-1-\varepsilon/2} dx < 2 \frac{C_2 C_1^2}{\varepsilon}$$

or taking into account inequality 11:

$$\int_0^1 \left| x^{-\varepsilon/2} \frac{d}{dx} [x^{1/2}F(x)] \right| dx < \infty \tag{68}$$

which is the sought estimate for $F(x)$.

It remains to show condition *b*. A first property may be found from the last inequality. In fact, one has:

$$\int_0^x \left| t^{-\varepsilon/2} \frac{d}{dt}[t^{1/2}F(t)] \right| dt = C_\varepsilon \qquad \text{for} \quad 0 < x \leqslant 1$$

where C_ε is a positive constant.

And

$$\left| \int_0^x dt \quad \frac{d}{dt}[t^{1/2}F(t)] \quad \right| \leqslant \int_0^x dt \left| \frac{d}{dt}[t^{1/2}F(t)] \right| \leqslant x^{\varepsilon/2}C_\varepsilon$$

One deduces:

$$| F(x) | \leqslant C_\varepsilon x^{\frac{\varepsilon-1}{2}} \tag{69}$$

Let us now introduce the function $Y_x(t)$ defined by Equation 64 when φ is replaced by F, and such as $Y_x(t)t^{\frac{\varepsilon-1}{2}}$ is summable on $0 < t < x \leqslant 1$. One has:

$$| Y_x(t) | \leqslant \int_0^x | Y_x(u) | C_\varepsilon(ut)^{\frac{\varepsilon-1}{2}} du \leqslant C_y t^{\frac{\varepsilon-1}{2}} \qquad t < x \tag{70}$$

We consider the integral equation with solution $Z_x(t)$:

$$Y_x(t) = Z_x(t) + \int_t^x K(\xi, t)Z_x(\xi)d\xi \tag{71}$$

where K is the kernel preceedingly defined.

The introduction of this equation into the equation defining $Y_x(t)$ leads to (cf. Fig. 1)

$$\begin{aligned} Z_x(t) + \int_t^x K(\xi, t)Z_x(\xi)d\xi &= \int_0^x Z_x(u)F(ut)du + \int_0^x F(ut)du \int_u^x K(\xi, u)Z_x(\xi)d\xi \\ &= \int_0^x Z_x(u)F(ut)du + \int_0^x Z_x(\xi)d\xi \int_0^\xi duK(\xi, u)F(ut) \end{aligned}$$

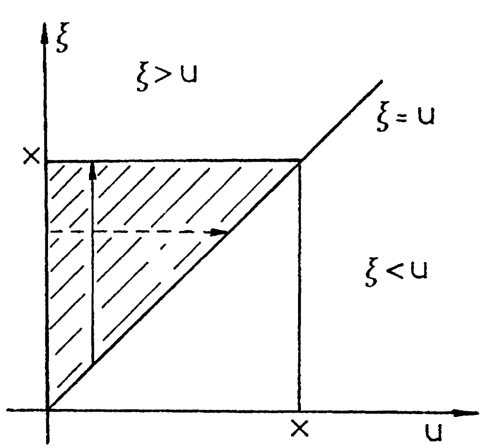

FIG. 1.

Insertion of the fundamental equation provides:

$$\int_0^x Z_x(\xi)d\xi \int_0^\xi duK(\xi, u)F(ut)du$$

$$= \int_0^t Z_x(\xi)d\xi \int_0^\xi duK(\xi, u)F(ut)du + \int_t^x Z_x(\xi)d\xi \int_0^\xi duK(\xi, u)F(ut)du$$

$$= \int_0^t Z_x(\xi)d\xi \int_0^\xi duK(\xi, u)F(ut)du + \int_t^x Z_x(\xi)[K(\xi, t) - F(\xi t)]d\xi$$

$$Z_x(t) = \int_0^t Z_x(\xi)d\xi \ \ F(\xi u) + \int_0^\xi duK(\xi, u)F(ut) \qquad t \leqslant x \tag{72}$$

Taking in account inequalities 20 and 69 one has:

$$\left| \int_0^\xi duK(\xi, u)F(ut) \right| \leqslant CC_\varepsilon \int_0^\xi (\xi u)^{\frac{\varepsilon-1}{2}} (ut)^{\frac{\varepsilon-1}{2}} du \leqslant C' t^{\frac{\varepsilon-1}{2}} \xi^{\frac{3\varepsilon-1}{2}}$$

So $Z_x(t)$ is solution of

$$Z_x(t) = \int_0^t Z_x(\xi)\bar{K}(\xi, t)d\xi \qquad \text{for} \quad t \leqslant x \tag{73}$$

This last equation is an homogeneous Volterra equation, the kernel $\bar{K}$ of which being finite. This equation admits no other solution than the solution identically zero. So $Z_x(t) = 0$ for $t \leqslant x$, and consequently $Y_x(t) = 0$ in the same domain by virtue of Equation 71.

THEOREM 4. — The inverse problem data (cf. § III.4) uniquely determine the operator $\mathscr{L}$.

This theorem deduces easily from theorem 3. The function $F(x)$ is obtained from the scattering data. Then the fundamental equation (62) uniquely determines the kernel $K(x, u)$, this kernel satisfying Equation 19. Having $K(x, u)$, one has $k(r, t)$ by means of Equation 18. So $U(r)$ is given by:

$$U(r) = -2\frac{d}{dr}k(r, r) = q(x) \tag{74}$$

so this last equation provides equally $q(x)$. One has directly:

$$q(x) = 2x\frac{d}{dx}[xK(x, x)] \tag{75}$$

And the operator $\mathscr{L}$ is obtained uniquely.

IV. RECONSTRUCTION OF $\mathscr{L}$ FROM THE INVERSE PROBLEM DATA

The aim of this paragraph will be to show that, given a set of coherent inverse problem data, one may build a differential operator $\mathscr{L}_1$, this operator belonging to the class we consider here. And to show that its scattering data are identical with the initial ones.

First let us consider $\mathcal{K}(x, u)$, solution of the fundamental equation:

$$\mathcal{K}(x, u) + \int_0^x \mathcal{K}(x, z)\mathcal{F}(zu)dz = \mathcal{F}(xu) \qquad 0 < u < x \leqslant 1 \tag{76}$$

where $\mathcal{F}$ is an arbitrary function of type (F_1, ε). From the transformation of a similar estimate of Ljance [3], deduced itself by the method of Agranovich and Marchenko [1], we are allowed to write:

$$|\mathcal{K}(x, u)| \leqslant C(xu)^{\frac{\varepsilon-1}{2}} \tag{77}$$

so as to differentiate Equation 77 under the integral sign (because $\mathcal{K}(x, u)$ possesses continuous partial derivatives with respect to x and u). We obtain the majoration:

$$\left|\frac{d}{dx}[x\mathcal{K}(x, x)] - \mathcal{F}(x^2) - 2x^2\frac{d\mathcal{F}(x^2)}{d(x^2)}\right| \leqslant Cx^{2\varepsilon-1} \tag{78}$$

(cf. § III.5). We now introduce the function $\mathcal{E}(x, v)$ defined by:

$$\mathcal{E}(x, v) = x^{v-1/2} + \int_0^x \mathcal{K}(x, u)u^{v-1/2}du \tag{79}$$

For $0 < x \leqslant 1$ and $\mathrm{Re}\, v > -\varepsilon/2$, this function satisfies the differential equation

$$\left\{-\frac{d}{dx}\left(x^2\frac{d}{dx}\right) - \frac{1}{4} + Q(x)\right\}\mathcal{E}(x, v) = -v^2\mathcal{E}(x, v) \tag{80}$$

where:

$$Q(x) = 2x\frac{d}{dx}[x\mathcal{K}(x, x)] \tag{81}$$

From inequalities 63 and 78, one obtains:

$$\int_0^1 \frac{|Q(x)|}{2}x^{-1-\varepsilon}dx \leqslant \int_0^1 \left|\frac{Q(x)}{2} - x\mathcal{F}(x^2) - 2x^3\frac{d\mathcal{F}(x^2)}{dx^2}\right|x^{-1-\varepsilon}dx$$
$$+ \int_0^1 x^{-1-\varepsilon}\left|\left[x\mathcal{F}(x^2) + 2x^3\frac{d\mathcal{F}(x^2)}{dx^2}\right]\right|dx \leqslant C\int_0^1 x^{2\varepsilon}x^{-1-\varepsilon}dx + C'$$

(for

$$\int_0^1 x^{-1-\varepsilon}\left|\left[x\mathcal{F}(x^2) + 2x^3\frac{d\mathcal{F}(x^2)}{dx^2}\right]\right|dx$$
$$= \int_0^1 x^{-\varepsilon/2}\left|\left[\frac{1}{2}x^{-1/2}\mathcal{F}(x) + x^{1/2}\frac{d\mathcal{F}(x)}{dx}\right]\right|dx$$

And:

$$\int_0^1 |Q(x)|\,x^{-1-\varepsilon}dx < \infty \tag{82}$$

Let us now introduce the following set of inverse problem data:

— a function $\mathcal{S}_1(v)$ of type (S_1) in the strip $|\operatorname{Re} v| < \varepsilon_0$, $\varepsilon_0 > 0$,

— some complex numbers $N_1, N_2, \ldots, N_\gamma$ such as $\operatorname{Re} N_k \geqslant \varepsilon_0$, $k = 1, \ldots, \gamma$,

— some polynomials $P_k(x)$ $(k = 1 \ldots \gamma)$.

As has been seen formerly, in order that these data could be considered as the inverse scattering data corresponding to a differential operator $\mathcal{L}_1$, it is necessary to impose the two conditions:

i) the equality

$$\operatorname{ind}_v \mathcal{S}_1 + 2(M_1 + \ldots + M_\gamma) + \frac{1}{2}[1 - \mathcal{S}_1(0)] = 0 \tag{83}$$

holds, where ind_v has been defined in § III.1 and M_i is the degree of the polynomial $P_i(x)$ $(i = 1, 2, \ldots, \gamma)$,

ii) the function

$$\mathcal{F}(x) = \mathcal{F}_s(x) - \sum_{k=1}^{\gamma} P_k\,[\operatorname{Log} x]x^{N_k - 1/2} = \mathcal{F}_s(x) - \sum_{k=1}^{\gamma} \bar{P}_k(x)x^{N_k - 1/2}$$

$$\mathcal{F}_s(x) = \frac{i}{2\pi}\mathcal{L}_2 \lim_{a\to\infty} \int_{ia+\eta}^{-ia+\eta} [\mathcal{S}_1(v) - 1]x^{v-1/2}dv \tag{85}$$

is a function of type (F_1, ε) for some $\varepsilon > 2\varepsilon_0$.

Conditions *i*) and *ii*) are necessary in order to handle coherent inverse problem data, i. e. data corresponding to a differential operator $\mathcal{L}_1$. This operator is then given by Equations 80 and 81 via the fundamental equation 76 where $\mathcal{F}$ is deduced from equations 84 and 85. The inequality 82 secures that $\mathcal{L}_1$ belongs to the desired class of operators. We call $s_1(v)$, $v_1, v_2, \ldots, v_\alpha$ and $p_1(x), p_2(x), \ldots, p_\alpha(x)$ the inverse problem data of $\mathcal{L}_1$. Now we show that these inverse problem data are the initial ones. The fundamental equation 76 may be written with the help of equation 84:

$$\mathcal{K}(x, u) - \int_0^x \mathcal{K}(x, z)\mathcal{F}_s(zu)dz - \mathcal{F}_s(xu)$$
$$+ \sum_{k=1}^{\gamma} \left\{ \bar{P}_k(xu)(xu)^{v-1/2} + \int_0^x \mathcal{K}(x, z)\bar{P}_k(zu)(zu)^{v-1/2}dz \right\}_{v=N_k} = 0 \tag{86}$$

As seen before, the function $\mathcal{E}(x, v)$ is a solution of $\mathcal{L}_1 y = -v^2 y$. $s_1(v)$ is defined by:

$$s_1(v) = \frac{\mathcal{E}(1, -v)}{\mathcal{E}(1, v)} \tag{87}$$

$$f_s(x) = \frac{i}{2\pi}\mathcal{L}_2 \lim_{a\to\infty} \int_{ia+\eta}^{-ia+\eta} [s_1(v) - 1]x^{v-1/2}dv \tag{88}$$

As the kernel defining $\mathscr{E}(x, v)$ is $\mathscr{K}(x, u)$, Equation 62 may be written for $\mathscr{K}(x, u)$ and the function $f(x)$ corresponding to the operator $\mathscr{L}_1$:

$$\mathscr{K}(x, u) - \int_0^x \mathscr{K}(x, z) f_s(zu)dz - f_s(xu) + \sum_{k=1}^{\alpha} \left\{ \bar{p}_k(xu)(xu)^{v-1/2} + \int_0^x \mathscr{K}(x, z)\bar{p}_k(zu)(zu)^{v-1/2}dz \right\}_{v=v_k} = 0 \quad 0 < u \leqslant x \leqslant 1 \tag{89}$$

We extend the function $\mathscr{K}(x, u)$ by setting:

$$\mathscr{K}(x, u) = 0 \qquad \text{for} \quad u > x \tag{90}$$

Then:

$$\mathscr{E}(x, v) - x^{v-1/2} = \int_0^\infty \mathscr{K}(x, u)u^{v-1/2}du \tag{91}$$

and the inverse Mellin transform may be written

$$\mathscr{K}(x, u) = \frac{i}{2\pi}\mathscr{L}_2 \lim_{a\to\infty} \int_{ia}^{-ia} [\mathscr{E}(x, -v) - x^{-v-1/2}]u^{v-1/2}dv \quad 0<u<+\infty$$

$$\mathscr{K}(x, u) = \frac{i}{2\pi}\mathscr{L}_2 \lim_{a\to\infty} \int_{ia\pm\eta}^{ia\pm\eta} [\mathscr{E}(x, -v) - x^{-v-1/2}]u^{v-1/2}dv \quad 0<u<+\infty \quad 0<\eta<\varepsilon_0 \tag{92}$$

for both functions $\mathscr{E}(x, -v)$ and $x^{-v-1/2}$ are holomorphic in the strip $|\operatorname{Re} v| < \varepsilon_0$. Let us now introduce:

$$\mathscr{K}^-(x, u) = \frac{1}{u}\mathscr{K}(x, u) = \frac{i}{2\pi}\mathscr{L}_2 \lim_{a\to\infty} \int_{ia-\eta}^{-ia-\eta} [\mathscr{E}(x, -v) - x^{-v-1/2}]u^{-v-1/2}dv \qquad 0<u<+\infty$$

We have:

$$\int_0^\infty \mathscr{K}(x, u)\mathscr{F}_s(ut)du = \frac{1}{t}\int_0^\infty \mathscr{K}\ \ x, \frac{v}{t}\ \ \mathscr{F}_s(v)dv = \int_0^\infty \mathscr{K}^-\ \ x, \frac{t}{v}\ \ \mathscr{F}_s(v)\frac{dv}{v}$$

and this last expression shows that it is nothing else than the convolution product for the Mellin transform. The properties of such products allow us to write:

$$\int_0^\infty \mathscr{K}(x, u)\mathscr{F}_s(ut)du = \frac{i}{2\pi}\mathscr{L}_2 \lim_{a\to\infty} \int_{ia+\eta}^{-ia+\eta} [\mathscr{E}(x, v) - x^{v-1/2}][\mathscr{S}_1(v) - 1]t^{v-1/2}dv = \int_0^\infty \mathscr{K}(x, u)\mathscr{F}_s(ut)du \tag{93}$$

because of property 90. So:

$$\int_0^x \mathscr{K}(x, u)\mathscr{F}_s(ut)du$$

$$= \frac{i}{2\pi}\mathscr{L}_2 \lim_{a\to\infty} \int_{ia+\eta}^{-ia+\eta} [\mathscr{E}(x, v) - x^{v-1/2}]\mathscr{S}_1(v)t^{v-1/2}dv - t\mathscr{K}\left(x, \frac{1}{t}\right) \quad (94)$$

But for $0 < t \leqslant x \leqslant 1$, $\mathscr{K}(x, 1/t) = 0$. So equality 86 may be written:

$$\frac{i}{2\pi}\mathscr{L}_2 \lim_{a\to\infty} \int_{ia+\eta}^{-ia+\eta} \{ \mathscr{E}(x, -v) - x^{-v-1/2} - \mathscr{S}_1(v)\mathscr{E}(x, v) + x^{v-1/2} \} u^{v-1/2}dv$$
$$+ \sum_{k=1}^{\gamma} \left\{ \bar{\mathrm{P}}_k(xu)(xu)^{v-1/2} + \int_0^x \mathscr{K}(x, z)\bar{\mathrm{P}}_k(zu)(zu)^{v-1/2}dz \right\}_{v=\mathrm{N}_k} = 0 \quad (95)$$
$$0<u\leqslant x\leqslant 1$$

If we consider $x = 1$ we obtain:

$$\frac{i}{2\pi}\mathscr{L}_2 \lim_{a\to\infty} \int_{ia+\eta}^{-ia+\eta} \{ \mathscr{E}(1, -v) - \mathscr{S}_1(v)\mathscr{E}(1, v) \} u^{v-1/2}dv$$
$$+ \sum_{k=1}^{\gamma} \left\{ \bar{\mathrm{P}}_k(u)u^{v-1/2} + \int_0^1 \mathscr{K}(1, z)\bar{\mathrm{P}}_k(zu)(zu)^{v-1/2}dz \right\}_{v=\mathrm{N}_k} = 0 \quad (96)$$
$$0 < u \leqslant 1$$

or with the symbolic writing:

$$\bar{\mathrm{P}}_k(x)x^{v-1/2} = \mathrm{P}_k\,(\mathrm{Log}\ x)x^{v-1/2} = \mathrm{P}_k\left(-\frac{d}{dv}\right)x^{v-1/2}$$

$$\frac{i}{2\pi}\mathscr{L}_2 \lim_{a\to\infty} \int_{ia+\eta}^{-ia+\eta} \{ \mathscr{E}(1, -v) - \mathscr{S}_1(v)\mathscr{E}(1, v) \} u^{v-1/2}dv$$
$$+ \sum_{k=1}^{\gamma} \left\{ \bar{\mathrm{P}}_k\left(-\frac{d}{dv}\right)u^{v-1/2} + \int_0^1 \mathscr{K}(1, z)\bar{\mathrm{P}}_k\left(-\frac{d}{dv}\right)(zu)^{v-1/2}dz \right\}_{v=\mathrm{N}k} = 0$$
$$0 < u \leqslant 1$$

But d/dv does not work on $\mathscr{K}(1, z)$. We recognize the expression of $\mathscr{E}(1, v)$ in the summation:

$$\frac{i}{2\pi}\mathscr{L}_2 \lim_{a\to\infty} \int_{ia+\eta}^{-ia+\eta} \{ \mathscr{E}(1, -v) - \mathscr{S}_1(v)\mathscr{E}(1, v) \} u^{v-1/2}dv$$
$$+ \sum_{k=1}^{\gamma} \left\{ \mathrm{P}_k\left(-\frac{d}{dv}\right)\mathscr{E}(1, v)u^{v-1/2} \right\}_{v=\mathrm{N}_k} = 0 \qquad 0 < u \leqslant 1 \quad (97)$$

This last summation is a linear combination of functions $(-\mathrm{Log}\ u)^{j_k}u^{\mathrm{N}_k-1/2}$.

These functions ($k = 1, \ldots, \gamma ; j_k = 0, \ldots, M_k - 1$) can be replaced by inverse Mellin transforms. In fact we have:

$$\left(\frac{d}{dv}\right)^n \left[\int_0^1 u^{N_k - v - 1} du \right] = \int_0^1 (- \operatorname{Log} u)^n u^{N_k - v - 1} du = \frac{n!}{(N_k - v)^{n+1}}$$

So

$$(- \operatorname{Log} u)^n u^{N_k - 1/2} = \frac{i}{2\pi} \mathscr{L}_2 \lim_{a \to \infty} \int_{ia}^{-ia} \frac{n!}{(N_k - v)^{n+1}} u^{v - 1/2} dv \qquad 0 < u \leqslant 1$$

And as every N_k is such that $\operatorname{Re} N_k \geqslant \varepsilon_0$, the function $\frac{1}{(N_k - v)^{n+1}}$ is holomorphic when $\operatorname{Re} v \leqslant \eta$. So we can write:

$$\frac{i}{2\pi} \mathscr{L}_2 \lim_{a \to \infty} \int_{ia+\eta}^{-ia+\eta} \left\{ [\mathscr{E}(1, - v) - \mathscr{S}_1(v)\mathscr{E}(1, v)] + \sum_{k=1}^{\gamma} \sum_{j=1}^{M_k} \frac{A_{kj}}{(N_k - v)^j} \right\} u^{v - 1/2} dv = 0 \quad (98)$$

where the A_{kj} are numbers.

We now introduce:

$$\tilde{\mathscr{H}}(v) = \mathscr{E}(1, - v) - \mathscr{S}_1(v)\mathscr{E}(1, v) + \sum_{k=1}^{\gamma} \sum_{j=1}^{M_k} \frac{A_{kj}}{(N_k - v)^j} \qquad (99)$$

and if we set

$$\mathscr{E}(x, v) = x^{v - 1/2} + \tilde{\mathscr{K}}(x, - v) \qquad (100)$$

we know (cf. § II.1) that $\tilde{\mathscr{K}}(x, - v)$ is holomorphic in v for $\operatorname{Re} v > - \varepsilon/2$ for each value of x such as $0 < x \leqslant 1$. Inequality 23 shows that:

$$\mathscr{E}(1, v) = 1 + \mathscr{O}\left(\frac{1}{v}\right) \qquad |v| \to \infty \qquad \text{uniformly for} \quad \operatorname{Re} v \geqslant -\delta \qquad \delta < \frac{\varepsilon}{2}$$

$$\mathscr{E}(1, -v) = 1 + \mathscr{O}\left(\frac{1}{v}\right) \qquad |v| \to \infty \qquad \text{uniformly for} \quad \operatorname{Re} v \leqslant \delta \qquad \delta < \frac{\varepsilon}{2}$$

Besides $\mathscr{S}_1(v)$ is a function of type (S_1) in the strip $|\operatorname{Re} v| \leqslant \varepsilon_0$, so it is meromorphic in this strip and

$$\mathscr{S}_1(v) = 1 + \mathscr{O}\left(\frac{1}{v}\right) \qquad |v| \to \infty \qquad \text{uniformly in this strip} \quad |\operatorname{Re} v| \leqslant \eta, \quad \eta < \varepsilon_0$$

So:

$$\tilde{\mathscr{H}}(v) = \mathscr{O}\left(\frac{1}{v}\right) \qquad |v| \to \infty \qquad \text{uniformly in the strip} \quad |\operatorname{Re} v| \leqslant \eta, \quad \eta < \varepsilon_0 \qquad (101)$$

and $\tilde{\mathscr{H}}(v)$ may be considered as the Mellin transform of a function $\mathscr{H}(t)$:

$$\mathscr{H}(t) = \frac{i}{2\pi} \mathscr{L}_2 \lim_{a \to \infty} \int_{ia+\eta}^{-ia+\eta} \tilde{\mathscr{H}}(v) u^{v - 1/2} dv \qquad 0 < t < \infty \qquad (102)$$

But Equation 98 implies that for $0 < t \leqslant 1$ $\mathcal{H}(t) = 0$. So:

$$\tilde{\mathcal{H}}(v) = \int_1^\infty \mathcal{H}(t)t^{-v-1/2}dt \qquad 0 < \text{Re } v < \varepsilon_0 \tag{103}$$

We choose η such as $0 < \eta < \varepsilon_0$ and we set $\sigma = \text{Re } v$. We have first:

$$|\tilde{\mathcal{H}}(v)| \leqslant \int_1^\infty |\mathcal{H}(t)|\, t^{-\sigma-1/2}dt$$

Then we apply Schwarz inequality:

$$|\tilde{\mathcal{H}}(v)| \leqslant \left\{ \int_1^\infty |\mathcal{H}(t)t^{-\eta}|^2 dt \right\}^{1/2} \left\{ \int_1^\infty t^{2(\eta-\sigma)-1}dt \right\}^{1/2}$$

$$\leqslant \left\{ \int_1^\infty |\mathcal{H}(t)t^{-\eta}|^2 dt \right\}^{1/2} \frac{1}{\sqrt{2(\sigma - \eta)}} \qquad \text{for} \quad \sigma > \eta$$

At last Parseval equality gives:

$$\int_1^\infty |\mathcal{H}(t)t^{-\eta}|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\tilde{\mathcal{H}}(\eta - iy)|^2 dy$$

$$|\tilde{\mathcal{H}}(v)| \leqslant \frac{C_\eta}{\sqrt{\text{Re } v - \eta}} \qquad \text{for} \quad \text{Re } v > \eta \tag{104}$$

This last inequality shows that $\tilde{\mathcal{H}}(v)$, function holomorphic in v for $\text{Re } v > 0$, admits an analytic continuation which is holomorphic in the half plane $\text{Re } v > 0$ and bounded in any half plane $\text{Re } v \geqslant \eta$, $\eta > 0$.

Let us call $w(v)$ the solution of the problem of symmetric factorization corresponding to the initial set of inverse problem data ($\mathcal{S}_1$, N_1, N_2 ... N_γ, $P_k(x)$).

Then: $\mathcal{S}_1(v) = \dfrac{w(-v)}{w(v)}$, and 99 becomes:

$$w(v)\mathcal{E}(1, -v) - w(-v)\mathcal{E}(1, v) = w(v)\tilde{\mathcal{H}}(v) - w(v)\sum_{k=1}^{\gamma}\sum_{j=1}^{M_k} \frac{A_{kj}}{(N_k - v)^j} \tag{105}$$

The properties of functions w and $\mathcal{E}$ show that the left side is holomorphic for $|\text{Re } v| < \varepsilon_0$ and is equal to $\mathcal{O}(1/v)$ uniformly in each strip $|\text{Re } v| \leqslant \eta$, $\eta < \varepsilon_0$. In the right side, we remark that the poles of the fractions are the roots of $w(v)$ with the same multiplicity. So the right side is holomorphic too. It is bounded for $\text{Re } v \geqslant \eta$, $0 < \eta < \varepsilon_0$. As the left side is an odd function, Liouville's theorem imposes that its analytic continuation should be an identically zero entire function

$$w(v)\mathcal{E}(1, -v) - w(-v)\mathcal{E}(1, v) = 0$$

or $\dfrac{w(-\nu)}{w(\nu)} = \dfrac{\mathscr{E}(1, -\nu)}{\mathscr{E}(1, \nu)} = s_1(\nu)$ by definition (Equation 87). So:

$$\mathscr{S}_1(\nu) = s_1(\nu) \tag{106}$$

we deduce immediately from Equations 85 and 88 that:

$$\mathscr{F}_s(x) = f_x(x) \tag{107}$$

And a comparison between Eq. 86 and Eq. 89 provides:

$$\sum_{k=1}^{\gamma} \left\{ \bar{\mathrm{P}}_k(xu)(xu)^{\nu-1/2} + \int_0^x \mathscr{K}(x, z)\bar{\mathrm{P}}_k(zu)(zu)^{\nu-1/2}dz \right\}_{\nu=\mathrm{N}_k}$$
$$= \sum_{k=1}^{\alpha} \left\{ \bar{p}_k(xu)(xu)^{\nu-1/2} + \int_0^x \mathscr{K}(x, z)\bar{p}_k(zu)(zu)^{\nu-1/2}dz \right\}_{\nu=\nu_k} \qquad 0<u\leqslant x\leqslant 1$$

or:

$$\sum_{\mu j} \mathrm{A}_{\mu j} \left\{ (-\operatorname{Log} xu)^j (xu)^{\mu-1/2} + \int_0^x \mathscr{K}(x, z)(-\operatorname{Log} zu)^j (zu)^{\mu-1/2}dz \right\} = 0 \qquad 0 < u \leqslant x \leqslant 1$$

where μ may be equal to any N_k or ν_k, and j is an integer $\leqslant \max(\mathrm{M}_k, m_k) - 1$. Inequality 76 allows us to majore the second term:

$$\left| \int_0^x \mathscr{K}(x, z)(-\operatorname{Log} zu)^j (zu)^{\mu-1/2}dz \right| \leqslant \mathrm{C} \int_0^x (xz)^{\frac{\varepsilon-1}{2}} (zu)^\delta (zu)^{\sigma-1/2}dz$$

where $\sigma = \operatorname{Re} \mu$ and $\delta > 0$ is arbitrarily small

$$\left| \int_0^x \mathscr{K}(x, z)(-\operatorname{Log} zu)^j (zu)^{\mu-1/2}dz \right| \leqslant \mathrm{C}x^{\frac{\varepsilon-1}{2}} u^{\delta+\sigma-1/2} \int_0^x z^{\varepsilon/2-1+\delta+\sigma}dz$$
$$\leqslant \mathrm{C}'x^\varepsilon (xu)^{\delta+\sigma-1/2}$$
$$\leqslant \mathrm{C}'x^{\varepsilon_1-1/2}(xu)^\delta$$

with $\varepsilon_1 = \varepsilon + \min \sigma$. So:

$$\sum_{\mu j} \mathrm{A}_{\mu j} \{ (-\operatorname{Log} xu)^j (xu)^{\mu-1/2} \} \mathrm{C}'x^{\varepsilon_1-1/2}(xu)^\delta$$

We now choose j maximal and μ with minimal imaginary part. The summation is reduced to $\sum_{\mu j}'$ and the previous inequality remains valid for $\sum'$.

In the particular case $u = x^2$, we obtain

$$\sum_{\mu j}{}' \mathrm{A}_{\mu j} \{ (- \mathrm{Log}\ x^3)^j x^{3(\mu - 1/2)} \} \leqslant \mathrm{C}' x^{\varepsilon_1 - 1/2} x^{3\delta}$$

$$\sum_{\mu j}{}' \mathrm{A}_{\mu j} x^{3\,\mathrm{Im}\,\mu - 3/2} \leqslant \mathrm{C}' x^{\varepsilon_1 + 1 - 3\,\mathrm{Re}\,\mu + 3\delta} (- 3\ \mathrm{Log}\ x)^{-j}$$

And the second member goes to zero with x.

So necessarily $\mathrm{A}_{\mu j} \equiv 0$. One shows similarly that the other $\mathrm{A}_{\mu j}$ are equal to zero.

This completes the proof that:

$$\alpha = \gamma \qquad \nu_k = \mathrm{N}_k \qquad p_k(x) = \mathrm{P}_k(x) \qquad k = 1, 2, \ldots, \alpha$$

for a suitable enumeration of the singular numbers and normalization polynomials. So we can write the following theorem.

THEOREM 5. — Suppose given a function $\mathrm{S}_1(\nu)$ of type S_1 in the half plane $\mathrm{Re}\ \nu > -\varepsilon_0$, $\varepsilon_0 > 0$, numbers $\mathrm{N}_1, \ldots, \mathrm{N}_\gamma$ such as $\mathrm{Im}\ \mathrm{N}_i \geqslant \varepsilon_0$ ($i = 1, \ldots, \gamma$) and corresponding polynomials $\mathrm{P}_1(x), \ldots, \mathrm{P}_\gamma(x)$, the degree of the polynomial P_i being $\mathrm{M}_i - 1$. These data will be the inverse problem data of a certain non selfadjoint differential operator $\mathscr{L}_1$ if they are coherent, i. e. if the two following conditions are realized:

i) $\quad \mathrm{ind}_\nu\, \mathrm{S}_1 + 2(\mathrm{M}_1 + \mathrm{M}_2 + \ldots + \mathrm{M}_\gamma) + \dfrac{1}{2}[1 - \mathrm{S}_1(0)] = 0$

ii) The function $\mathscr{F}(x)$ defined by:

$$\mathscr{F}(x) = \frac{i}{2\pi} \mathscr{L}_2 \lim_{a \to \infty} \int_{ia + \eta}^{-ia + \eta} [\mathrm{S}_1(\nu) - 1] x^{\nu - 1/2} d\nu - \sum_{k=1}^{\gamma} \mathrm{P}_k\, (\mathrm{Log}\ x) x^{\mathrm{N}_k - 1/2} \qquad 0 < x \leqslant 1$$

is a function of type $(\mathrm{F}_1, \varepsilon)$ for some $\varepsilon > 2\varepsilon_0$. Then the fundamental equation possesses a unique solution $\mathrm{K}(x, t)$, from which it is possible to deduce $q(x)$ satisfying the following inequality:

$$\int_0^1 x^{-1-\varepsilon} \mid q(x) dx < \infty$$

and the inverse problem data of the operator $\mathscr{L}_1$ corresponding to $q(x)$ are identical with the initial ones.

CONCLUSION

The next step of this work would be to deduce the inverse scattering data from the experimental data, i. e. from the phase shifts for every entire value of $l = \nu - 1/2$. Loeffel was able to show that this deduction was unique in the real case. However a numerical calculation would contain the extrapolation to the entire imaginary axis of a function given by discrete points on the real axis. Such a numerical procedure gives rise to instability problems [*8*]. So till now, no effective calculation has ever been done. One will easily be convinced that in the complex case the situation is still more complicated.

APPENDIX

Normalization multipliers in the real case

Let us consider the function $\Phi(x, -v^2)$ defined by Equations 26. It satisfies Eq. 28 for $|v| > v_0$, Re $v > 0$. In this domain the wronskian of $\Phi(x, -v^2)$ and of $\varphi(x, v)$ is readily obtained from Eq. 16:

$$W[\Phi(x, -v^2), \varphi(x, v)] = \frac{1}{2v}\{ w(v)W[\varphi(x, v), \psi(x, v)]\} = \frac{w(v)}{x^2} \tag{A.1}$$

Let us differentiate (A.1) with respect to v (this differentiation being indicated by a dot):

$$\dot{w}(v) = x^2W[\dot{\Phi}(x, -v^2), \varphi(x, v)] + x^2W[\Phi(x, -v^2), \dot{\varphi}(x, v)] \tag{A.2}$$

Now we choose $v = v_k$, root of $w(v)$. As we limit ourselves in this appendix to the real case, such a root is real and simple. It is positive, and such as:

$$v_k > \varepsilon_0$$

So it is in the domain considered. Equation (28) reduces then to:

$$\Phi(x, -v_k^2) = \frac{w_1(v_k)\varphi(x, v_k)}{2v_k} \tag{A.3}$$

and the two solutions are multiples of one another. Eq. A.2 becomes:

$$\dot{w}(v_k) = \frac{2v_k}{w_1(v_k)}x^2W[\Phi(x, -v_k^2), \dot{\Phi}(x, -v_k^2)] + \frac{w_1(v_k)}{2v_k}x^2W[\dot{\varphi}(x, v_k), \varphi(x, v_k)] \tag{A.4}$$

Let us consider now Equation 8, with the solution $f(x, v)$ corresponding to the eigenvalue v. Multiplication of this equation by $f(x, v')$, then substraction from that for $f(x, v')$ multiplied by $f(x, v)$ leads to:

$$\frac{d}{dx}\{ x^2W[f(x, v), f(x, v')]\} = (v^2 - v'^2)f(x, v)f(x, v') \tag{A.5}$$

or after differentiation with respect to v

$$\frac{d}{dx}\{ x^2W[\dot{f}(x, v), f(x, v')]\} = (v^2 - v'^2)\dot{f}(x, v)f(x, v') + 2vf(x, v)f(x, v') \tag{A.6}$$

If now we choose $v = v' = v_k$ this last equation reduces to:

$$\frac{d}{dx}\{ x^2W[\dot{f}(x, v_k), f(x, v_k)]\} = 2v_kf^2(x, v_k) \tag{A.7}$$

By integration this last equation writes, when $f \equiv \Phi$

$$x^2W[\dot{\Phi}(x, -v_k^2), \Phi(x, -v_k^2)] = 2v_k\int_1^x \Phi^2(x, v_k)dx \tag{A.8}$$

For, in the domain considered Equations 26 and 28 lead to:

$$W[\dot{\Phi}(1, -\nu^2), \Phi(1, -\nu^2)] = \dot{\Phi}(1, -\nu^2)$$
$$= \frac{w_1(\nu)\dot{\varphi}(1, \nu)}{2\nu} - \frac{w(\nu)\dot{\psi}(1, \nu)}{2\nu} + \frac{w_1(\nu)\varphi(1, \nu)}{2\nu} - \frac{\dot{w}(\nu)\psi(1, \nu)}{2\nu} - \frac{w_1(\nu)\varphi(1, \nu) - w(\nu)\psi(1, \nu)}{2\nu^2} = 0$$

Equation (A.7) becomes when $f \equiv \varphi$:

$$x^2 W[\dot{\varphi}(x, \nu_k), \varphi(x, \nu_k)] = 2\nu_k \int_0^x \varphi^2(x, \nu_k)dx \tag{A.9}$$

For Equation 6 implies that

$$x^2 W[\dot{\varphi}(x, \nu), \varphi(x, \nu)] = \mathcal{O}(x^{2\nu} \operatorname{Log} x)$$

when $x \to 0$. When $\nu = \nu_k > \varepsilon_0 > 0$, this quantity is zero when x is zero. So Eq. A.4 may be written:

$$\dot{w}(\nu_k) = w_1(\nu_k) \int_0^1 \varphi^2(x, \nu_k)dx \tag{A.10}$$

One easily deduces tath:

$$\operatorname{Re} s_{\nu=\nu_k} \frac{w_1(\nu)}{w(\nu)} x^{\nu - 1/2} = \frac{w_1(\nu_k)}{\dot{w}(\nu_k)} x^{\nu_k - 1/2} = \frac{x^{\nu_k - 1/2}}{\int_0^1 \varphi^2(x, \nu_k)dx} \tag{A.11}$$

or with notations of paragraph II.3

$$C_k = \left[\int_0^1 \varphi^2(x, \nu_k)dx\right]^{-1} \tag{A.12}$$

The polynomial $\overline{p_k(x)}$ reduces to the constant C_k.

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(Manuscrit reçu le 3 avril 1978)