ANNALES DE L'I. H. P., SECTION A

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Annales de l'I. H. P., section A, tome 28, nº 4 (1978), p. 379-397
<http://www.numdam.org/item?id=AIHPA_1978__28_4_379_0>

Ann. Inst. Henri Poincaré,
Vol. XXVIII, n° 4, 1978, p. 379-397.

Section A :
Physique théorique.

Predictive relativistic mechanics of gravitating masses (*)

by

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ABSTRACT. — In order to describe gravitational interaction, within the framework of Predictive Relativistic Mechanics, the slow motion approximation of General Relativity is considered. The covariance of the field equations in General Relativity allows to choose a coordinate condition (the harmonic gauge) in which the equations of motion up to order c^{-4} are Poincaré invariant and therefore approximate solutions of Predictive Relativistic Mechanics.

RÉSUMÉ. — Nous considérons l'approximation de mouvement lent en Relativité Générale, afin de décrire l'interaction gravitationelle en Mécanique Relativiste Prédictive. La covariance des équations du champ en Relativité Générale permet d'élire une condition de coordonnées (gauge harmonique) dans laquelle les équations du mouvement approchées sont invariantes jusqu'à l'ordre c^{-4} sous les transformations de Poincaré et par conséquent elles sont des solutions approchées de la Mécanique Relativiste Prédictive.

I. INTRODUCTION

Predictive Relativistic Mechanics (PRM) is a theory of isolated systems of structureless particles which motion is governed by second order differential equations.

(*) Work supported by the Instituto de Estudios Nucleares (Madrid, Spain).

The basis of PRM were established by Currie [1], Hill [2] and Bel [3]. The equations of PRM can be written in a manifestly predictive formalism or in a manifestly covariant one [4] [5]. Up to now only approximate, physically meaningful, solutions of the equations of PRM are known for the following interactions: the scalar or vector interaction of long or short range.

The description of the electromagnetic interaction in the manifestly predictive formalism was considered by Hill [2] following a Kerner's [6] scheme. The equations of PRM for the same interaction were solved by Bel, Salas and Sanchez [7] using the manifestly covariant formalism which is more appropriate for expansion calculations. The method used was later generalized by Bel and Martin [8] for the short-range scalar interaction [9].

The two main ingredients used to solve the PRM equations for these interactions are: *a*) Poincaré invariant field equations (field produced by source particles) and *b*) a generalized Lorentz force (force acting on the test particle) which is also Poincaré invariant.

However, the gravitational interaction as described by General Relativity, is based on field equations which are covariant respect to any change of coordinates but are not Poincaré invariant, nor are the equations of motion. That is why the gravitational interaction needs to be discussed in a different way.

In General Relativity the problem of motion can be taken up by different approximation methods. One of them, the « slow motion » approximation was developed by Einstein, Infeld and Hoffmann [10] (EIH). The equations of motion for gravitating point masses up to terms of order c^{-2} were found in this scheme. This is called the first post-Newtonian approximation (first PNA).

Chandrasekhar and Contopoulos [11] (CC) have shown that the post-Newtonian metric tensor for point masses can be made invariant under a post-Galilean transformation.

The invariance of the metric tensor guarantees that the equations of EIH which follow are similarly invariant under the post-Galilean transformation of Chandrasekhar and Contopoulos.

In the slow motion approximation in General Relativity the components of the metric tensor are instantaneous potentials, and the equations of motion (geodesic equations) can be written up to order N, using the following notation $\overset{(N)}{A} \equiv \sum_{L=0}^{N} c^{-L}\overset{L}{A}$, as

$$\frac{d^2\vec{x}_a}{dt^2} \simeq \overset{(N)}{\vec{a}_a}\left(\vec{x}_b, \frac{d\vec{x}_c}{dt} = \vec{v}_c\right)$$

where the arguments (given at the same time t) are only positions and velocities. The higher order time derivatives of the positions are removed using the equations of motion at lower orders.

Whenever a transformation leaves the metric tensor invariant up to the order N, the new equations of motion will be

$$\frac{d^2\vec{\xi}_a}{d\tau^2} \simeq \overset{(N)}{\vec{a}_a}\left(\vec{\xi}_b, \frac{d\vec{\xi}_c}{d\tau} = \vec{w}_c\right)$$

the $\overset{(N)}{\vec{a}_a}$'s being the same functions as before with the arguments given at the time τ. The family of particle trajectories is the same for both systems.

The structure of the post-Galilean transformation of CC ensures that the equations of motion of order $N = 2$ are invariant under a Poincaré transformation. Therefore the accelerations $\overset{(2)}{\vec{a}_a}$ satisfy Currie-Hill conditions up to order c^{-2}, and the gravitational interaction can be described up to this order within PRM [*12*].

The problem we want to discuss in this paper is the following one: is it possible to preserve at higher orders in the slow motion approximation the Poincaré invariance of the theory?

The first apparent difficulty cames from the structure of the post-Galilean transformation of CC. However we must keep in mind that the gravitational term depends on the choice of the gauge or coordinate conditions needed to compute the post-Newtonian metric tensor. For instance, if harmonic coordinates are used the corresponding post-Galilean transformation becomes a Poincaré transformation.

Moreover, as it will be shown, when these coordinates are used the equations of motion for gravitating point masses up to c^{-4} terms are also Poincaré invariant. That is, the gravitational interaction can be included in the PRM framework up to c^{-4} terms. (The use of harmonic coordinates up to this order was suggested by Hirondel [*13*]).

Here, following the slow motion scheme of General Relativity the metric tensor for an N-point mass system is evaluated at the second PNA using two different gauges. Then the coordinate transformations leaving invariant those metric tensors are evaluated. In harmonic coordinates the equations of motion are found to be Poincaré invariant.

In Section II, a summary is made of the slow motion approximation in General Relativity and special care is devoted to the selection of the gauge. Although we are interested in the metric tensor for an isolated system of point masses we begin with the energy-momentum tensor for a perfect fluid. The evaluation of the metric tensor for point masses can be carried out integrating the equations of the metric for a perfect fluid composed of spherical bodies with null radius. We belive that this procedure is more satisfactory for the PNA than to begin with the energy-momentum tensor for point masses. In Subsection II A the Chandrasekhar and harmonic coordinate conditions are explicitly given, up to the second PNA. The metric tensor for an N-point mass system is explicitly given in harmonic coordinates up to the same order in Subsection II B.

In Section III a method is given to evaluate the coordinate transformation leaving invariant the metric tensor and in Subsection III A, we discuss the invariance of the equations of motion.

The CC method is revised in Subsection III B, and the results and physical interpretation for the first post-Galilean transformation in the usual and the harmonic gauges are revised in Subsection III C. At the next order, the second post-Galilean transformations are evaluated in Chandrasekhar's gauge in Subsection III D and in harmonic gauge in Subsection III E.

Subsection III F is devoted to the extension at higher orders.

Conclusions and comments are made in Section IV.

II. SLOW MOTION APPROXIMATION IN GENERAL RELATIVITY

In General Relativity, the physical description of a system is fixed by choosing a suitable energy momentum tensor $T_{\mu\nu}$. For a perfect fluid, the energy momentum tensor is [*14*]:

$$T_{\mu\nu} = \rho(c^2 + \Pi + p/\rho)u_\mu u_\nu - pg_{\mu\nu} \qquad (\mu, \nu = 0, 1, 2, 3) \tag{1}$$

where $\rho\Pi$ denotes the internal energy, $\rho(c^2 + \Pi)$ the energy-density, p is the pressure, u_μ the covariant four-velocity of the fluid and $g_{\mu\nu}$ the metric tensor.

The bahaviour of the system is then described by Einstein's field equations:

$$R_{\mu\nu} = -\frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right) \tag{2}$$

where G is the gravitational constant, $R_{\mu\nu}$ the Ricci tensor and $T = T^\mu_\mu$.

These field equations can be solved by approximation methods. One of these is based on the expansion in powers of c^{-1} which is adequate only in the « near zone » [*15*] for the study of slow motion (fluid velocity $\ll c$). The metric tensor can be written in the form:

$$g_{\mu\nu} \simeq \sum_{L=0}^{N} c^{-L}\overset{L}{g}_{\mu\nu} \equiv \overset{(N)}{g}_{\mu\nu} \qquad \text{with} \quad \overset{0}{g}_{\mu\nu} = \eta_{\mu\nu} \tag{3}$$

where $\eta_{\mu\nu} = \text{diag}\,(1, -1, -1, -1)$ is the Minkowskian metric tensor.

With the above expansion, the field equations (2) can now be written, up to order N, in the form:

$$\Delta \overset{(N)}{g}_{ij} - \overset{(N)}{W}_{i,j} - \overset{(N)}{W}_{j,i} = \overset{(N)}{S}_{ij} \qquad (i, j = 1, 2, 3) \tag{4 a}$$

$$\Delta \overset{(N+1)}{g}_{0i} - \overset{(N+1)}{W}_{,i} - \overset{(N)}{W}_{i,0} = \overset{(N+1)}{S}_{0i} \tag{4 b}$$

$$\Delta \overset{(N+2)}{g}_{00} - 2\overset{(N+1)}{W}_{,0} + \overset{(N)}{g}_{ij}\overset{2}{g}_{00,ij} + \overset{(N)}{W}_{i}\overset{2}{g}_{00,i} = \overset{(N+2)}{S}_{00} \tag{4 c}$$

where the following notations have been used:

$$\overset{N}{W}_i \equiv \overset{N}{g}_{ik,k} - \frac{1}{2}\overset{N}{g}_{kk,i} \tag{5 a}$$

$$\overset{N+1}{W} \equiv \overset{N+1}{g}_{0k,k} - \frac{1}{2}\overset{N}{g}_{kk,0} \tag{5 b}$$

$,i$ stands for $\partial/\partial x^i$ and $,0$ for $\partial/c\partial t$ (in the near zone $,0$ is of higher order in c^{-1} than $,i$), $\Delta = \partial^2/\partial x^i \partial x^i$ is the Laplace operator for flat space. The functions $\overset{(N)}{S}_{ij}$, $\overset{(N+1)}{S}_{0i}$ and $\overset{(N+2)}{S}_{00}$ appearing in (4) depend on $\overset{(L)}{g}_{ij}$, $\overset{(L+1)}{g}_{0i}$ and $\overset{(L+2)}{g}_{00}$ for $L < N$.

The integrability conditions of the equations (4 *a*) and (4 *b*) are, respectively:

$$\overset{(N)}{S}_{kj,k} - \frac{1}{2}\overset{(N)}{S}_{kk,j} = 0 \tag{6 a}$$

$$\overset{(N+1)}{S}_{0k,k} - \frac{1}{2}\overset{(N)}{S}_{kk,0} = 0 \tag{6 b}$$

It can readily be shown that, when these equations are verified, the functions $\overset{N}{W}_i$ and $\overset{N+1}{W}$ can be considered as *arbitrary functions*.

In order to solve equations (4) the functions $\overset{N}{W}_i$ and $\overset{N+1}{W}$ must be previously chosen, and this is equivalent to choose (up to order N) a coordinate condition or gauge.

Once the metric tensor, up to the desired order, has been calculated, the equations of motion up to the same order can then be obtained.

From the field equations, as a consequence of Bianchi identities, we have:

$$T^{\mu\nu};\, \nu = 0 \tag{7}$$

which gives the equations of motion for the fluid.

II A. Gauge conditions in the second PNA for a perfect fluid

At the zero order (N = 0) equations (4) can be written as:

$$\overset{2}{g}_{00} = -2\mathcal{U} \qquad \text{where} \quad \Delta\mathcal{U} = -4\pi G\rho \tag{8 a, b}$$

Equations (7) reduce, in this case, to the Eulerian and continuity equations of hydrodynamics for a perfect fluid. This is, therefore, the Newtonian approximation. Successive approximations are called post-Newtonian (PNA).

At the first order, N = 1, the field equations are homogeneous, therefore the corresponding metric components can be made to vanish ($\overset{1}{g}_{ij} = \overset{2}{g}_{0i} = \overset{3}{g}_{00} = 0$) by a suitable choice of gauge [*16*].

If this prescription is used, at the next order, N = 2 (first PNA), we have

$$\overset{2}{\mathrm{S}}_{ij} = \overset{2}{g}_{00,ij} + 8\pi\mathrm{G}\rho\delta_{ij} \tag{9 a}$$

$$\overset{3}{\mathrm{S}}_{0i} = -\,16\pi\mathrm{G}\rho v^i \tag{9 b}$$

$$\overset{4}{\mathrm{S}}_{00} = 16\pi\mathrm{G}\rho\left(v^2 - \mathcal{U} + \frac{1}{2}\Pi + \frac{3}{2}p/\rho\right) + 2(\mathcal{U}_{,i})^2 \tag{9 c}$$

where $v^i = dx^i/dt$.

In this case, equation (6 *a*) is automatically satisfied, as it is a consequence of equation (8 *b*) (field equation for the lowest order : N = 0). Equation (6 *b*) reduces to the Newtonian continuity equation. Thus the arbitrary functions $\overset{2}{\mathrm{W}}_i$ and $\overset{3}{\mathrm{W}}$ (the gauge) can be chosen arbitrarily.

There are two usual ways to choose a gauge in the first PNA. The more usual is gauge (L) [*17*] [*18*] defined by :

$$\overset{2}{\mathrm{W}}_i(\mathrm{L}) = \mathcal{U}_{,i} \qquad \overset{3}{\mathrm{W}}(\mathrm{L}) = 0\,; \tag{10}$$

it has the advantage of simplifying the equations for the metric because it does not include the term $\dfrac{\partial^2}{\partial t^2}$. This obviates the use of Euler equations. The solution for the metric components is

$$\overset{2}{g}_{ij}(\mathrm{L}) = -\,2\mathcal{U}\delta_{ij} \qquad \overset{3}{g}_{0i}(\mathrm{L}) = \mathrm{P}_i \qquad \overset{4}{g}_{00}(\mathrm{L}) = 2\mathcal{U}^2 - 4\Phi \tag{11}$$

where

$$\mathrm{P}_i = 4\mathcal{U}_i - \frac{1}{2}\chi_{,ti} \tag{12 a}$$

$$\Delta\mathcal{U}_i = -\,4\pi\mathrm{G}\rho v^i \qquad \Delta\chi = -\,2\mathcal{U} \tag{12 b}$$

$$\Delta\Phi = -\,4\pi\mathrm{G}\rho\left(v^2 + \mathcal{U} + \frac{1}{2}\Pi + \frac{3}{2}\,p/\rho\right) \tag{12 c}$$

The second choice, gauge (H), called the harmonic coordinate condition, has been used by several authors (ref. [*14*] [*19*]) and was obtained from the de Donder condition :

$$(\sqrt{-\,g}\;g^{\mu\nu})_{,\nu} = 0 \tag{13}$$

which, up to the first PNA, in our expansion gives:

$$\overset{2}{\mathrm{W}}_i(\mathrm{H}) = \mathcal{U}_{,i} \qquad \overset{3}{\mathrm{W}}(\mathrm{H}) = -\mathcal{U}_{,t} \tag{14}$$

the solution for the metric components is now:

$$\overset{2}{g}_{ij}(\mathrm{H}) = \overset{2}{g}_{ij}(\mathrm{L}) \qquad \overset{3}{g}_{0i}(\mathrm{H}) = 4\mathcal{U}_i \qquad \overset{4}{g}_{00}(\mathrm{H}) = \overset{4}{g}_{00}(\mathrm{L}) + \chi_{,tt} \tag{15}$$

Once the metric components are known ((11) or (15)) we can readily obtain from equation (7) the relativistic extensions including c^{-2} terms of Euler equations (the first PNA equations of hydrodynamics [*18*]) and the continuity ones.

It must be noted that in the two gauges we have mentioned, the equations of motion have the same form, since the two systems of coordinates only differ in the c^{-4} term in the time coordinate.

We can now proceed to the next order.

At third order, N = 3, we have $\overset{3}{\mathrm{S}}_{ij} = \overset{4}{\mathrm{S}}_{0i} = \overset{6}{\mathrm{S}}_{00} = 0$ so that the field equations are homogeneous and the usual choice of gauge can be made: $\overset{3}{g}_{ij} = \overset{4}{g}_{0i} = \overset{5}{g}_{00} = 0$.

At the fourth order, N = 4 (second PNA), the metric equations have been given by Chandrasekhar and Nutku [*20*], starting with gauge (L) in the first PNA, and by Anderson and Decanio [*19*], beginning with gauge (H). In ref. [*20*], the corresponding functions $\overset{4}{\mathrm{S}}_{ij}(\mathrm{L})$, $\overset{5}{\mathrm{S}}_{0i}(\mathrm{L})$ and $\overset{6}{\mathrm{S}}_{00}(\mathrm{L})$ are found.

It can easily be shown that the equation (6 *a*) is, at this order, the Newtonian Euler equation and (6 *b*) the continuity equation in the first PNA [*18*]. So the functions $\overset{4}{\mathrm{W}}_i$ and $\overset{5}{\mathrm{W}}$ can also be arbitrarily chosen.

The gauge chosen by Chandrasekhar at this order, hereafter denoted as gauge (C), is defined by:

$$\overset{4}{\mathrm{W}}_i(\mathrm{C}) = \overset{5}{\mathrm{W}}(\mathrm{C}) = 0 \tag{16}$$

Similarly, in gauge (H) up to the order N = 2, we have: $\overset{4}{\mathrm{S}}_{ij}(\mathrm{H}) = \overset{4}{\mathrm{S}}_{ij}(\mathrm{L})$

$$\overset{5}{\mathrm{S}}_{0i}(\mathrm{H}) = -16\pi\mathrm{G}\rho[(v^2 + 4\mathcal{U} + \Pi + p/\rho)v^i - 2\mathcal{U}_i] \\ - 12\mathcal{U}_{,i}\mathcal{U}_{,t} + 8\mathcal{U}_j\mathcal{U}_{,ij} - 8\mathcal{U}_{j,i}\mathcal{U}_j . \tag{17 a}$$

$$\overset{6}{\mathrm{S}}_{00}(\mathrm{H}) = 16\pi\mathrm{G}\rho\left[v^2(v^2 + 4\mathcal{U} + \Pi + p/\rho) - (\mathcal{U}^2 + 2\Phi) + \frac{1}{2}\chi_{,tt}\right] \\ + 12\mathcal{U}_{,i}\Phi_{,i} - 8(\mathcal{U}_{,t})^2 + 8\mathcal{U}_{,i}\mathcal{U}_{i,t} \\ - 12\mathcal{U}(\mathcal{U}_{,i})^2 - 16\mathcal{U}_{i,j}(\mathcal{U}_{i,j} - \mathcal{U}_{j,i}) - 3\mathcal{U}_{,i}\chi_{,tti} \tag{17 b}$$

If the de Donder condition (13) up to the order N = 4 is imposed we have:

$$\overset{4}{\mathrm{W}}_i(\mathrm{H}) = 4\mathcal{U}_{i,t} + 2\Phi_{,i} - \frac{1}{2}\chi_{,tti} + 2\mathcal{U}\mathcal{U}_{,i} \tag{18 a}$$

$$\overset{5}{\mathrm{W}}(\mathrm{H}) = -2\Phi_{,t} + \frac{1}{2}\chi_{,ttt} - \mathcal{U}\mathcal{U}_{,t} - 8\mathcal{U}_i\mathcal{U}_{,i} - \frac{1}{2}\mathcal{U}_{,i}\chi_{,ti} \tag{18 b}$$

II B. Metric tensor for point masses up to the second PNA

We are interested in the metric components for an isolated system of point masses in gauges (C) and (H).

These components can be obtained through Einstein equations for a perfect fluid. In refs. [*21*] and [*22*] the metric components for an N-body system composed of spherical and isotropic bodies of small dimensions and slow rotation velocity were evaluated using gauge (C). The metric components $\overset{4}{g}_{ij}(\mathrm{C})$, $\overset{5}{g}_{0i}(\mathrm{C})$ and $\overset{6}{g}_{00}(\mathrm{C})$ can be divided in three groups: « rotation », « structure » and « pointlike ». « Rotation » terms are null when the angular velocity is null or when the dimension of the bodies is null (point limit). « Structure » terms are functions of Π, p and the Newtonian self-potential. These terms stem from the definition of mass of the body (M_a) [*23*]. The elimination of these terms in the point limit, is equivalent to the use of Infeld and Plebanski's « good » δ-functions [*24*]. « Pointlike » terms are functions of masses of the bodies, of their velocities and of their relative positions. These last terms are the only ones which interest us. (An analogous classification was also given by Spyrou [*25*]).

These metric components have a Minkowskian form at great distance from the bodies [*22*] (and are given in a different gauge from that used by Ohta *et al.* [*26*]). According to their dependence on the powers of the gravitational constant G, « pointlike » terms can be expressed as:

$$\begin{aligned}
\overset{4}{g}_{ij}(\mathrm{C}) &= \overset{4}{g}_{ij}(\mathrm{C}\,;\,\mathrm{G}) + \overset{4}{g}_{ij}(\mathrm{C}\,;\,\mathrm{G}^2)\\
\overset{5}{g}_{0i}(\mathrm{C}) &= \overset{5}{g}_{0i}(\mathrm{C}\,;\,\mathrm{G}) + \overset{5}{g}_{0i}(\mathrm{C}\,;\,\mathrm{G}^2)\\
\overset{6}{g}_{00}(\mathrm{C}) &= \overset{6}{g}_{00}(\mathrm{C}\,;\,\mathrm{G}) + \overset{6}{g}_{00}(\mathrm{C}\,;\,\mathrm{G}^2) + \overset{6}{g}_{00}(\mathrm{C}\,;\,\mathrm{G}^3)
\end{aligned}$$

The term $\overset{6}{g}_{00}(\mathrm{C}\,;\,\mathrm{G}^3)$ does not depend on the velocities of the bodies and it will not be needed here.

The metric tensor components in gauge (H), up to the second PNA, can be worked out in an analogous fashion. The results at the point a where the body is located are:

$$\begin{aligned}
\overset{4}{g}_{ij}(\mathrm{H}) = &-\left[1+\frac{2\mathrm{G}}{c^2}\sum_{b\neq a}\frac{\mathrm{M}_b}{r_{ab}}\right]\delta_{ij}+\frac{1}{c^4}\left\{\mathrm{G}\sum_{b\neq a}\frac{\mathrm{M}_b}{r_{ab}}\left[-4v_b^i v_b^j+(\vec{v}_b\vec{n}_{ab})^2\delta^{ij}\right]\right.\\
&-\mathrm{G}^2\sum_{b\neq a}\frac{\mathrm{M}_b^2}{r_{ab}^2}\left[\delta^{ij}+n_{ab}^i n_{ab}^j\right]-2\mathrm{G}^2\sum_{b\neq a}\sum_{\substack{c\neq b\\ a}}\frac{\mathrm{M}_b\mathrm{M}_c}{r_{ab}r_{ac}}\delta^{ij}+2\mathrm{G}^2\sum_{b\neq a}\sum_{c\neq b}\frac{\mathrm{M}_b\mathrm{M}_c}{r_{ab}r_{ac}}\delta^{ij}\\
&\left.-\mathrm{G}^2\sum_{b\neq a}\sum_{c\neq b}\frac{\mathrm{M}_b\mathrm{M}_c}{r_{bc}^2}(\vec{n}_{ab}\vec{n}_{bc})\delta^{ij}+4\mathrm{G}^2\sum_{b\neq a}\sum_{c\neq b}\mathrm{M}_b\mathrm{M}_c\frac{\partial^2}{\partial x_b^i\partial x_c^j}\ln\mathrm{S}\right\}.
\end{aligned}$$

$$\overset{(5)}{g}_{0i}(\mathrm{H}) = \frac{4\mathrm{G}}{c^3} \sum_{b \neq a} \frac{\mathrm{M}_b}{r_{ab}} v_b^i + \frac{1}{c^5} [\overset{5}{g}_{0i}(\mathrm{H}\,;\ \mathrm{G}) + \overset{5}{g}_{0i}(\mathrm{H}\,;\ \mathrm{G}^2)]$$

$$\overset{5}{g}_{0i}(\mathrm{H}\,;\ \mathrm{G}) = \mathrm{G} \sum_{b \neq a} \frac{\mathrm{M}_b}{r_{ab}} [4v_b^2 v_b^i - 2(\vec{v}_b \vec{n}_{ab})^2 v_b^i]$$

$$\begin{aligned}
\overset{5}{g}_{0i}(\mathrm{H}\,;\ \mathrm{G}^2) = {} & \mathrm{G}^2 \sum_{b \neq a} \frac{\mathrm{M}_b^2}{r_{ab}} \left[-\frac{3}{2} v_b^i + 2(\vec{v}_b \vec{n}_{ab}) n_{ab}^i \right] \\
& + \mathrm{G}^2 \sum_{b \neq a} \sum_{c \neq b} \frac{\mathrm{M}_b \mathrm{M}_c}{r_{bc}^2} [2v_b^i (\vec{n}_{ab} \vec{n}_{bc}) + 4n_{bc}^i (\vec{v}_b \vec{n}_{ab})] \\
& + \mathrm{G}^2 \sum_{b \neq a} \sum_{c \neq b} \frac{\mathrm{M}_b \mathrm{M}_c}{r_{bc}^3} r_{ab} (v_b^j - v_c^j)(-2\delta^{ij} + 6n_{bc}^i n_{bc}^j) \\
& + \mathrm{G}^2 \sum_{b \neq a} \sum_{c \neq b} \frac{\mathrm{M}_b \mathrm{M}_c}{r_{ab} r_{bc}} [4v_b^i - 8v_c^i] \\
& + \mathrm{G}^2 \sum_{b \neq a} \sum_{c \neq b} \frac{\mathrm{M}_b \mathrm{M}_c}{r_{ab}^2} \left[\frac{1}{4} (\vec{v}_c \vec{n}_{ac}) n_{ab}^i - \frac{1}{4} (\vec{v}_b \vec{n}_{bc}) n_{ab}^i \right] \\
& + \mathrm{G}^2 \sum_{b \neq a} \sum_{\substack{c \neq b \\ a}} \frac{\mathrm{M}_b \mathrm{M}_c}{r_{ab} r_{ac}} \left[-\frac{1}{4} v_b^i + \frac{1}{4} (\vec{v}_b \vec{n}_{ab}) n_{ab}^i \right] \\
& + \mathrm{G}^2 \sum_{b \neq a} \sum_{c \neq b} \mathrm{M}_b \mathrm{M}_c v_b^j \left[-16 \frac{\partial^2}{\partial x_b^i \partial x_c^j} + \frac{23}{2} \frac{\partial^2}{\partial x_c^i \partial x_b^j} - \frac{1}{2} \frac{\partial^2}{\partial x_b^i \partial x_b^j} \right] \ln \mathrm{S}.
\end{aligned}$$

$$\begin{aligned}
\overset{(6)}{g}_{00}(\mathrm{H}) = {} & 1 - \frac{2\mathrm{G}}{c^2} \sum_{b \neq a} \frac{\mathrm{M}_b}{r_{ab}} + \frac{1}{c^4} \overset{4}{g}_{00}(\mathrm{H}) \\
& + \frac{1}{c^6} [\overset{6}{g}_{00}(\mathrm{H}\,;\ \mathrm{G}) + \overset{6}{g}_{00}(\mathrm{H}\,;\ \mathrm{G}^2) + \overset{6}{g}_{00}(\mathrm{H}\,;\ \mathrm{G}^3)]
\end{aligned}$$

$$\begin{aligned}
\overset{4}{g}_{00}(\mathrm{H}) = {} & \mathrm{G} \sum_{b \neq a} \frac{\mathrm{M}_b}{r_{ab}} [-4v_b^2 + (\vec{v}_b \vec{n}_{ab})^2] + 2\mathrm{G}^2 \sum_{b \neq a} \frac{\mathrm{M}_b^2}{r_{ab}^2} \\
& + 2\mathrm{G}^2 \sum_{b \neq a} \sum_{\substack{c \neq b \\ a}} \frac{\mathrm{M}_b \mathrm{M}_c}{r_{ab} r_{ac}} + 2\mathrm{G}^2 \sum_{b \neq a} \sum_{c \neq b} \frac{\mathrm{M}_b \mathrm{M}_c}{r_{ab} r_{bc}} - \mathrm{G}^2 \sum_{b \neq a} \sum_{c \neq b} \frac{\mathrm{M}_b \mathrm{M}_c}{r_{bc}^2} (\vec{n}_{ab} \vec{n}_{bc}).
\end{aligned}$$

$$\overset{6}{g}_{00}(\mathrm{H}\,;\,\mathrm{G})=\mathrm{G}\sum_{b\neq a}\frac{\mathrm{M}_b}{r_{ab}}\left[-4v_b^4+3v_b^2(\vec{v}_b\vec{n}_{ab})^2-\frac{3}{4}(\vec{v}_b\vec{n}_{ab})^4\right]$$

$$\begin{aligned}
\overset{6}{g}_{00}(\mathrm{H}\,;\,\mathrm{G}^2)=&\,\mathrm{G}^2\sum_{b\neq a}\frac{\mathrm{M}_b^2}{r_{ab}^2}[2v_b^2-5(\vec{v}_b\vec{n}_{ab})^2]\\
&+\mathrm{G}^2\sum_{b\neq a}\sum_{c\neq b,a}\frac{\mathrm{M}_b\mathrm{M}_c}{r_{ab}r_{ac}}\left[\frac{21}{2}v_b^2-10(\vec{v}_b\vec{v}_c)-\frac{5}{2}(\vec{v}_b\vec{n}_{ab})^2\right]\\
&+\mathrm{G}^2\sum_{b\neq a}\sum_{c\neq b}\frac{\mathrm{M}_b\mathrm{M}_c}{r_{ab}r_{bc}}\left[-8v_b^2+\frac{7}{2}v_c^2+\frac{17}{2}(\vec{v}_b\vec{v}_c)-(\vec{v}_b\vec{n}_{ab})^2\right.\\
&\left.-\frac{1}{2}(\vec{v}_c\vec{n}_{bc})^2-\frac{1}{2}(\vec{v}_c\vec{n}_{bc})(\vec{v}_b\vec{n}_{bc})\right]\\
&+\mathrm{G}^2\sum_{b\neq a}\sum_{c\neq b}\frac{\mathrm{M}_b\mathrm{M}_c}{r_{ac}^2}\left[-\frac{1}{2}(\vec{v}_b\vec{n}_{ab})(\vec{v}_c\vec{n}_{ac})-\frac{1}{2}(\vec{v}_b\vec{n}_{bc})(\vec{v}_c\vec{n}_{ac})\right]\\
&+\mathrm{G}^2\sum_{b\neq a}\sum_{c\neq b}\mathrm{M}_b\mathrm{M}_c\left\{v_b^iv_b^j\left[-7\frac{\partial^2}{\partial x_b^i\partial x_b^j}-8\frac{\partial^2}{\partial x_c^i\partial x_c^j}\right]\right.\\
&\left.+v_b^iv_c^j\left[-\frac{11\partial^2}{\partial x_b^i\partial x_c^j}+\frac{16\partial^2}{\partial x_c^i\partial x_b^j}+\frac{16\partial^2}{\partial x_c^i\partial x_c^j}\right]\right\}\ln \mathrm{S}\\
&+\mathrm{G}^2\sum_{b\neq a}\sum_{c\neq b}\frac{\mathrm{M}_b\mathrm{M}_c}{r_{bc}^2}n_{bc}^k\left\{v_b^iv_b^i\left[-3\delta^{jk}n_{ab}^i-4\delta^{ij}n_{ab}^k+\frac{3}{2}n_{ab}^in_{ab}^jn_{ab}^k\right]\right.\\
&\left.+v_b^iv_c^j[-4\delta^{ik}n_{ab}^j-5\delta^{kj}n_{ab}^i+4\delta^{ij}n_{ab}^k]+v_c^iv_c^j\left[3\delta^{ik}n_{ab}^j-2\delta^{ij}n_{ab}^k+\frac{3}{2}n_{bc}^in_{bc}^jn_{bc}^k\right]\right\}\\
&+\mathrm{G}^2\sum_{b\neq a}\sum_{c\neq b}\frac{\mathrm{M}_b\mathrm{M}_c}{r_{bc}^3}r_{ab}\{\,v_b^iv_b^j[3\delta^{ij}+n_{ab}^in_{ab}^j-9n_{bc}^in_{bc}^j-3n_{ab}^in_{bc}^j(\vec{n}_{ab}\vec{n}_{bc})]\\
&+v_b^iv_c^j[-2\delta^{ij}-n_{ab}^in_{ab}^j+6n_{bc}^in_{bc}^j+3n_{ab}^in_{bc}^j(\vec{n}_{ab}\vec{n}_{bc})]+v_c^iv_c^j[-\delta^{ij}+3n_{bc}^in_{bc}^j]\,\}\\
&+\mathrm{G}^2\sum_{b\neq a}\sum_{c\neq b}\frac{\mathrm{M}_b\mathrm{M}_c}{r_{bc}^4}r_{ab}^2n_{ab}^i(-v_b^jv_b^k+2v_b^kv_c^j-v_c^jv_c^k)\\
&\left(-\frac{3}{4}\delta^{ik}n_{bc}^j-\frac{3}{4}\delta^{jk}n_{bc}^i-\frac{3}{4}\delta^{ij}n_{bc}^k+\frac{9}{4}n_{bc}^in_{bc}^jn_{bc}^k\right).
\end{aligned}$$

where $\vec{r}_{ab}=\vec{x}_a-\vec{x}_b$, $\vec{n}_{ab}=\vec{r}_{ab}/r_{ab}$, $\mathrm{S}=r_{ab}+r_{ac}+r_{bc}$ ($\vec{x}_a$ and $\vec{x}_b$ are the positions of bodies a and b and $\vec{v}_b=\dfrac{d\vec{x}_b}{dt}$ is the velocity of body b).

These metric components diverge at large distances from the bodies (carrying particle a to infinity). This divergence, due to the gauge, does not preclude the use of harmonic coordinates up to order N = 4 because it has been pointed out by Burke and Thorne [*15*] [*27*] (see also Ehlers *et al.* [*28*]) that the metric solutions in the « near zone » (« inner » expansion, eqs. (3) and (4)) must be matched with the solutions valid in the « far zone » (« outer » expansion).

III. METRIC TENSOR INVARIANCE AND EQUATIONS OF MOTION

In this Section, the transformations leaving the metric tensor (second PNA) invariant will be evaluated. The connection between them and the transformations that leave the equations of motion invariant will also be presented.

The evaluation of the coordinate transformation connecting the variables $x^\mu : (\vec{x}, ct)$ and $\xi^\mu : (\vec{\xi}, c\tau)$ shall be made from the standard transformation law

$$g_{\mu\nu}(\xi^\sigma) = \frac{\partial x^\alpha}{\partial \xi^\mu}\frac{\partial x^\beta}{\partial \xi^\nu} g_{\alpha\beta}(x^\sigma) \tag{19}$$

where the metric tensor is assumed to have the same form in the two coordinate systems.

According to the expansion in powers of c^{-1} given in (3) for the metric tensor this tensor can be splitted in two parts: $g_{\mu\nu} = g_{0\mu\nu} + h_{\mu\nu}$ where $g_{0\mu\nu}$ is the Minkowskian metric tensor and where $h_{\mu\nu}$ are functions depending on the gravitational constant G. Hereafter these functions and their derivatives that depend on G will be called « potential functions ».

The previous decomposition of the metric tensor in $\eta_{\mu\nu}$ plus potential functions suggests to separate the linear part of the transformation from the rest: $x^\mu = \mathrm{A}^\mu{}_\nu \xi^\nu + \phi^\mu$, where $\mathrm{A}^\mu{}_\nu$ is a constant matrix which does not depend on G. Then equation (19) becomes two independent equations, one of them contains $\eta_{\mu\nu}$ and the linear part of the transformation, the other contains the potential functions. The first equation shows that the linear part of the transformation is a Poincaré transformation (equation (19) is not modified by additive constants in the coordinate transformation).

In accordance with the power expansion in c^{-1} of the metric tensor, the coordinate transformation shall be written as:

$$\begin{aligned} x^i &\simeq \sum_{\mathrm{L}=0}^{\mathrm{N}} c^{-\mathrm{L}} \overset{\mathrm{L}}{\psi}{}^i \equiv \overset{(\mathrm{N})}{\psi}{}^i \qquad (\overset{0}{\psi}{}^i = \xi^i - \mathrm{V}^i\tau) \\ t &\simeq \sum_{\mathrm{L}=0}^{\mathrm{N}+2} c^{-\mathrm{L}} \overset{\mathrm{L}}{\eta} \equiv \overset{(\mathrm{N}+2)}{\eta} \qquad (\overset{0}{\eta} = \tau) \end{aligned} \tag{20}$$

where only the pure Galilean transformations are considered. As it shall be seen this restriction does not mean a restriction in our results. A transformation for which $\mathrm{N} > 0$ will be called post-Galilean.

With an obvious notation, from (20) we can write

$$\begin{aligned}
&\frac{\partial t}{\partial \tau} \simeq \overset{(N+2)}{\eta_\tau} \quad (\overset{0}{\eta}_\tau = 1) \qquad && \frac{\partial x^i}{\partial \tau} \simeq \overset{(N)}{\psi^i_\tau} \quad (\overset{0}{\psi^i_\tau} = -\,V^i) \\
&\frac{\partial t}{\partial \xi^i} \simeq \overset{(N+2)}{\eta_i} \quad (\overset{0}{\eta}_i = 0) \qquad && \frac{\partial x^i}{\partial \xi^j} \simeq \overset{(N)}{\psi^i_j} \quad (\overset{0}{\psi^i_j} = \delta^i_j)
\end{aligned} \tag{21}$$

Now, in order to write eq. (19) in terms of the new coordinates in the power expansion of c^{-1}, the following notation is introduced: let f be a function of x^μ, we shall call f' the same functional form in terms of the new coordinates ξ^μ. According to the transformation (20) f can be written in terms of the new coordinates as

$$f \simeq \sum_{L=0}^{N} c^{-L} \underset{L}{\tilde{f}} \equiv \underset{(N)}{\tilde{f}}, \qquad \underset{0}{\tilde{f}} \equiv f' + f'' \tag{22}$$

the first expression defines $\underset{L}{\tilde{f}}$ and the second f''. This last function depends only on the zero order of the transformation (20).

According to these notations we write eq. (19) as:

$$\begin{aligned}
\overset{(N+2)}{g'_{00}} = \sum_{K,L,M=0}^{N+2} \overset{(N+2-L-K)}{\underset{(L)}{\tilde{g}_{00}}} \overset{(K-M)}{\eta_\tau} \overset{(M)}{\eta_\tau} &+ \frac{2}{c} \sum_{K,L,M=0}^{N+1} \overset{(N+1-L-K)}{\underset{(L)}{\tilde{g}_{0i}}} \overset{(K-M)}{\eta_\tau} \overset{(M)}{\psi^i_\tau} \\
&+ \frac{1}{c^2} \sum_{K,L,M=0}^{N} \overset{(N-L-K)}{\underset{(L)}{\tilde{g}_{ij}}} \overset{(K-M)}{\psi^i_\tau} \overset{(M)}{\psi^j_\tau}
\end{aligned} \tag{23 a}$$

$$\begin{aligned}
\overset{(N+1)}{g'_{0i}} = c \sum_{K,L,M=0}^{N+2} \overset{(N+2-L-K)}{\underset{(L)}{\tilde{g}_{00}}} \overset{(K-M)}{\eta_\tau} \overset{(M)}{\eta_i} &+ \sum_{K,L,M=0}^{N+1} \overset{(N+1-L-K)}{\underset{(L)}{\tilde{g}_{ij}}} \Big[\overset{(K-M)}{\eta_\tau} \overset{(M)}{\psi^j_i} + \overset{(K-M)}{\eta_i} \overset{(M)}{\psi^j_\tau} \Big] \\
&+ \frac{1}{c} \sum_{K,L,M=0}^{N} \overset{(N-L-K)}{\underset{(L)}{\tilde{g}_{kl}}} \overset{(K-M)}{\psi^k_\tau} \overset{(M)}{\psi^l_i}
\end{aligned} \tag{23 b}$$

$$\begin{aligned}
\overset{(N)}{g'_{ij}} = c^2 \sum_{K,L,M=0}^{N+2} \overset{(N+2-L-K)}{\underset{(L)}{\tilde{g}_{00}}} \overset{(K-M)}{\eta_i} \overset{(M)}{\eta_j} &+ c \sum_{K,L,M=0}^{N+1} \overset{(N+1-L-K)}{\underset{(L)}{\tilde{g}_{0k}}} \Big[\overset{(K-M)}{\eta_i} \overset{(M)}{\psi^k_j} + \overset{(K-M)}{\eta_j} \overset{(M)}{\psi^k_i} \Big] \\
&+ \sum_{K,L,M=0}^{N} \overset{(N-L-K)}{\underset{(L)}{\tilde{g}_{kl}}} \overset{(K-M)}{\psi^k_i} \overset{(M)}{\psi^l_j}
\end{aligned} \tag{23 c}$$

where we have taken into account that, up to order c^{N+2}, $g_{00} \simeq \overset{(N+2)}{g_{00}}$ and that in the new coordinates: $g_{00} \simeq \sum_{L=0}^{N+2} \overset{(N+2-L)}{\underset{(L)}{g_{00}}}$. Analogous results for the remaining coefficients have been used.

The following comments on equations (23) must be made. Equation (23 *a*) includes all the components of the metric tensor up to $\overset{N}{g'_{ij}}$, $\overset{N+1}{g'_{0i}}$ and $\overset{N+2}{g''_{00}}$ (but not $\overset{N+2}{g'_{00}}$) and gives an expression for the transformation term $\overset{N+2}{\eta_\tau}$. Equation (23 *b*) contains the components of the metric tensor up to $\overset{N}{g'_{ij}}$ and $\overset{N+1}{g''_{0i}}$ (but neither $\overset{N+1}{g'_{0i}}$ nor $\overset{N+2}{g''_{00}}$) and gives a relation for the term $\overset{N+2}{\eta_i}$. Both equations contain the transformation term $\overset{N}{\psi}{}^i$. Finally, equation (23 *c*) contains the metric tensor components up to $\overset{N}{g''_{ij}}$ (but neither $\overset{N+1}{g''_{0i}}$ nor $\overset{N+2}{g''_{00}}$) and gives an expression for the term $\overset{N}{\psi}{}^i_j$. Note that $\overset{N+2}{\eta}$ does not appear in this equation.

Therefore, if system (23) is integrable, the functions $\overset{N}{\psi}{}^i$ and $\overset{N+2}{\eta}$ can be obtained by means of an iterative method from the metric tensor up to the order N. The integrability conditions will be examined later.

III A. **Invariance of the equations of motion**

Since the equations of motion are derived from the metric, a transformation which leaves the metric invariant also leaves the corresponding equations of motion invariant.

It has been noted that to give a meaning to the invariance of the metric tensor up to order N needs the consideration of the transformation up to terms $\overset{N}{\psi}{}^i$ and $\overset{N+2}{\eta}$. However it can readily be shown that the invariance of the equations of motion up to order N needs only to consider the transformation up to terms $\overset{N}{\psi}{}^i$ and $\overset{N}{\eta}$.

III B. **Post-Galilean transformations**

Since we are interested in post-Galilean transformations up to order $N = 4$, the following notation is introduced:

$$\begin{aligned} &\overset{2}{\psi}{}^i = \frac{1}{2} V^i(\vec{\xi}\vec{V} - V^2\tau) + Y^i \qquad \overset{4}{\psi}{}^i = \psi^i \\ &\overset{2}{\eta} = \frac{1}{2} V^2\tau - \vec{\xi}\vec{V} + Z \qquad \overset{4}{\eta} = \zeta \qquad \overset{6}{\eta} = \eta \end{aligned} \tag{24}$$

where Z, ζ and Y^i are the functions introduced by CC. $\overset{2}{\psi}{}^i$ and $\overset{2}{\eta}$ are decomposed in two parts: the first one corresponding to a pure Lorentz transformation (with parameter $\vec{V}$).

From equations (23) and from the post-Newtonian metric tensor, we conclude that functions $\overset{1}{\psi}{}^i$, $\overset{3}{\psi}{}^i$, $\overset{1}{\eta}$, $\overset{3}{\eta}$ and $\overset{5}{\eta}$ are constants. Since space-time translations are not considered here, we must take:

$$\overset{1}{\psi}{}^i = \overset{3}{\psi}{}^i = \overset{1}{\eta} = \overset{3}{\eta} = \overset{5}{\eta} = 0 \tag{25}$$

III C. **The first post-Galilean transformation: gauges (L) and (H)**

The first post-Galilean transformation, that leaves the metric tensor invariant in the first PNA, was calculated in gauge (L) by CC. The main difficulty in the calculation lies in the fact that all the functions appearing in the metric are given at the same time t (resp. τ). Thus $\vec{x}_a - \vec{x}_b = \vec{r}_{ab}$, $\vec{n}_{ab} = \vec{r}_{ab}/r_{ab}$ and $\vec{v}_b$, simultaneous in t, must be written in terms of the functions: $\vec{\xi}_a - \vec{\xi}_b = \vec{\sigma}_{ab}$, $\vec{v}_{ab} = \vec{\sigma}_{ab}/\sigma_{ab}$ and $\vec{w}_b$, simultaneous in τ. Since transformation (20) does not conserve simultaneity, the passage from one system to another requires further calculation.

At the Newtonian order ($N = 0$), from equations (23) the solution $Z_a = 0$ can be chosen and the transformation is Lorentz-like.

If we choose appropriate integration constants we obtain in gauge (L) the following expressions which define the first post-Galilean transformation:

$$\zeta_a(\mathrm{L}) = \frac{3}{8}\mathrm{V}^4\tau - \frac{1}{2}(\vec{\xi}_a\vec{\mathrm{V}})\mathrm{V}^2 - \frac{1}{2}\mathrm{G}\sum_{b\neq a}\mathrm{M}_b(\vec{v}_{ab}\vec{\mathrm{V}}) \tag{26 a}$$

$$\vec{\mathrm{Y}}_a(\mathrm{L}) = q(\vec{\xi}_a \times \vec{\mathrm{V}}) \qquad (q = \text{arbitrary constant}) \tag{26 b}$$

where a G dependent term of order c^{-4} appears in addition to the Lorentz-like term.

However, as has been noted in Subsection III A, this term does not play any role in the transformation of the equations of motion. (This has been proved by CC using the EIH Lagrangian).

A physical interpretation of $\vec{V}$, as the velocity of system (x^μ) with respect to system (ξ^μ) can now be obtained up to order c^{-2}

$$\vec{\xi} = \vec{\mathrm{V}}\tau + 0(c^{-4}) \quad (\text{when } \vec{x} = 0), \qquad \vec{x} = -\vec{\mathrm{V}}t + 0(c^{-4}) \quad (\text{when } \vec{\xi} = 0) \tag{27}$$

The situation changes in gauge (H) where an analogous calculation must be made to find $\zeta_a(\mathrm{H})$ and $\vec{\mathrm{Y}}_a(\mathrm{H})$. The result being now:

$$\zeta_a(\mathrm{H}) = \frac{3}{8}\mathrm{V}^4\tau - \frac{1}{2}(\vec{\xi}_a\vec{\mathrm{V}})\mathrm{V}^2, \quad \vec{\mathrm{Y}}_a(\mathrm{H}) = \vec{\mathrm{Y}}_a(\mathrm{L}) \tag{28}$$

Thus, the first PNA metric in gauge (H) is invariant under a Lorentz transformation. The equations of motion in this gauge have the same form as those of the EIH.

III D. The second post-Galilean transformation : gauge (C)

Using the equations (23), solutions $\psi^i_a(C)$ and $\eta_a(C)$ can be obtained for $N = 4$. The method is the same as for $N = 2$, although much more complicated. Since $\overset{6}{g}{}''_{00}$ and not $\overset{6}{g}{}'_{00}$ is used, the term $\overset{6}{g}_{00}(C\,;G^3)$ is not necessary. This term does not contains any velocity and therefore its contribution to $\overset{6}{g}{}''_{00}$ (which only includes the zero order of the transformation) is null.

A lengthy calculation leads to the following solution of equation (23 c)

$$\begin{aligned}\vec{\psi}_a(C) = \vec{a}(\tau) + \vec{\xi}_a \times \vec{B}(\tau) + \frac{3}{8}(\vec{\xi}_a\vec{V})V^2\vec{V} - \frac{q^2}{2}[V^2\vec{\xi}_a - (\vec{\xi}_a\vec{V})\vec{V}] \\ + G\sum_{b\neq a} M_b\left\{\frac{9}{4}(\vec{w}_b\vec{V})\vec{v}_{ab} - \frac{5}{4}(\vec{w}_b\vec{v}_{ab})\vec{V} - \frac{7}{4}\vec{w}_b(\vec{v}_{ab}\vec{V})\right. \\ \left. - \frac{1}{4}\vec{v}_{ab}(\vec{w}_b\vec{v}_{ab})(\vec{v}_{ab}\vec{V}) - \frac{9}{8}V^2\vec{v}_{ab} + \frac{7}{4}\vec{V}(\vec{v}_{ab}\vec{V}) + \frac{1}{8}\vec{v}_{ab}(\vec{v}_{ab}\vec{V})^2\right\} \end{aligned} \tag{29}$$

Similarly, if we define

$$\begin{aligned}F_a(C) \equiv \eta_a(C) + \frac{V^2}{2}\zeta_a(L) - \frac{1}{8}V^4(\vec{\xi}_a\vec{V}) + G\sum_{b\neq a} M_b\left[\frac{7}{4}w_b^2(\vec{v}_{ab}\vec{V})\right. \\ \left. + \frac{5}{8}V^2(\vec{v}_{ab}\vec{w}_b) - \frac{7}{4}(\vec{w}_b\vec{V})(\vec{v}_{ab}\vec{V}) - \frac{1}{8}(\vec{v}_{ab}\vec{V})^2(\vec{v}_{ab}\vec{w}_b) + \frac{1}{2}V^2(\vec{v}_{ab}\vec{V})\right] \\ + \frac{1}{2}G^2\sum_{b\neq a}\sum_{c\neq b} M_bM_c\left[\frac{\vec{v}_{ab}\vec{V}}{\sigma_{bc}} + \frac{\vec{v}_{ab}\vec{V} + \vec{v}_{bc}\vec{V}}{\sigma_{ac}}\right] - G^2\sum_{b\neq a} M_b^2\,\frac{(\vec{v}_{ab}\vec{V})}{\sigma_{ab}} \\ + G^2\sum_{b\neq a}\sum_{c\neq b,a} M_bM_cV^j\left(-4\frac{\partial}{\partial\xi^j_a} + 5\frac{\partial}{\partial\xi^j_b}\right)\ln S'. \end{aligned} \tag{30}$$

where $S' = \sigma_{ab} + \sigma_{ac} + \sigma_{bc}$, it is possible to write equations (23 a) and (23 b) as:

$$\frac{\partial}{\partial\xi^i_a}[F_a(C) + \vec{V}\vec{\psi}_a(C)] = \frac{da^i(\tau)}{d\tau} + \varepsilon^{ikl}\xi^k_a\frac{dB^l}{d\tau} \tag{31 a}$$

$$\frac{\partial}{\partial\tau}[F_a(C) + \vec{V}\vec{\psi}_a(C)] = \frac{1}{8}V^6 \tag{31 b}$$

The integrability condition of this system gives $\vec{B}$ as a constant vector and $\vec{a} = \vec{A}\tau + \vec{C}$ where $\vec{A}$ and $\vec{C}$ are constants. If we take $\vec{A} = -\frac{3}{8}V^4\vec{V}$, $\vec{C} = 0$ and $\vec{B} = q'\vec{V}$ (q' = arbitrary constant), from (31),

$$F_a(C) = \frac{1}{8}V^6\tau - \frac{3}{8}V^4(\vec{\xi}_a\vec{V}) - \vec{V}\vec{\psi}_a(C) \tag{32}$$

Then, the functions $\vec{\psi}_a(C)$ and $\eta_a(C)$ take the following form:

$$\psi_a^i(C) = -\frac{3}{8}V^4V^i\tau + \frac{3}{8}(\vec{\xi}_a\vec{V})V^2V^i + q'(\vec{\xi}_a \times \vec{V})^i - \frac{q^2}{2}[V^2\xi_a^i - (\vec{\xi}_a\vec{V})V^i] + \psi_a^i(C\,;G). \tag{33 a}$$

$$\eta_a(C) = \frac{5}{16}V^6\tau - \frac{3}{8}(\vec{\xi}_a\vec{V})V^4 + \eta_a(C\,;G) + \eta_a(C\,;G^2). \tag{33 b}$$

where, $\psi_a^i(C\,;G)$ is given in (29) and $\eta_a(C\,;G)$ and $\eta_a(C\,;G^2)$ can be obtained from (30) substituting $F_a(C)$ by its expression (32).

Unlike what happened at the preceding order (N = 2) the parameter $\vec{V}$ can not be interpreted as the relative velocity of one system with respect to the other.

In addition the potential terms $\zeta_a(L)$, $\psi_a^i(C)$ appear now explicitly in the transformation of the equations of motion, and therefore these equations are not invariant under a Lorentz-like transformation. This situation changes radically in gauge (H).

III E. The second post-Galilean transformation : gauge (H)

A similar calculation to that of the preceding Subsection yields the following second post-Galilean terms in gauge (H):

$$\psi_a^i(H) = -\frac{3}{8}V^4V^i\tau + \frac{3}{8}V^2(\vec{\xi}_a\vec{V})V^i + q'(\vec{\xi}_a \times \vec{V})^i - \frac{q^2}{2}[V^2\xi_a^i - (\vec{\xi}_a\vec{V})V^i]. \tag{34 a}$$

$$\eta_a(H) = \frac{5}{16}V^6\tau - \frac{3}{8}(\vec{\xi}_a\vec{V})V^4 + \eta_a(H\,;G^2). \tag{34 b}$$

where

$$\eta_a(H\,;G^2) = -\frac{G^2}{2}\sum_{b\neq a}M_b^2\frac{(\vec{v}_{ab}\vec{V})}{\sigma_{ab}} - \frac{G^2}{4}\sum_{b\neq a}\sum_{c\neq b}M_bM_c\frac{(\vec{v}_{ab}\vec{V} + \vec{v}_{bc}\vec{V})}{\sigma_{ac}} + \frac{G^2}{2}\sum_{b\neq a}\sum_{c\neq b}M_bM_cV^j\frac{\partial}{\partial\xi_b^j}\ln S'. \tag{34 c}$$

Thus the metric tensor is invariant under a Lorentz-like transformation except for terms of order c^{-6} in the time coordinate transformation. However the corresponding equations of motion are invariant under a Lorentz-like transformation at this order because this result does not depend on those terms.

A physical interpretation of $\vec{V}$ is given at this order by the following result $\vec{x} = -\vec{V}t + 0(c^{-6})$ (when $\vec{\xi} = 0$), $\vec{\xi} = \vec{V}\tau + 0(c^{-6})$ (when $\vec{x} = 0$).

Since the coefficients $\overset{(6)}{g}_{00}$, $\overset{(5)}{g}_{0i}$ and $\overset{(4)}{g}_{ij}$ are respectively a scalar, a vector and a tensor under rotations, it can be stated in general that the equations of motion in harmonic coordinates up to c^{-4} are invariant under any Poincaré transformations.

III F. **Extension to Higher order**

When the harmonic gauge up to second PNA is used the G dependent term of order c^{-6} of the coordinate transformation, in eq. (34 *b*) makes it impossible to obtain Poincaré invariant equations of motion at higher orders. The situation is similar to what happened in the first PNA using gauge (L).

However this G dependent term can be eliminated keeping the condition (18 *a*) but using, instead of condition (18 *b*), the following one:

$$\overset{5}{W} = -2\Phi_{,t} + \frac{1}{2}\chi_{,ttt} + 6\mathcal{U}\mathcal{U}_{,t} \tag{35}$$

In this modified harmonic gauge the metric is Poincaré invariant up to second PNA and the equations of motion are the same that those obtained in the harmonic gauge; therefore they can be evaluated from the metric tensor expressions given in Subsection II B.

This gauge can be used in the second PNA if we want to obtain a metric tensor which is Poincaré invariant up to order higher than second PNA.

The gauge condition (35) can be obtained by assuming that the coordinate transformation must be Poincaré-like, then from equation (19) or (23) the gauge condition is reached.

IV. CONCLUSIONS

In Section II, following Chandrasekhar's post-Newtonian scheme [*16*], the equations giving the metric tensor, up to the second PNA for a perfect fluid were found and the metric tensor for an isolated system of point masses was also given.

In the Chandrasekhar gauge the metric tensor was a Minkowskian behaviour at great distance from the bodies, and in the harmonic coordinates the metric tensor diverges at great distance. This divergence is due to the gauge and does not posses a physical meaning because the post-Newtonian scheme is only valid in the « near zone ».

In Section III, the second post-Galilean transformations leaving the second PNA metric tensor invariant were evaluated using the Chandrasekhar gauge and the harmonic gauge.

In the Chandrasekhar gauge, the transformation could be splitted in a Poincaré-like term plus a G dependent term, but the parameter $\vec{V}$ appearing in the Poincaré-like part could not be interpreted as the velocity parameter of a pure Lorentz transformation, and therefore nor the metric tensor at the second PNA, neither the equations of motion at the same order are Poincaré invariant.

In the harmonic gauge, the parameter $\vec{V}$ can be interpreted as the velocity parameter of a pure Lorentz transformation, and as the G dependent terms in this gauge are of order c^{-6}, appearing only in the time coordinate transformation it can easily be seen that the equations of motion are Poincaré invariant. The metric tensor, of course, is not Poincaré invariant.

If a modified harmonic gauge is used, no G dependent terms will appear in the second post-Galilean transformation, and as the $\vec{V}$ parameter can be interpreted as a velocity parameter, both the metric tensor and the equations of motion are Poincaré invariant at the second PNA.

Therefore, it can be concluded that the typical arbitrariness of the gauge in General Relativity permits the construction of equations of motion for a system of point masses up to order c^{-4} that are invariant under Poincaré transformations. The accelerations are therefore approximate solutions of PRM for the gravitational interaction. (The explicit form of these accelerations are rather cumbersome).

Within the framework of PRM the study of conserved quantities and the definition of a Hamiltonian form for a point mass gravitating system can, in principle, be made according to Bel and Martin [*29*].

It is usually admitted that effects due to the gravitational radiation appear at the order c^{-5} of the equations of motion. To include this effect in the slow motion approximation of General Relativity the post-Newtonian scheme must be altered by including the Sommerfeld radiation condition in the « far zone » [*30*].

V. ACKNOWLEDGMENTS

I am grateful to Dr. L. Mas and Dr. X. Fustero for many helpful discussions.

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(Manuscrit reçu le 3 novembre 1977)