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Maximizing Properties of Extremal Surfaces in General Relativity (*)

by

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ABSTRACT. — In earlier work we have discussed the uniqueness and local maximizing properties of maximal surfaces. We continue this study in the present paper, including surfaces of constant mean curvature in spaces of non-vanishing matter content and with arbitrary cosmological constant. The nature of the extremum is characterized by means of the eigenvalues of an elliptic differential operator defined on the surface. To illustrate the different possibilities, a universe of the Taub type with cosmological constant is constructed, and this example suggests a conjecture that the index of these surfaces is less than 2.

0. INTRODUCTION

Spacelike surfaces of prescribed mean curvature, especially maximal surfaces, have important physical as well as mathematical properties in spacetime. They can be helpful in solving the Einstein equations and under certain conditions they indicate the presence of singularities [*1*].

In a previous paper [*2*] we have discussed the uniqueness of surfaces of constant mean curvature and the local maximizing properties of maximal

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surfaces. In the present work we shall generalize the earlier investigations in two respects: (*a*) We study the second derivative or variation of a weighted volume functional, whose extremals are surfaces of constant mean curvature (k-surfaces for short). (*b*) We consider spacetimes with non-vanishing matter content.

The k-surfaces extremize a function W(T), closely related to the volume functional V(T) (extremized by maximal surfaces T). The weighted volume W(T) of a spacelike surface T with respect to the fixed compact spacelike surface S is defined to be

$$\mathrm{W(T)} = \mathrm{V(T)} + k\mathrm{H(S, T)} \tag{0.1}$$

with H(S, T) the 4-dimensional measure of the set between S and T. Since we are only interested in local properties of W we can restrict our attention to those T contained in a fixed (compact) tubular neighbourhood of S. The constant k can be considered a Lagrange multiplier. For T a variation of S it is easy to see [*3*] [*4*] that the variational principle

$$\frac{d}{d\mathrm{T}}(\mathrm{V(T)} + k\mathrm{H(S, T)})_{\mathrm{T=S}} = 0 \tag{0.2}$$

yields the Euler-Lagrange equation:

$$\text{mean curvature (S)} = k\,.$$

Thus the variational principle characterizes k-surfaces.

The second variation of the weighted volume gives rise to a differential operator $\underset{\sim}{\mathrm{N}}$, necessarily elliptic, the eigenvalues of which are related to the character of the extremum at a k-surface. If Einstein's equations hold, it is natural to compare these eigenvalues to the cosmological constant. We therefore consider spacetimes with both matter and cosmological constant.

In Section 2 we discuss differential operators on spacelike surfaces, and show some general properties of the operator $\underset{\sim}{\mathrm{N}}$. In Section 3 we apply these results to see the extent that k-surfaces imply incompleteness. Of particular interest is the extension to spaces with matter content of the situation already treated in [*2*] for empty spacetime where it is possible to conclude incompleteness both to the future and to the past of a maximal surface. The final section contains an example, the Taub universe with cosmological constant, where all quantities can be computed explicitly to illustrate the general theorems derived earlier. This section also contains a conjecture based on a surprising feature of the Taub universe.

Detailed discussions of the weighted volume functional presented in this introduction can be found in slightly different settings in the Oxford thesis of A. J. Goddard or in the article of M. Miranda cited in the references [*3*] [*4*].

1. DIFFERENTIAL OPERATORS ON SPACELIKE HYPERSURFACES

Let S be a compact spacelike hypersurface in a Lorentz manifold M (signature $-, +, +, +$). The inner product $\langle\ ,\ \rangle$ induced on the bundle of normal vectors to S from the bundle of tangent vectors of M is then negative definite. Recall [5] that if A^X represents the second fundamental form on S in direction X, normal to S, the operator A^2 is defined by

$$\langle A^2(X), X \rangle := \text{trace}\,(A^X)^2 = \Sigma \langle A^X(e_i), A^X(e_i) \rangle = \Sigma K_{ij}K^{ij} \langle X, X \rangle \langle n, n \rangle \qquad (1.1)$$

for (e_i) a local frame field on S, and n the unit normal.

The Laplace operator in the bundle of normal vectors to S is defined with the aid of the covariant derivative D:

$$\nabla^2 X := \text{trace}\,((u, v) \mapsto D_u(D_v X) - D_{D_u v} X)$$

with u, v tangent to S.

Using Stokes' theorem and assuming that ∂S is empty or that X vanishes on the boundary of S we find

$$\int_S \langle \nabla^2 X, X \rangle dV = -\Sigma \int_S \langle D_{e_i}X, D_{e_i}X \rangle dV \qquad (1.2)$$

where the (e_i) are again a local frame field. If we define the first order operator ∇X by

$$\langle \nabla X, \nabla X \rangle := \Sigma \langle D_{e_i}X, D_{e_i}X \rangle$$

we can write (1.2) as

$$\int_S \langle \nabla^2 X, X \rangle dV = -\int_S \langle \nabla X, \nabla X \rangle dV\,. \qquad (1.3)$$

In previous work [2] we pointed out the properties of the differential operator

$$L(X) = \nabla^2 X + A^2(X)\,.$$

Here we consider the generalization appropriate to the case where matter is present. If the energy density is interpreted as a type (1, 1) tensor on the normal bundle

$$\langle T(X), X \rangle = T(X, X)$$

we can define

$$\underset{\sim}{N}(X) := L(X) + 8\pi\left(T(X) - \frac{1}{2}(\text{trace } T)X\right). \qquad (1.4\,a)$$

Explicitly

$$\langle \underset{\sim}{N}(X), X \rangle = \langle \nabla^2 X, X \rangle + \text{trace}\,(A^X)^2 + 8\pi\left(T(X, X) - \frac{1}{2}\text{trace } T \langle X, X \rangle\right). \qquad (1.4\,b)$$

The differential operator $\underset{\sim}{N}$ is a strongly elliptic operator on a compact manifold, that is, its characteristic polynomial [6] is a positive definite quadratic form, hence N can be diagonalized with eigenvalues

$$(\nu_k) : \nu_1 < \nu_2 < \ldots, \nu_k \to \infty .$$

In case S is a compact manifold with boundary ∂S, we confine attention to normal vector fields which vanish on ∂S. More precisely:

THEOREM 1. — The symmetric differential operator $\underset{\sim}{N}$ on the space of normal vector fields to S, vanishing on the boundary of S, can be diagonalized with respect to the inner product $\int_S \langle X, X \rangle dV$; has distinct eigenvalues (ν_k); and the dimension of each eigenspace is finite.

The operator $\underset{\sim}{N}$ differs from the Morse index form of Riemannian geometry in that the curvature terms are replaced by the more physically meaningful energy-momentum tensor. Moreover, if Einstein's equations with vanishing cosmological constant hold, $\underset{\sim}{N}$ is identical with the index form [7]. The eigenvalues of $\underset{\sim}{N}$ allow us to formulate a condition that a k-surface represents a true maximum for the weighted volume, as opposed to other types of extremal point; thus in the sequel S will always denote a k-surface.

THEOREM 2. — Suppose that M satisfies the Einstein equations with cosmological constant λ coupled to a source satisfying the strong energy condition. If S is a k-surface and λ is smaller than the least eigenvalue ν_1 of $\underset{\sim}{N}$ then S locally maximizes the weighted volume.

Proof. — From the strong energy condition it follows that

$$\int_S \langle \underset{\sim}{N}(X), X \rangle dV = \int_S \Big(\langle \nabla^2 X, X \rangle + \langle A^2(X), X \rangle + 8\pi\Big(T(X, X) - \frac{1}{2}\langle X, X \rangle \operatorname{trace} T \Big)\Big) dV$$

is non-negative. Thus $\underset{\sim}{N}$ is positive semi-definite, and hence the lowest eigenvalue ν_1 of the equation

$$\underset{\sim}{N}(Y) + \nu_1 Y = 0$$

is non-negative. Furthermore, for all non-zero X,

$$-\int_S \langle \underset{\sim}{N}(X), X \rangle dV \leq \nu_1 \int_S \langle X, X \rangle dV .$$

The usual second variation formula [2] [4] holds unchanged for the weighted volume W if S is a k-surface:

$$\ddot{W}_X(0) = -\int_S (\langle \nabla^2 X, X \rangle + \langle A^2(X), X \rangle + \operatorname{Ricci}(X, X)) dV . \quad (1.5)$$

Into this expression we substitute the Einstein equations

$$\text{Ricci}\,(X, X) - \lambda \langle X, X \rangle = 8\pi\left(T(X, X) - \frac{1}{2}\langle X, X \rangle \operatorname{trace} T\right) \quad (1.6)$$

and find

$$\ddot{W}_X(0) = -\int_S (\langle \underset{\sim}{N}(X), X \rangle + \lambda \langle X, X \rangle) dV \quad (1.7\,a)$$

$$\leq (\nu_1 - \lambda) \int_S \langle X, X \rangle dV \leq 0 . \quad (1.7\,b)$$

This proof was based on the positivity of $\underset{\sim}{N}$. To characterize these and more general cases we study the kernel of $\underset{\sim}{N}$, first for manifolds with boundary, and then for closed S.

PROPOSITION 3. — If S is a k-surface with boundary and T(X, X) satisfies the strong energy condition then the kernel of $\underset{\sim}{N}$ is trivial.

Proof. — Suppose that N(X) = 0 and X vanishes on the boundary of S. Then

$$0 = \int_S \langle \underset{\sim}{N}(X), X \rangle dV = \int_S \Big(- \langle \nabla X, \nabla X \rangle + \langle A^2(X), X \rangle + 8\pi\Big(T(X, X) - \frac{1}{2}\operatorname{trace} T \langle X, X \rangle\Big)\Big) dV$$

which is a sum of three non-negative terms, so that ∇X must vanish. Thus, X has constant length on S, hence X vanishes on S.

Remark. — Theorem 2 and proposition 3 combine to show that no conjugate boundaries can exist on a maximal or k-surface with boundary, when $\lambda < \nu_1$ and strong energy holds. See Simons [*8*] for a thorough explanation of conjugate boundaries. The modifications necessary for maximal and k-surfaces are easily interpolated.

PROPOSITION 4. — If T(X) satisfies the strong energy condition and S is a closed manifold ($\partial S = \emptyset$) then kernel N is non-trivial iff S is totally geodesic (time symmetric) and $T(X) = \frac{1}{2}(\operatorname{trace} T)X$.

Proof. — If the conditions on S and T are satisfied then $\underset{\sim}{N}(X) = \nabla^2 X$, and kernel $\underset{\sim}{N}$ consists of all real multiples of a fixed unit normal to S. Conservely, suppose that X is a nowhere zero vector field in kernel $\underset{\sim}{N}$, then we have

$$\int_S \langle \underset{\sim}{N}(X), X \rangle dV = 0$$

and from equation (1) of proposition 3

$$\langle A^2(X), X \rangle = \operatorname{trace} (A^X)^2 = 0$$

$$T(X, X) - \frac{1}{2}\operatorname{trace} T \langle X, X \rangle = 0$$

so S is time symmetric as well as $T(X) = \frac{1}{2}(\operatorname{trace} T)X$.

Remark. — Negative pressure ($p = -(1/3)\rho$ in the isotropic case) is necessary to satisfy this last condition on the energy tensor, however none of the familiar energy conditions need be violated.

Recall that the Morse index form of a variational problem is the quadratic form arising from the second variation (second derivative test). In our case the form is $\ddot{W}_X(0)$ of Equation (1.7 *a*). The index (or extended index) of S is defined to be the dimension of a maximal subspace on which $\ddot{W}_X(0)$ is positive definite (or positive semi-definite).

PROPOSITION 5. — Suppose that M satisfies the Einstein equations (1.6) and S is a maximal or k-surface. If $\lambda = \nu_k$ then the extended index of S is at least k, and if $\nu_k < \lambda < \nu_{k+1}$ the index of S is at least k.

Proof. — Let X be a normalized eigenfunction of $\underset{\sim}{N}$, then $N(X) + \nu X = 0$, $\int \langle X, X \rangle dV = -1$, and

$$\ddot{W}_X(0) = -\int_S \langle \underset{\sim}{N}(X, X) \rangle dV - \lambda \int_S \langle X, X \rangle dV$$
$$= \lambda - \nu .$$

Hence if $\lambda = \nu_k$ then $\ddot{W}_X(0)$ is positive semi-definite for $X_1, \ldots, X_k$. Similarly if $\lambda > \nu_k$, $\ddot{W}_X(0)$ is positive for $X_1, \ldots, X_k$. Roughly the index tells us in how many directions the weighted volume functional is increasing.

2. APPLICATIONS: INDEX AND INCOMPLETENESS

In this section we apply the theorems and techniques of the first section together with results from [*2*] to identify the nature of the extremum of the weighted volume functional. An extremum is called stable if the weighted volume functional is a local maximum, more precisely, if the second variation is negative semi-definite. It will be seen here that under certain conditions the extremum will be unstable. Later in the section we prove a singularity theorem using the Raychaudhuri equation.

First we summarize Section 1 in the following:

THEOREM 6. — Suppose that S is a compact spacelike hypersurface of constant mean curvature in a spacetime satisfying the Einstein equations with cosmological constant λ coupled to a strong energy source. Then either S is a strong local maximum for the volume or weighted volume functional, or S has flat Cauchy development, or the extended index of S is at least one.

Proof. — This theorem is proved using theorem 2, propositions 4 and 5, and theorem 4.3 of [2]. To see the various possibilities we outline the theorem with a table:

S compact k-surface, λ cosmological constant, (ν_j) eigenvalues of N			
Compare λ to eigenvalues of N	Examine $\mathcal{R}$ and T	Quantity	Value on S
$\lambda < \nu_1$	no effect	weighted volume	strong local maximum
$\lambda = \nu_1 = 0$	one or the other not identically zero	volume	strong local maximum
	both identically zero	Cauchy development	flat
$\lambda = \nu_j \neq 0$	no effect	extended index	at least j
$\nu_j < \lambda < \nu_{j+1}$	no effect	index	at least j

THEOREM 7. — Suppose that M satisfies the above conditions on the Einstein equations with strong energy condition and with λ strictly less than the smallest positive eigenvalue ν_+ of $\underset{\sim}{N}$. Then if M contains a closed k-surface with $k = -\sqrt{3\lambda}$ $(k = \sqrt{3\lambda})$ the future (past) of S contains an incomplete timelike geodesic ray.

Proof. — From the « Raychaudhuri » equation, A.6 in [2] [14] for an eigenfunction X with eigenvalue ν_+, we have

$$\langle X, D_X H \rangle \leq (\nu_+ - \lambda) \langle X, X \rangle < 0$$

hence S can be deformed slightly so that the convergence θ is strictly smaller than $-\sqrt{3\lambda}$. Now using the standard Raychaudhuri equation along a geodesic orthogonal to S we obtain

$$\frac{d\theta}{ds} + \frac{\theta^2}{3} - \lambda \leq 0 \tag{2.1}$$

and if $\theta(s_0) < \sqrt{3\lambda}$ we can analyze the Raychaudhuri equation in the same manner as Hawking and Ellis [9]. Namely choose $t > s_0$ so that $\sqrt{3\lambda}$ coth $\sqrt{\frac{\lambda}{3}}(s_0 - t) > \theta(s_0)$. Then (2.1) implies $\theta(s) < \sqrt{3\lambda}$ coth $\sqrt{\frac{\lambda}{3}}(s - t)$ for all s and hence $\theta(s) \to -\infty$ as $s \to t$. Without repeating the argument we now follow Theorem 4 of Chapter 8 in [9] to obtain geodesic incompleteness to the future of S.

If the cosmological constant is non-positive it is necessarily less than ν_+. We then can conclude:

THEOREM 8. — Suppose that M satisfies the above conditions on the Einstein equations with $\lambda \leq 0$. If M contains a closed maximal surface then M is geodesically incomplete in the past of S as well as the future of S.

Proof. — The case of empty spacetime, $\lambda = 0$, requires analysis of the fourth variation of the volume, see [*2*]. If either $\lambda < 0$ or matter is present, the theorem follows from an analysis as above of Equation (2.1). For this case, as well as for the other properties of spacetimes with singularities in past and future, see F. Tipler [*10*].

Remarks. — 1. The idea of a focal point is weaker than that of a conjugate point in the sense that a space may have no conjugate points (Minkowski flat space) but can contain submanifolds with focal points (the spacelike hyperboloids in Minkowski space).

2. The example of the de Sitter space shows that the bounds in Theorem 7 are the best possible, because for the equator we have $\nu_1 = 0$ and $\lambda = \nu_2 = \nu_+$.

3. In the next section we show that Theorem 8 is false when the cosmological constant is allowed to be positive. Thus Theorem 4.4 of [*2*] is false when $\lambda > 0$.

3. EXAMPLE: THE TAUB UNIVERSE

The simplest cosmological solutions have too much symmetry to allow significant application of the theorems derived above. For example, in Friedmann-Robertson-Walker universes the global spacelike k-surfaces are totally umbillic ($K_{ij} = (k/3)g_{ij}$). By contrast, spacelike homogeneous but non-isotropic universes can illustrate the general case.

We consider a spacetime of the type first introduced by Taub [*11*] [*12*] and write the spacelike homogeneous metric

$$ds^2 = \sum_{\mu} (\omega^\mu)^2$$

in terms of the orthonormal 1-forms

$$\omega^0 = dt\,, \quad \omega^1 = A(t)\sigma^1\,, \quad \omega^2 = B(t)\sigma^2\,, \quad \omega^3 = B(t)\sigma^3$$

where σ^i are the orthonormal pfaffians in the bi-invariant metric on the unit 3-sphere which satisfy the Maurer-Cartan equations [*13*]:

$$d\sigma^3 = \sigma^1 \wedge \sigma^2 \qquad \text{and cyclically}. \tag{3.1}$$

The independent Einstein equations for empty spacetime with cosmological constant follow immediately from equation (19) of [*12*],

$$2\lambda = R^i_i - R^0_0 = 2\left(\frac{\dot{B}}{B}\right)^2 + 4\frac{\dot{A}\dot{B}}{AB} + \frac{2}{B^2} - \frac{A^2}{2B^4} \tag{3.2 a}$$

$$0 = R^0_0 - R^z_z = 2\left(\frac{\dot{B}}{B}\right)^{\cdot} + 2\left(\frac{\dot{B}}{B}\right)^2 - 2\frac{\dot{A}\dot{B}}{AB} - \frac{A^2}{2B^4} \tag{3.2 b}$$

$$2\lambda = R^z_z + R^x_x = \left(\frac{\dot{A}}{A}\right)^{\cdot} + \left(\frac{\dot{A}}{A}\right)^2 + \left(\frac{\dot{B}}{B}\right)^2 + 3\frac{\dot{A}\dot{B}}{AB} + \frac{1}{B^2}. \tag{3.2 c}$$

Due to spacelike homogeneity the surfaces S given by setting t = constant are maximal or k-surfaces. If $\lambda \leq 0$ then by the uniqueness theorem 3.3 of [*2*], these are the only k-surfaces of the Taub universe. The second fundamental form of any such surface has the components, in the orthonormal coframe field ω^μ,

$$K_{ij} = \text{diag}\,(\dot{A}/A, \dot{B}/B, \dot{B}/B).$$

The two invariants that are of interest here are

$$k = K^i_i = (\dot{A}/A) + 2(\dot{B}/B)$$

$$K_{ij}K^{ij} = (\dot{A}/A)^2 + 2(\dot{B}/B)^2.$$

On some initial surface S these quantities cannot be chosen independently of the 3-geometry of S and the cosmological constant λ, due to the « Hamiltonian » constraint (3.2 *a*),

$$K_{ij}K^{ij} - k^2 = (2/B^2) - (A^2/2B^4) - 2\lambda. \tag{3.2 d}$$

For the hypersurface S in this empty universe (T(X) = 0), the eigenvalues of $\underset{\sim}{N} = \nabla^2 - K_{ij}K^{ij}$ acting on the (one-dimensional) normal bundle are the same as the eigenvalues of this operator acting on scalars in S, with ∇^2 the ordinary 3-dimensional Laplace operator on S. The first few eigenvalues of ∇^2 are

$$l_1 = 0, \quad l_2 = (1/4A^2 + 1/2B^2),$$
$$l_3 = 2/B^2, \quad l'_3 = (1/A^2) + (1/B^2). \tag{3.3}$$

The lowest non-zero eigenvalue is the lower of l_2 or l'_3. To show that no other eigenvalue is lower, let l_x be the lowest eigenvalue other than those of (3.3). Then l_x can be estimated from the Ritz principle,

$$l_x = \inf\left(\int(\nabla\varphi)^2 dV \Big/ \int \varphi^2 dV\right).$$

Here ∇ is the gradent on S, φ is a scalar function on S, and the infimum is taken over all functions φ orthogonal to the eigenfunctions corresponding to (3.3). Let s_i be the dual vector basis to the 1-forms σ^i, viewed as differential operators on functions. Then we have

$$\begin{aligned}(\nabla\varphi)^2 &= (s_1\varphi)^2/A^2 + [(s_2\varphi)^2 + (s_3\varphi)^2]/B^2 \\ &\geq [(s_1\varphi)^2 + (s_2\varphi)^2 + (s_3\varphi)^2]/C^2 \\ &= (\nabla_0\varphi)^2/C^2\end{aligned}$$

where $C^2 = \max(A^2, B^2)$ and the subscript 0 denotes operators on the round sphere with $A = B = 1$ (which, because of Equation (3.1), has radius 2). Since the lowest eigenvalue of ∇_0^2 orthogonal to those of (3.3) is at least 2, we have

$$l_x \geq 2/C^2 \geq \min(l_2, l'_3).$$

Hence the lowest eigenvalues of $\underset{\sim}{N}$ are

$$\nu_1 = K_{ij}K^{ij}, \quad \nu_2 = \begin{cases} K_{ij}K^{ij} + (1/4\,A^2) + (1/2\,B^2) & A^2 \geq B^2/6 \\ K_{ij}K^{ij} + 2/B^2 & A^2 < B^2/6 \end{cases}$$

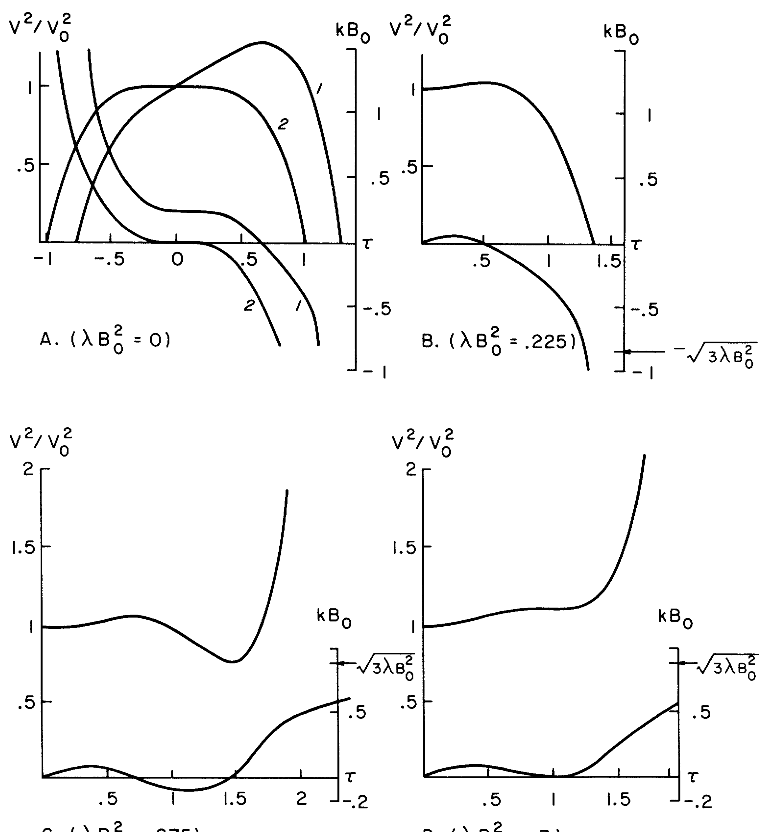

FIG. 1.

We note that the constraint (3.2 *d*) implies

$$\nu_2 - \lambda = \begin{cases} (3K_{ij}K^{ij} - k^2)/2 + (B^2 - A^2)^2/4\,A^2B^2 & A^2 \geq B^2/6 \\ (3K_{ij}K^{ij} - k^2)/2 + 1/B^2 + A^2/4\,B^4 & A^2 < B^2/6 \end{cases}$$

The first term is $\sum_{i \neq j} (k_i - k_j)^2/4$, where k_i are the eigenvalues of K_{ij}. Thus we have $\lambda \leq \nu_2$ for all S, and $\lambda = \nu_2$ only if S is totally umbillic ($K_{ij} = (k/3)g_{ij}$) and a round sphere ($A^2 = B^2$).

To illustrate our theorems about the behavior of the volume at a k-surface we distinguish several cases, depending on the value of λ:

(1) $\lambda < 0$. Since then $\lambda < \nu_1$, $\ddot{W}_X$ and hence $\ddot{V}_X$ is always negative and bounded away from zero (for $X \neq 0$). Since k is essentially the logarithmic derivative of V, it is easily seen that k-surfaces exist for all values of k. One of them ($k = 0$) is maximal; hence by Theorem 8 the spacetime is incomplete in the past and in the future (singularity symmetric in the sense of Tipler [*10*]).

FIG. 1. — Time development of total volume V and mean curvature k of t = constant surfaces in the Taub universe. The curves show the square of V (normalized to unity at $\tau = 0$) as functions of the time parameter τ of Equation (3.6). Except for curves 1 of figure A, where $C_0 = 2$, the constant C_0 of Equation (3.7) is set to zero, making $\tau = 0$ a moment of time symmetry, V an even function of τ, and k an odd function of τ. Only $\tau > 0$ is shown in figures B-D. The cosmological constant λB_0^2 has the value 0 in figure A, .225 in figure B, .275 in figure C, and .3 in figure D. For $\lambda B_0^2 < .25$ there is incompleteness and collapse in future and past, and k ranges in $(-\infty, \infty)$. If λB_0^2 is close to .25 (Fig. B), k stays below $\sqrt{3\lambda}$ for most of the universe's life, during which time Theorem 7 cannot predict the incompleteness. For $\lambda B_0^2 > .25$ spacetime is complete and k ranges in $(-\sqrt{3\lambda}, \sqrt{3\lambda})$; or, if C differs sufficiently from 0 (not shown) it is incomplete in only one time direction. In the direction of infinite expansion the model approaches the de Sitter universe. For $\lambda B_0^2 = .75$ it is exactly the de Sitter universe. The volume extrema correspond to maximal surfaces and illustrate the different cases discussed in the text:

Case illustrated	Figure	*Value at maximal surface* τ	$\nu_1 B_0^2$	λB_0^2
2 (generic)	A *1*	.66	.8	0
2 (time symmetric)	A *2*	0	0	0
3 (local maximum)	B	.5	.5	.225
	C	.68	.48	.275
4 (one neutral direction)	D	1	.3	.3
5 (one expanding direction)	C	1.46	.08	.275

In addition, the maximal surfaces at $\tau = 0$ in figures B-D have $\nu_1 = 0$ and $\lambda > 0$, illustrating case 5.

(2) $\lambda = 0$, the standard Taub universe. Generically, we have $\nu_1 > 0$ on all $t =$ constant surfaces, so the same conclusions hold as in case (1). If $\nu_1 = 0$, Equation (3.4) implies that S is a surface of time-symmetry, but then Theorem 8 shows that the universe must still be singularity-symmetric.

(3) $0 < \lambda < \nu_1$. Here $\ddot{W}_X$ is still negative in a neighbourhood of S, but not necessarily for all other S, since ν_1 can vary in time. If S is maximal, it is a true volume maximum (Theorem 2), but the spacetime may or may not be incomplete. (The complete case as in figure C is a counterexample to a part of Theorem 4.4 in [2]). If it is complete the k-surfaces are limited to the range $-\sqrt{3\lambda} < k < \sqrt{3\lambda}$, and there are at least two other surfaces of type (5) below.

(4) $0 < \nu_1 = \lambda$. $\ddot{W}_X$ vanishes for the parallel normal deformation (X = unit normal), and is negative for all orthogonal deformations (Proposition 5).

(5) $0 < \nu_1 < \lambda < \nu_2$. $\ddot{W}_X$ is positive for the parallel deformation, and negative for all orthogonal deformations (Proposition 5).

(6) $\lambda = \nu_2$. From Equation (3.5) we find that S is a round totally umbilic 3-sphere. Then the initial data are isotropic as well as homogeneous, hence lead to the unique solution of the de Sitter universe. For the deformations described by the eigenfunctions of ν_2, $\ddot{W}_X$ vanishes; for the parallel deformation $\ddot{W}_X$ is negative (eigenfunction corresponding to ν_1); and $\ddot{W}_X$ is positive for all deformations orthogonal to these two. For a maximal, hence time-symmetric, surface the parallel deformation corresponds to the expansion of the de Sitter universe away from the time-symmetric moment, and the deformations in the neutral directions correspond to the rotations which transform the surfaces of time symmetry (equators) in the de Sitter universe into each other.

We note that, according to Equation (3.5), $\lambda > \nu_2$ is not possible in the Taub universe. That is, there can be at most one independent deformation of a maximal (or k −) surface which corresponds to a (weighted) volume increase. This suggests that even when $\lambda \neq 0$, all gravitational wave and matter degrees of freedom in a closed universe still contribute to contraction. We formulate this idea as a

CONJECTURE. — If a closed universe with positive cosmological constant contains a k-surface S, then either $\lambda < \nu_2$ at S, that is the index of the weighted volume extremum is at most one; or the universe is the de Sitter universe.

The general exact solution of Equations (3.2 b, c) (except for time-translation) can be most conveniently expressed in terms of a new time variable τ, and the values of A_0 and B_0 of A and B at $\tau = 0$, where

$$d\tau = (A/2B_0^2)dt. \tag{3.6}$$

The solution is

$$B^2/B_0^2 = 1 + \tau^2$$
$$A^2B^2/B_0^4 = (8\lambda B_0^2 - 4)\tau^2 + (4\lambda B_0^2/3)\tau^4 + C_0\tau + (A_0^2/B_0^2) \qquad (3.7)$$

where the constants A_0, B_0, C_0 have to satisfy the constraint (3.2 *a*) which takes the simple form at $\tau = 0$

$$\lambda = (1/B_0^2) - (A_0^2/4B_0^2).$$

The total 3-volume of the spacelike surface is $V = (4\pi)^2AB^2$. If we normalize V to unity at $\tau = 0$, the set of solutions can be parametrized by λ and C_0. In figure 1 we show plots of V^2 and k vs τ for several values of λ and C_0, which illustrate the cases discussed above.

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[5] The operators which we define on the normal bundle would correspond to operators acting on scalars in the usual [1] « 3 + 1 decomposition ». See appendix of [2] for more detail. Among the advantages of using the normal bundle are that e. g. the mean curvature vector is independent of the choice of normal direction, and that the approach can more easily be generalized to hypersurfaces of higher codimension.

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