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Dynamics of relative motion of test particles in general relativity (1)

by

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ABSTRACT. — This paper presents several variational principles which lead to the first and the second geodesic deviation equations, recently formulated by the author and used for the description of the relative motion of test particles in general relativity. Relations between these principles have been investigated and exhibited here. The paper contains also a study of the Hamilton-Jacobi equation for these generalized deviations and a discussion of the conservations laws appearing here.

RÉSUMÉ. — L'article présente plusieurs principes de variation menant aux premières et deuxièmes équations de la déviation des géodésiques. Ces équations ont été récemment formulées par l'auteur et utilisées pour la description du mouvement des particules d'essai en relativité générale. Les relations entre ces principes sont étudiées et démontrées.

L'article contient également une étude de l'équation de Hamilton-Jacobi pour ces déviations généralisées, ainsi qu'une discussion des lois de conservation qu'on obtient dans le cas présenté.

INTRODUCTION

As it has been shown in a previous article [*1*] the relative motion of test bodies in general relativity can be described by means of an infinite sequence of suitably defined general geodesic deviation vectors forming fields along

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a chosen geodesic Γ. Truncating this sequence gives one an approximate description with a desired order of accuracy.

This paper, in turn, is devoted to a study of variational principles which lead to the equations describing the evolution of the generalized deviation vectors along Γ.

There are two types of such variational principles. Existence of the first type, called the « accessoric » variational principles, is a consequence of a theorem in variational calculus, ascribed to Carathéodory [*2*]. In the case of the geodesic deviation in general relativity this approach has been also used by Plebanski [*3*]. According to the accessoric point of view one formulates the variational principle for a deviation vector field along a given Γ (and for fixed values of deviation of the lower order—but this second fact will not, in this introductory comment, always be explicitly mentioned) and one considers this principle as being independent from the geodesic variational principle for Γ (and for the lower deviations as well). Thus, in this approach one has two independent dynamics:

i) of the motion of a test body in a gravitational field;

ii) of the relative motion of two of such bodies in the same field. The form of the second action functional, although postulated, is suggested by the second variation of the first action.

The second type of variational principles is based on a theorem about generalized Jacobi fields in variational calculus [*5*]. It permits one to formulate action principles which unify both dynamics (or n dynamics when generalized deviations up to the n-th order are discussed) in a single variational principle based on a single action functional. This action, when varied, leads to both the geodesic—and the geodesic deviation equations simultaneously.

Some of the main properties of the Hamiltonian formalisms corresponding to the discussed action functionals are briefly reviewed. In the general case they constitute singular cannonical formalisms with constraints of the Dirac type [*6*]. Therefore, the Hamilton-Jacobi equations which appear here form, in general, a set of partial differential equations in involution. In this connection, it is interesting to observe that at least one of the equations in such a set is usually linear in the first partial derivatives of the Hamilton-Jacobi principal function. It thus has a form which one would expect for the eikonal equation derived from field equations of the type of the Dirac relativistic wave equation for the electron. The process of integration of the Hamilton-Jacobi equations for the first and the second geodesic deviation vectors can always be reduced to the integration of the geodesic Hamilton-Jacobi equations and of these linear equations.

This paper contains also a discussion of conservation laws which follow, due to the Noether theorem (cf. [*4*]), from all the variational principles considered. In the case of so-called dynamic variational principles [*9*], this discussion enables one to interpret the constraints which select the natural deviations (cf. [1]) as describing the relative energy of falling particles. This energy is necessarily equal to zero. Thus, the constraints have gained a dynamical interpretation: two observers, freely falling in a given gravitational field, when using to determine their proper times two respectively comoving ideal clocks, must find that their relative total energy is equal to zero.

The study has been completed here in some detail, because of the computational reasons, only for the first and the second geodesic deviation. There are, however, no obstacles in principle to continue it up to the geodesic deviations of any higher order.

The previous paper [1] is referred to as I; similarly, references like, for instance, (I.2.7), etc., apply to corresponding formulae in [1].

1. GEODESIC PRINCIPLE

As is known, the geodesic equations in an arbitrary parametrization (I.1.2), can be derived from a variational principle characterized by the action

$$W = \int_{\tau_0}^{\tau_1} \sqrt{g_{\alpha\beta}u^\alpha u^\beta}\, d\tau, \tag{1.1}$$

where τ is an arbitrary parameter, $u^\alpha = \dfrac{d\xi^\alpha}{d\tau}$ and is, together with $g_{\alpha\beta}$, evaluated along a curve from a one-parametric family of curves

$$x^\alpha = \xi^\alpha(\tau, \varepsilon)$$

such that all curves of it intersect each other in two points corresponding to the values τ_0 and τ_1 of the parameter. The procedure leading from the above variational principle to its Euler-Lagrange equations is described in every standard text-book. Here we shall outline a somewhat simpler procedure, equivalent to the standard one, based on the concept of covariant variation of geometric objects, used by Plebanski [3]. This second approach would not be worth mentioning, if one were considering the formalism with first variations only. Later we shall, however, be also discussing the variations of higher order for which the covariant approach is considerably simpler.

Along a chosen curve of the family, *e. g.* along the curve $x^\alpha = \xi^\alpha(\tau, 0)$, one defines in the standard way the variations

$$\delta x^\alpha : = \varepsilon\left(\frac{\partial\xi^\alpha}{\partial\varepsilon}\right)_{\varepsilon=0} \tag{1.2}$$

which are components of a vector tangent to the curve: $x^\alpha = \xi^\alpha(\tau, \varepsilon)$, $\tau = \text{const}$, at $\varepsilon = 0$; ε in (1.2) is arbitrary. To evaluate the variation of geometric objects it is convenient to complete (1.2) by the definition of covariant variation which *e. g.* for a tensor field $t^{\alpha_1\ldots\alpha_n}$ is given by

$$\Delta t^{\alpha_1\ldots\alpha_n} : = \varepsilon\left(\frac{Dt^{\alpha_1\ldots\alpha_n}}{\partial\varepsilon}\right)_{\varepsilon=0}, \tag{1.3}$$

i. e. is proportional to the absolute derivative of $t^{\alpha_1\ldots\alpha_n}$ evaluated along the curve: $x^\alpha = \xi^\alpha(\tau, \varepsilon)$, $\tau = \text{const}$; at $\varepsilon = 0$.

Thus, if calculating

$$\delta W : = \varepsilon\left(\frac{dW}{d\varepsilon}\right)_{\varepsilon=0},$$

we replace the ordinary derivative of the scalar Lagrange function in (1.1) by the absolute one and take into account that $\Delta g_{\alpha\beta} = 0$. Thence

$$\delta W = \int_{\tau_0}^{\tau_1} d\tau \frac{g_{\alpha\beta}u^\alpha \Delta u^\beta}{\sqrt{g_{\mu\nu}u^\mu u^\nu}}.$$

Due to the rule of commutation of derivatives (cf. (I.2.3)) the covariant variation of u^α can be expressed by means of δx^α as

$$\Delta u^\alpha = \frac{D}{d\tau}\delta x^\alpha \tag{1.4}$$

and therefore

$$\delta W = -\int_{\tau_0}^{\tau_1} d\tau \delta x^\alpha \frac{D}{d\tau}\frac{g_{\alpha\beta}u^\beta}{\sqrt{g_{\mu\nu}u^\mu u^\nu}} + \frac{g_{\alpha\beta}u^\beta}{\sqrt{g_{\mu\nu}u^\mu u^\nu}}\delta x^\alpha \Big|_{\tau_0}^{\tau_1}. \tag{1.5}$$

From this and from the stationary action principle: $\delta W = 0$ for δx^α satisfying the conditions $\delta x^\alpha(\tau_0) = \delta x^\alpha(\tau_1) = 0$ and otherwise arbitrary, we get the geodesic equations

$$\frac{D}{ds}\frac{u^\alpha}{\sqrt{u_\rho u^\rho}} = 0, \tag{1.6}$$

equivalent to (I.1.2).

The action (1.1) is invariant with respect to arbitrary transformations of the parameter τ. This invariance, due to the second Noether theorem (cf. *e. g.* [4]), leads to a strong identity fulfilled by the l. h. side of (1.6), *i. e.*

$$u^\alpha \frac{D}{d\tau}\frac{g_{\alpha\beta}u^\alpha}{\sqrt{u_\mu u^\mu}} = 0 \tag{1.7}$$

for any function $x^\alpha = \xi^\alpha(\tau)$.

2. THE GEODESIC DEVIATION

a) Second variation approach

To calculate the second variation of W, defined as

$$\delta^2 W : = \varepsilon^2\left(\frac{d^2W}{d\varepsilon^2}\right)_{\varepsilon=0},$$

the concept of the second covariant variation is needed. It is defined as

$$\Delta^2 x^\alpha := \varepsilon^2 \left(\frac{D}{\partial\varepsilon} \frac{\partial \xi^\alpha}{\partial \varepsilon} \right)_{\varepsilon=0} ;$$
$$\Delta^2 t^{\alpha_1 \ldots \alpha_n} := \varepsilon^2 \left(\frac{D^2}{\partial \varepsilon^2} t^{\alpha_1 \ldots \alpha_n} \right)_{\varepsilon=0} . \tag{2.1}$$

Then

$$\delta^2 W = \int_{\tau_0}^{\tau_1} d\tau (g_{\mu\nu} u^\mu u^\nu)^{-1/2} \left[\left(g_{\alpha\beta} - \frac{u_\alpha u_\beta}{u_\rho u^\rho} \right) \Delta u^\alpha \Delta u^\beta + g_{\alpha\beta} u^\alpha \Delta^2 u^\beta \right] ,$$

where $u_\alpha := g_{\alpha\beta} u^\beta$. Because of the Ricci identity

$$\Delta^2 u^\alpha = \frac{D}{d\tau} \Delta^2 x^\alpha + R^\alpha{}_{\mu\nu\rho} \delta x^\mu \delta x^\nu u^\rho . \tag{2.2}$$

(This identity can be considered as a counterpart of (1.4) for the second variation in a curved manifold). Thus

$$\begin{aligned} \delta^2 W = & \int_{\tau_0}^{\tau} d\tau (u_\mu u^\mu)^{-1/2} \left[\left(g_{\alpha\beta} - \frac{u_\alpha u_\beta}{u_\rho u^\rho} \right) \Delta u^\alpha u^\beta - R_{\alpha\beta\gamma\delta} \delta x^\alpha u^\beta \delta x^\gamma u^\delta \right. \\ & \left. + u_\alpha \frac{D}{d\tau} \Delta^2 x^\alpha \right] = - \int_{\tau_0}^{\tau_1} d\tau \left\{ \left[\frac{D}{d\tau} \left((u^\mu u_\mu)^{-1/2} \left(g_{\alpha\beta} - \frac{u_\alpha u_\beta}{u_\rho u^\rho} \right) \frac{D}{d\tau} \delta x^\beta \right) \right. \right. \\ & \left. \left. + (u^\mu u_\mu)^{-1/2} R_{\alpha\beta\gamma\delta} u^\beta \delta x^\gamma u^\delta \right] \delta x^\alpha + \frac{D}{d\tau} \left(\frac{u_\alpha}{\sqrt{u_\rho u^\rho}} \right) \Delta^2 x^\alpha \right\} \\ & + \left[\frac{\delta x^\alpha}{\sqrt{u_\rho u^\rho}} \left(g_{\alpha\beta} - \frac{u_\alpha u_\beta}{u_\rho u^\rho} \right) \frac{D}{d\tau} \delta x^\beta + \frac{u_\alpha}{\sqrt{u_\rho u^\rho}} \Delta^2 x^\alpha \right]_{\tau_0}^{\tau_1} . \end{aligned} \tag{2.3}$$

From (1.5) and (2.3) it follows that when $x^\alpha = \xi^\alpha(\tau)$ satisfies the geodesic equations (1.6) and the first variation δx^α fulfils the general geodesic deviation equations

$$\frac{D}{d\tau} \left[\frac{1}{\sqrt{u_\lambda u^\lambda}} \left(\delta^\alpha_\beta - \frac{u^\alpha u_\beta}{u_\mu u^\mu} \right) \frac{Dr^\beta}{d\tau} \right] + \frac{1}{\sqrt{u_\lambda u^\lambda}} R^\alpha_{\beta\gamma\delta} u^\beta r^\gamma u^\delta = 0 \tag{2.4}$$

(cf. (I.2.7)) then the variation $\delta W + \frac{1}{2} \delta^2 W$ vanishes for δx^α and $\Delta^2 x^\rho$ vanishing at τ_0 and τ_1. This is a particular case of a general property fulfilled by the Jacobi equations of any variational principle, cf. [5].

b) The accessoric variational principle

According to the general procedure, discussed *e. g.* in [5], the form of the integrand of $\delta^2 W$ in (2.3) implies the following form of the so-called « accessoric » Lagrange function

$$\mathcal{L}\left(r^\alpha, \frac{Dr^\alpha}{d\tau}, \tau \right) = \frac{1}{2\sqrt{u^\mu u_\mu}} \left[\left(g_{\alpha\beta} - \frac{u_\alpha u_\beta}{u_\rho u^\rho} \right) \frac{Dr^\alpha}{d\tau} \frac{Dr^\beta}{d\tau} - R_{\alpha\beta\gamma\delta} r^\alpha u^\beta r^\gamma u^\delta \right] \tag{2.5}$$

in which all the functions u^α, $g_{\alpha\beta}$, $R_{\alpha\beta\gamma\delta}$ are evaluated along a given geodesic Γ, with a fixed parametrization, described in the local coordinate system $\{ x^\alpha \}$ by given functions ξ^α. The accessoric Lagrangian (2.5) defines the accessoric, or the Carathéodory action functional

$$\mathscr{W}[r^\alpha] = \int_{\tau_0}^{\tau_1} \mathscr{L}\left(r^\alpha, \frac{Dr^\alpha}{d\tau}, \tau\right) d\tau \tag{2.6}$$

which is a functional depending on vector fields defined along Γ. The stationary action principle: $\delta\mathscr{W} = 0$ for variations (²) δr^α fulfilling

$$\delta r^\alpha(\tau_0) = \delta r^\alpha(\tau_1) = 0$$

and otherwise arbitrary, implies the general geodesic deviation equation (2.4) along Γ. In this sense the geodesic deviation equations are independent dynamical equations and several of their properties could be deduced as a result of application of appropriate general theorems of analytical dynamics to (2.6). Let us illustrate this feature by two examples.

As one can immediately verify, $\mathscr{W}[r^\alpha]$ is invariant under the transformation: $r^\alpha \mapsto r^\alpha + \kappa(\tau)u^\alpha$, generated by an arbitrary, differentiable function κ. This invariance implies, due to the second Noether theorem applied to (2.6), the strong identity

$$u^\alpha \left\{ \frac{D}{d\tau} \frac{1}{\sqrt{u_\rho u^\rho}} \left(g_{\alpha\beta} - \frac{u_\alpha u_\beta}{u_\mu u^\mu} \right) \frac{Dr^\beta}{d\tau} + \frac{1}{\sqrt{u_\rho u^\rho}} R_{\alpha\beta\gamma\delta} u^\beta r^\gamma u^\delta \right\} = 0$$

satisfied by the l. h. side of the general geodesic deviation equations (2.4). This identity was crucial for Proposition 2.2 in I.

As the second example let us consider the canonical formalism corresponding to the Lagrangian (2.5). According to a general remark about the accessoric Hamiltonian made in [5], it should be a singular canonical formalism with constraints. And indeed, when one defines the canonical momentum π_μ in a standard way as

$$\pi_\mu := \frac{\partial \mathscr{L}}{\partial \dot{r}^\mu} = \frac{1}{\sqrt{u^\rho u_\rho}} \left(g_{\mu\nu} - \frac{u_\mu u_\nu}{u^\lambda u_\lambda} \right) \frac{Dr^\nu}{d\tau}, \tag{2.7}$$

$\left(\text{here } \dot{r}^\mu = \frac{dr^\mu}{d\tau}\right)$ we get at once the following constraint relation

$$\pi_\mu u^\mu = 0. \tag{2.8}$$

(²) Here the variation δr^α is equal to the linear part of the difference $r^\alpha(\tau, \varepsilon) - r^\alpha(\tau, 0)$ of two vectors defined at the same point of the manifold V_n. We have, of course,

$$\delta \frac{Dr^\alpha}{d\tau} = \frac{D\delta r^\alpha}{d\tau}.$$

The « accessoric » Hamiltonian which, due to the Legendre transformation, corresponds to the Lagrangian $\mathscr{L}$ is given by

$$\mathscr{H}(r^\alpha, \pi_\beta, \tau) = \frac{\sqrt{u^\rho u_\rho}}{2}\left(g^{\alpha\beta}\pi_\alpha\pi_\beta + \frac{1}{u_\rho u^\rho}\mathrm{R}_{\alpha\beta\gamma\delta}r^\alpha u^\beta r^\gamma u^\lambda - \frac{2}{\sqrt{u_\rho u^\rho}}\Gamma^\alpha_{\mu\nu}u^\mu r^\nu \pi_\alpha\right) \tag{2.9}$$

In all of the above expressions quantities like u^α, $g_{\alpha\beta}$, $\mathrm{R}_{\alpha\beta\gamma\delta}$, etc., are taken along a fixed Γ and are given functions of τ. In Dirac's terminology, cf. [*6*], (2.8) describes primary constraints. One can easily verify that there are no secondary constraints in this theory. Dirac's canonical equations of motion with the Hamiltonian (2.9) contain one Lagrange multiplier. They are equivalent to the equations (I.2.13) from Proposition 2.2 in I. The arbitrary function $\mu(\tau)$ in (I.2.13) is now seen to be the first derivative with respect to τ of this Lagrange multiplier.

In a fairly standard way one can derive the Hamilton-Jacobi equation corresponding to (2.9). It reads as

$$\frac{\partial \mathrm{S}}{\partial \tau} = \mathscr{H}\left(r^\nu, \frac{\partial \mathrm{S}}{\partial r^\mu}, \tau\right), \tag{2.10}$$

where $\mathrm{S} = \mathrm{S}(r^\mu, \tau)$ is the principal function which now besides (2.10) must also satisfy the equation

$$\frac{\partial \mathrm{S}}{\partial r^\mu} u^\mu = 0 \tag{2.11}$$

being a consequence of (2.8). Thus S should now be the complete integral of partial differential equations (2.10) and (2.11). Since these equations form an involutive system, such an S exists, cf. [7]. This could be checked by showing that their Poisson bracket vanishes (which in Dirac's terminology means that $\mathscr{H}$ is a first class quantity).

The consistency of the system (2.10)-(2.11) of the Hamilton-Jacobi equations can be also demonstrated directly. If we namely change the variables, replacing r^μ by $r^\mu_\perp$ and κ:

$$r^\mu_\perp := r^\mu - \frac{r^\rho u_\rho}{u^\sigma u_\sigma} u^\mu;$$
$$\kappa := r^\rho u_\rho;$$

(only three among the $r^\mu_\perp$ are independent) and put

$$\mathrm{S}(r^\mu, \tau) = \mathrm{S}_0(r^\mu_\perp, \kappa, \tau)$$

then

$$\frac{\partial \mathrm{S}}{\partial r^\mu} = \frac{\partial \mathrm{S}_0}{\partial r^\nu_\perp}\left(\delta^\nu_\mu - \frac{u^\nu u_\mu}{u^\sigma u_\sigma}\right) + \frac{\partial \mathrm{S}_0}{\partial \kappa} u_\mu;$$

$$\frac{\partial \mathrm{S}}{\partial \tau} = - r^\rho \frac{\partial \mathrm{S}_0}{\partial r^\mu_\perp}\frac{\mathrm{D}}{d\tau}\left(\frac{u_\rho u^\mu}{u^\sigma u_\sigma}\right) + \frac{\partial \mathrm{S}_0}{\partial \kappa} r^\rho \frac{\mathrm{D}u_\rho}{d\tau} + \frac{\partial \mathrm{S}_0}{\partial \tau}.$$

The equation (2.11) implies then $\frac{\partial S_0}{\partial \kappa} = 0$ and this together with the geodesic equation (1.6) means that $\frac{\partial S}{\partial \tau} = \frac{\partial S_0}{\partial \tau}$. We are thus left with only one Hamilton-Jacobi equation

$$\frac{\partial S_0}{\partial \tau} = \mathscr{H}\left(r_\perp^\nu, \frac{\partial S_0}{\partial r_\perp^\mu}, \tau\right) \tag{2.12}$$

for the principal function S_0 depending solely on $r_\perp^\mu$ and τ (the skew symmetry of $R_{\alpha\beta\gamma\delta}$ in (2.10) allowed the replacement r^ν by $r_\perp^\nu$). Thus the overdetermination of the system (2.10)-(2.11) physically means that the true dynamical variables are not the r^μ's, but rather the $r_\perp^\mu$'s.

c) The simultaneous variational principle

As it has been shown in a previous paper by the author [5], there is always a unified variational principle simultaneously leading to both the Lagrange and the Jacobi equations. In particular, for the geodesic problem, the unified action $W^{(1)}$ is given by the functional

$$W^{(1)}[\xi^\alpha ; r^\alpha] = \int_{\tau_0}^{\tau_1} L^{(1)}\left(\xi^\alpha, \frac{d\xi^\alpha}{d\tau}, r^\alpha, \frac{Dr^\alpha}{d\tau}\right) d\tau \tag{2.13}$$

with

$$L^{(1)} := g_{\alpha\beta} \frac{u^\alpha}{\sqrt{u^\rho u_\rho}} \frac{Dr^\beta}{d\tau}, \tag{2.14}$$

where $u^\alpha := \frac{d\xi^\alpha}{d\tau}$, $u_\rho = g_{\rho\sigma} u^\sigma$, $g_{\alpha\beta}$ is a given metric tensor of V_n taken at the given point x on a line Γ described by $x^\alpha = \xi^\alpha(\tau)$ and $\frac{Dr^\alpha}{d\tau}$ is the absolute derivative of a vector field r^α in the direction of Γ. Computing the variation of (2.13) caused by independent variations of ξ^α and r^α we obtain

$$\delta W^{(1)} = \int_{\tau_0}^{\tau_1} \left[\frac{1}{\sqrt{u_\rho u^\rho}} \left(g_{\alpha\beta} - \frac{u_\alpha u_\beta}{u_\rho u^\rho} \right) \Delta u^\alpha \frac{Dr^\beta}{d\tau} + \frac{u_\alpha}{\sqrt{u_\rho u^\rho}} \Delta \frac{Dr^\alpha}{d\tau} \right] d\tau$$

Making then use of (1.4) and of

$$\Delta \frac{Dr^\alpha}{d\tau} = \frac{D}{d\tau} \Delta r^\alpha + R^\alpha{}_{\lambda\mu\nu} r^\lambda \delta x^\mu u^\nu, \tag{2.15'}$$

we find

$$\delta W^{(1)} = \int_{\tau_0}^{\tau_1} \left\{ \left[-\frac{D}{d\tau} \left(\frac{1}{\sqrt{u_\rho u^\rho}} \left(g_{\alpha\beta} - \frac{u_\alpha u_\beta}{u^\rho u_\rho} \right) \frac{Dr^\beta}{d\tau} \right) - \frac{1}{\sqrt{u_\rho u^\rho}} R_{\alpha\beta\gamma\delta} u^\beta r^\gamma u^\delta \right] \delta x^\alpha - \Delta r^\alpha \frac{D}{d\tau} \frac{u_\alpha}{\sqrt{u_\rho u^\rho}} \right\} d\tau + \frac{1}{\sqrt{u_\rho u^\rho}} \left(g_{\alpha\beta} - \frac{u_\alpha u_\beta}{u_\rho u^\rho} \right) \frac{Dr^\beta}{d\tau} \delta x^\alpha \Bigg|_{\tau_0}^{\tau_1} + \frac{u_\alpha}{\sqrt{u_\rho u^\rho}} \Delta r^\alpha \Bigg|_{\tau_0}^{\tau}.$$

Taking now into account that due to (1.2) and (1.3)

$$\Delta r^\alpha = \delta r^\alpha + \Gamma^\alpha_{\beta\gamma} r^\beta \delta x^\gamma, \tag{2.15}$$

we arrive at the following.

THEOREM 2.1. — The variation of the functional (2.13) $\delta W^{(1)} = 0$, for δx^α and δr^α vanishing at τ_0 and τ_1 and being otherwise arbitrary, if and only if $x^\alpha = \xi^\alpha(\tau)$ and $r^\alpha = r^\alpha(\tau)$ fulfil the system of geodesic (1.6) and geodesic deviation equations (2.4).

Now within the framework based on $W^{(1)}$ instead of on $\mathscr{W}$, the « dynamical » properties of the geodesic deviation are not considered as separate from those of the geodesic motion. The generalized momenta associated to (2.13) are defined as

$$\pi^{(1)}_\alpha := \frac{\partial L^{(\alpha)}}{\partial u^\alpha} = \frac{1}{\sqrt{u_\alpha u^\alpha}}\left(g_{\alpha\beta} - \frac{u_\alpha u_\beta}{u_\rho u^\rho}\right)\frac{Dr^\beta}{d\tau} + g_{\mu\nu}\frac{u^\mu \Gamma^\nu_{\sigma\alpha} r^\sigma}{\sqrt{u_\rho u^\rho}} ;$$

$$p_\alpha := \frac{\partial L^{(1)}}{\partial \dot{r}^\alpha} = \frac{g_{\alpha\beta}u^\beta}{\sqrt{u_\rho u^\rho}}.$$

They fulfil the following constraint conditions:

$$\begin{gathered} g^{\alpha\beta}p_\alpha p_\beta = 1, \\ g^{\alpha\beta}p_\alpha\left(\pi^{(1)}_\beta - \Gamma^\mu_{\beta\nu} r^\nu p_\mu\right) = 0. \end{gathered} \tag{2.16}$$

The Hamiltonian corresponding by means of the Legendre transformation to $L^{(1)}$ is identically equal to zero. When one defines the principal Hamilton function $S^{(1)}$ as being the value of (2.13) at the end point on a true trajectory of (1.6) and (2.4), one gets in the standard way ([3]) that

$$\frac{\partial S^{(1)}}{\partial x^\alpha} = \pi^{(1)}_\alpha, \qquad \frac{\partial S^{(1)}}{\partial r^\alpha} = p_\alpha. \tag{2.17}$$

From here, due to vanishing of the Hamiltonian and to (2.16), one obtains the following set of Hamilton-Jacobi equations

$$\begin{gathered} \frac{\partial S^{(1)}}{\partial \tau} = 0 ; \\ g^{\alpha\beta}\frac{\partial S^{(1)}}{\partial r^\alpha}\frac{\partial S^{(1)}}{\partial r^\beta} = 1 ; \\ g^{\alpha\beta}\frac{\partial S^{(1)}}{\partial r^\alpha}\left(\frac{\partial S^{(1)}}{\partial x^\beta} - \Gamma^\mu_{\beta\sigma} r^\sigma \frac{\partial S^{(1)}}{\partial r^\mu}\right) = 0 \end{gathered} \tag{2.18}$$

which form a system of partial differential equations for the function $S^{(1)}$

([3]) Comp. [8], chap. II, § 9.

depending, because of the first of Eqs. (2.18), in general on eight variables x^α, r^α. The knowledge of the complete integral of Eqs. (2.18) allows one to determine in the standard way in a given Riemannian manifold a geodesic line and a geodesic deviation field along it. Let us observe that no components of the curvature tensor of the manifold enter into the equations (2.18), in contrast to the set (2.10)-(2.11). The equations (2.18) form an involutive system. That can be checked directly, since Eqs. (2.18) admit the separation of the variables x^μ and r^μ by representing $S^{(1)}$ in the form

$$S^{(1)}(x^\mu, r^\nu) = r^\alpha \frac{\partial U}{\partial x^\alpha} + V \tag{2.19}$$

in which U, V are functions of x^α only. Then Eqs. (2.18) imply (cf. Appendix A)

$$\begin{aligned} g^{\alpha\beta} \frac{\partial U}{\partial x^\alpha} \frac{\partial U}{\partial x^\beta} &= 1, \\ g^{\alpha\beta} \frac{\partial U}{\partial x^\alpha} \frac{\partial V}{\partial x^\beta} &= 0. \end{aligned} \tag{2.20}$$

d) The « dynamic » variational principles

The above considerations can be repeated for the so-called (in the terminology of Misner, Thorne and Wheeler [9]) dynamic variational principle which leads to the equations

$$\frac{Du^\alpha}{d\tau} = 0 \tag{2.21}$$

of a geodesic line parametrized by an affine parameter τ. Its action

$$I = \frac{1}{2} \int_{\tau_0}^{\tau_1} g_{\alpha\beta} u^\alpha u^\beta d\tau \tag{2.22}$$

is invariant under a one-parametric group of translations of the parameter τ: $\tau \mapsto \tau' = \tau + \varepsilon$, $\varepsilon = \text{const}$. This invariance leads, by means of the first Noether theorem, to the first integral $g_{\alpha\beta} u^\alpha u^\beta = \text{const}$ which in general relativity could therefore be interpreted as the energy integral of a test particle freely falling in a gravitational field. As we know, the constant here must be taken to be equal to one,

$$g_{\alpha\beta} u^\alpha u^\beta = 1, \tag{2.23}$$

if the world line is to be parametrized by the proper time (what, of course, means that the energy of a freely falling particle equals its rest mass).

The accessoric action corresponding to (2.22) is now defined as

$$\mathscr{I} = \frac{1}{2} \int_{\tau_0}^{\tau_1} \left(g_{\alpha\beta} \frac{Dr^\alpha}{d\tau} \frac{Dr^\beta}{d\tau} - R_{\alpha\beta\gamma\delta} r^\alpha u^\beta r^\gamma u^\delta \right) d\tau, \tag{2.24}$$

where all the quantities: $g_{\alpha\beta}$, $\Gamma^{\alpha}_{\beta\gamma}$, $R_{\alpha\beta\gamma\delta}$ and u^{α} should be evaluated along a given geodesic Γ parametrized by an affine parameter τ. Varing (2.24) with respect to r^{α} we get the geodesic deviation equations:

$$\frac{Dr^{\alpha}}{d\tau} + R^{\alpha}{}_{\beta\gamma\delta}u^{\beta}r^{\gamma}u^{\delta} = 0 \tag{2.25}$$

The action (2.24) is invariant $\left(\text{due to the assumption that along } \Gamma\text{: } \frac{Du^{\alpha}}{d\tau} = 0\right)$ under the group of transformations: $r^{\mu} \mapsto \tilde{r}^{\mu} = r^{\mu} + \lambda u^{\mu}$, where $\lambda = \text{const.}$ This invariance implies the existence of the first integral $u_{\alpha}\frac{Dr^{\alpha}}{d\tau} = \text{const.}$; its special form

$$u_{\alpha}\frac{Dr^{\alpha}}{d\tau} = 0 \tag{2.26}$$

must be accepted when Eqs. (2.25) and (2.26) determine the natural geodesic deviation, *i. e.* in the case, when both « neighbouring » geodesics are parametrized by the Riemannian proper time s, as it is usually done in general relativity. The action (2.24) is not any more invariant under the translations of the parameter τ. Therefore there is no first integral of the « energy » type. Instead one has the following equation

$$\frac{d}{d\tau}\frac{1}{2}\left(g_{\alpha\beta}\frac{Dr^{\alpha}}{d\tau}\frac{Dr^{\beta}}{d\tau} + R_{\alpha\beta\gamma\delta}r^{\alpha}u^{\beta}r^{\gamma}u^{\delta}\right) = \frac{1}{2}R_{\alpha\beta\gamma\delta;\varepsilon}r^{\alpha}u^{\beta}r^{\gamma}u^{\delta}u^{\varepsilon} \tag{2.27}$$

which could be interpreted as determining the rate of change of the « energy », cf. [*10*].

Also on the level of the « dynamic » approach one can formulate the simultaneous action principle, similar to (2.13). Its action ([4]) is given by

$$I^{(1)} = \frac{1}{2}\int_{\tau_0}^{\tau} g_{\alpha\beta}u^{\alpha}\frac{Dr^{\beta}}{d\tau}\,d\tau. \tag{2.28}$$

It is invariant both under the translations of the parameter τ and under the « gauge » transformations: $r^{\mu} \mapsto \tilde{r}^{\mu} = r^{\mu} + \lambda u^{\mu}$ ($\lambda = \text{const.}$) (These last transformations do not leave the integrand in (2.28) invariant, but add to it a complete differential). The first Noether theorem applied to translations of τ ($\delta\tau = \varepsilon = \text{const.}$; $\xi^{\alpha}(\tau) \mapsto \tilde{\xi}^{\alpha}(\tau+\varepsilon) = \xi^{\alpha}(\tau)$; $r^{\alpha}(\tau) \mapsto \tilde{r}^{\alpha}(\tau+\varepsilon) = r^{\alpha}(\tau)$) results now in the first integral $u_{\alpha}\frac{Dr^{\alpha}}{d\tau} = \text{const.}$ and the other invariance leads to $g_{\alpha\beta}u^{\alpha}u^{\beta} = \text{const.}$ Thus, the quantity $g_{\alpha\beta}u^{\alpha}\frac{Dr^{\beta}}{d\tau}$ could now be interpreted as being the energy—the kinetic energy of the relative motion described

([4]) An action of this form has also been discussed by Mitskevich [*11*].

by $r^\alpha(\tau)$. For reasons already mentioned before, one has to replace these constants correspondingly by such as in (2.26) and (2.23).

For all three actions (2.22), (2.24) and (2.28) the corresponding canonical formalism will be of the standard type with no constraints. The single Hamilton-Jacobi equations appearing in both approaches, based correspondingly on (2.24) and (2.28), will as a result of separation of variables lead again either to (2.12) or to (2.20) provided the choice of the constants of integration appearing in the first integrals will be such as exhibited in (2.23) and (2.26).

To be more explicit we shall illustrate this last point for the case based on (2.28). Here the Hamilton-Jacobi equation is of the form

$$\frac{\partial S^{(1)}}{\partial \tau} = g^{\alpha\beta} \frac{\partial S^{(1)}}{\partial r^\alpha} \left(\frac{\partial S^{(1)}}{\partial x^\beta} - \Gamma^\nu_{\beta\mu} r^\mu \frac{\partial S^{(1)}}{\partial r^\nu} \right). \tag{2.29}$$

Taking

$$S^{(1)}(x^\alpha, r^\beta, \tau) = (\tau - \tau_0)h + r^\mu \frac{\partial U}{\partial x^\mu} + V,$$

where U, V are functions of x^α only and h is a constant, we get from (2.29)

$$r^\mu \frac{\partial}{\partial x^\mu} \left(g^{\alpha\beta} \frac{\partial U}{\partial x^\alpha} \frac{\partial U}{\partial x^\beta} \right) + g^{\alpha\beta} \frac{\partial U}{\partial x^\alpha} \frac{\partial V}{\partial x^\beta} = h.$$

Since this equation should be satisfied for any r^μ and the $g^{\alpha\beta}$ depends solety on x^α, we must have

$$g^{\alpha\beta} \frac{\partial U}{\partial x^\alpha} \frac{\partial V}{\partial x^\beta} = h,$$

$$g^{\alpha\beta} \frac{\partial U}{\partial x^\alpha} \frac{\partial U}{\partial x^\beta} = c$$

with c being a new constant of integration. If we choose the constants here in agreement with (2.23) and (2.26) ($h = 0$, $c = 1$) this turns out to be equivalent to (2.20).

3. THE SECOND GEODESIC DEVIATION

The third variation of the action W is defined as

$$\delta^3 W : = \varepsilon^3 \left(\frac{d^3 W}{d\varepsilon^3} \right)_{\varepsilon=0}.$$

To calculate it one introduces the concept of the third covariant variation as

$$\Delta^3 x^\alpha : = \varepsilon^3 \left(\frac{D^2}{d\varepsilon^2} \frac{\partial \xi^\alpha}{\partial \varepsilon} \right)_{\varepsilon=0};$$
$$\Delta^3 t^{\alpha_1 \dots \alpha_k} : = \varepsilon^3 \left(\frac{D^3}{d\varepsilon^3} t^{\alpha_1 \dots \alpha_k} \right)_{\varepsilon=0}. \tag{3.1}$$

Then differentiating the integrand in $\delta^2 W$ one obtains

$$\delta^3 W = \int_{\tau_0}^{\tau_1} d\tau (u_\rho u^\rho)^{-\frac{1}{2}} \left[u_\alpha \Delta^3 u^\alpha + 3h_{\alpha\beta}\Delta u^\alpha \Delta^2 u^\beta - 3\frac{u_\mu \Delta u^\mu}{u_\rho u^\rho} h_{\alpha\beta}\Delta u^\alpha \Delta u^\beta \right],$$

where

$$h_{\alpha\beta} := g_{\alpha\beta} - \frac{u_\alpha u_\beta}{u_\rho u^\rho}.$$

The relations (1.4) and (2.2) must now be completed by the following

$$\begin{aligned} \Delta^3 u^\alpha = \frac{D}{d\tau}\Delta^3 x^\alpha &+ 2R^\alpha{}_{\beta\gamma\delta}\Delta^2 x^\beta \delta x^\gamma u^\delta + R^\alpha{}_{\beta\gamma\delta}\delta x^\beta \Delta^2 x^\gamma u^\delta \\ &+ R^\alpha{}_{\beta\gamma\delta}\delta x^\beta \delta x^\gamma \Delta u^\delta + R^\alpha{}_{\beta\gamma\delta;\varepsilon}\delta x^\beta \delta x^\gamma u^\delta \delta x^\varepsilon \end{aligned} \tag{3.2}$$

which results from (1.4) and (2.2) by means of the Ricci identity.

From (1.4), (2.2) and (3.2) one obtains

$$\begin{aligned} \delta^3 W = -\int_{\tau_0}^{\tau_1} d\tau \left(\Delta^3 x^\alpha \frac{Dp_\alpha}{d\tau} + 2\Delta^2 x^\alpha h_\alpha + \delta x^\alpha j_\alpha \right) \\ + \left(p_\alpha \Delta^3 x^\alpha + 2\pi_\alpha \Delta^2 x^\alpha + \tau_\alpha \delta x^\alpha \right) \Big|_{\tau_0}^{\tau_1}, \end{aligned} \tag{3.3}$$

where

$$p_\alpha := \frac{u_\alpha}{\sqrt{u_\lambda u^\lambda}}; \qquad h_\alpha := \frac{D}{d\tau}\frac{h_{\alpha\beta}\Delta u^\beta}{\sqrt{u_\lambda u^\lambda}} + \frac{1}{\sqrt{u_\lambda u^\lambda}} R_{\alpha\beta\gamma\delta}u^\beta \delta x^\gamma u^\delta;$$

$$\begin{aligned} j_\alpha := \frac{D}{d\tau}\left(\frac{h_{\alpha\beta}}{\sqrt{u_\lambda u^\lambda}} \frac{D\Delta^2 x^\beta}{d\tau} \right) &+ \frac{1}{\sqrt{u_\lambda u^\lambda}} R_{\alpha\beta\gamma\delta}u^\beta \Delta^2 x^\gamma u^\delta \\ &- \frac{1}{\sqrt{u_\lambda u^\lambda}} (R_{\alpha\beta\gamma\delta;\varepsilon} + R_{\alpha\varepsilon\gamma\delta;\beta}) u^\beta u^\gamma \delta x^\delta \delta x^\varepsilon \\ &- \frac{4}{\sqrt{u_\lambda u^\lambda}} R_{\alpha\beta\gamma\delta}\Delta u^\beta u^\gamma \delta x^\delta - \frac{2\Delta u_\alpha}{\sqrt{u_\lambda u^\lambda}} \frac{d}{d\tau}\left(\frac{u_\rho \Delta u^\rho}{u_\sigma u^\sigma} \right) \\ &- \frac{D}{d\tau}\frac{u_\alpha}{(u_\lambda u^\lambda)^{3/2}} \left(\Delta u^\mu \Delta u_\mu - R_{\mu\nu\rho\sigma}u^\mu \delta x^\nu u^\rho \delta x^\sigma - 2\frac{(u_\mu \Delta u^\mu)^2}{u_\rho u^\rho} \right) \\ &- \left(\frac{u_\mu \Delta u^\mu}{u_\rho u^\rho} \right)^2 \frac{Dp_\alpha}{d\tau} - \frac{2\Delta u^\mu u_\mu}{u_\rho u^\rho} h_\alpha; \end{aligned}$$

$$\begin{aligned} \pi_\alpha = \frac{h_{\alpha\beta}\Delta u^\beta}{\sqrt{u_\lambda u^\lambda}}; \qquad \tau_\alpha = \frac{h_{\alpha\beta}}{\sqrt{u_\lambda u^\lambda}} \frac{D\Delta^2 x^\beta}{d\tau} &- \frac{2\Delta u_\alpha u_\mu \Delta u^\mu}{(u_\lambda u^\lambda)^{3/2}} \\ &- \frac{u_\alpha}{(u_\lambda u^\lambda)^{3/2}} \left(\Delta u^\sigma \Delta u_\sigma - R_{\mu\nu\rho\sigma}u^\mu \delta x^\nu u^\rho \delta x^\sigma - 3\frac{(u_\mu \Delta u^\mu)^2}{u_\rho u^\rho} \right). \end{aligned}$$

From here now follows that if $x^\alpha = \xi^\alpha(\tau)$ satisfies the geodesic equations (1.6), δx^α the first geodesic equation (2.4) and $\Delta^2 x^\alpha$ the second geodesic equations (I.3.2), then $\delta^3 W$ is equal to the « boundary » terms which vanish when $\Delta^3 x^\alpha$, $\Delta^2 x^\alpha$, δx^α vanish at τ_0 and τ_1.

The accessoric Lagrangian $\mathscr{L}_2\left(w^\alpha, \frac{Dw^\alpha}{d\tau}, \tau\right)$ is then according to the general rule (cf. [5]) given by the expression

$$\mathscr{L}_2 = \frac{1}{\sqrt{u_\lambda u^\lambda}}\left[\mathscr{L}_2^{(d)} + 2w^\alpha \frac{Dr_\alpha}{d\tau}\frac{d}{d\tau}\frac{u_\mu \frac{Dr^\mu}{d\tau}}{u_\lambda u^\lambda} + w^\alpha u_\alpha \frac{d}{d\tau}\frac{1}{u_\lambda u^\lambda}\left(\frac{Dr^\mu}{d\tau}\frac{Dr_\mu}{d\tau} - R_{\mu\nu\rho\sigma}u^\mu r^\nu u^\rho r^\sigma - \frac{2}{u_\lambda u^\lambda}\left(u_\mu \frac{Dr^\mu}{d\tau}\right)^2\right)\right]. \tag{3.4}$$

where

$$\mathscr{L}_2^{(d)} = \frac{1}{2}\left(h_{\alpha\beta}\frac{Dw^\alpha}{d\tau}\frac{Dw^\beta}{d\tau} - R_{\alpha\beta\gamma\delta}w^\alpha u^\beta w^\gamma u^\delta\right) + w^\alpha(R_{\alpha\beta\gamma\delta;\varepsilon} + R_{\alpha\varepsilon\gamma\delta;\beta})r^\beta u^\gamma r^\delta u^\varepsilon + 4w^\alpha R_{\alpha\beta\gamma\delta}\frac{Dr^\beta}{d\tau}u^\gamma r^\delta. \tag{3.5}$$

In the expressions (3.4) and (3.5) all functions u^α, $g_{\alpha\beta}$, $R_{\alpha\beta\gamma\delta}$, etc., are evaluated along a given geodesic Γ and r^α is a fixed solution of the general first geodesic deviation equations. The accessoric action is defined as

$$\mathscr{W}^{(2)}[w^\alpha] = \int_{\tau_0}^{\tau_1}\mathscr{L}_2\left(w^\alpha, \frac{Dw^\alpha}{d\tau}, \tau\right)d\tau \tag{3.6}$$

and is a functional depending on vector fields w^α defined along Γ. The stationary action principle: $\delta\mathscr{W}^{(2)} = 0$ for variations δw^α fulfilling

$$\delta w^\alpha(\tau_0) = \delta w^\alpha(\tau_1) = 0$$

leads to the second geodesic deviation equations

$$\sqrt{u_\lambda u^\lambda}\frac{D}{d\tau}\left(\frac{h_{\alpha\beta}}{\sqrt{u_\rho u^\rho}}\frac{Dw^\beta}{d\tau}\right) + R_{\alpha\beta\gamma\delta}u^\beta w^\gamma u^\delta - (R_{\alpha\beta\gamma\delta;\varepsilon} + R_{\alpha\varepsilon\gamma\delta;\beta})u^\beta u^\gamma r^\delta r^\varepsilon - 4R_{\alpha\beta\gamma\delta}\frac{Dr^\beta}{d\tau}u^\gamma r^\delta - 2\frac{Dr_\alpha}{d\tau}\frac{d}{d\tau}\left(\frac{u_\rho}{u_\lambda u^\lambda}\frac{Dr^\rho}{d\tau}\right) - u_\alpha \frac{d}{d\tau}\frac{1}{u_\lambda u^\lambda}\left[\frac{Dr^\mu}{d\tau}\frac{Dr_\mu}{d\tau} - R_{\mu\nu\rho\sigma}u^\mu r^\nu u^\rho r^\sigma - \frac{2}{u_\lambda u^\lambda}\left(u_\rho\frac{Dr^\rho}{d\tau}\right)^2\right] = 0 \tag{3.7}$$

which are equivalent to (I.3.2). While performing the variations one must take into account that u^α and r^α in $\mathscr{L}_2$ fulfil correspondingly (1.6) and (2.4). Thus, in this approach the second geodesic deviation equations can be considered as independent dynamical equations.

The Lagrangian $\mathscr{L}_2$ is form-invariant under the transformations: $w^\alpha \mapsto w^\alpha + u^\alpha\psi(\tau)$ in which ψ is an arbitrary function. This invariance, implies due to the second Noether theorem, a strong identity of the form $i_\alpha u^\alpha = 0$, where i_α is the l. h. side of Eqs. (3.7).

The discussion of the corresponding cannonical formalism goes along the same lines as in section 2. The cannonical momentum

$$p_\mu := \frac{\partial \mathscr{L}_2}{\partial \dot{w}^\mu} = \frac{h_{\mu\nu}}{\sqrt{u_\lambda u^\lambda}} \frac{Dw^\nu}{d\tau}, \tag{3.8}$$

fulfils one primary constraint relation

$$p_\mu u^\mu = 0. \tag{3.9}$$

There are again no secondary constraints. Thus, the cannonical equations contain one Lagrange multiplier. It can be show that these cannonical equations are equivalent to Eqs. (I.3.7) in which $\nu(\tau)$ is the first derivative with respect to τ of the Lagrange multiplier.

The Hamilton-Jacobi equations again form a set of equations in involution:

$$\begin{aligned} &\frac{\partial \mathscr{S}^{(2)}}{\partial \tau} = \mathscr{H}^{(2)}\left(w^\mu, \frac{\partial \mathscr{S}^{(2)}}{\partial w^\mu}, \tau\right); \\ &\frac{\partial \mathscr{S}^{(2)}}{\partial w^\mu} u^\mu = 0; \end{aligned} \tag{3.10}$$

where $\mathscr{H}^{(2)}$ is the accessoric Hamiltonian defined by means of the Legendre transformation from (3.4) and (3.8). Introducing new variables

$$\begin{aligned} s^\mu &= \left(\delta^\mu_\nu - \frac{u^\mu u_\nu}{u^\sigma u_\sigma}\right) w^\nu; \\ \psi &= w^\mu u_\mu; \end{aligned}$$

one can show, as in section 2, that the system (3.10) reduces to one single equation.

Let us pass now to the unified variational principle which simultaneously leads to the equations of geodesics, of the first geodesic deviation and of the second geodesic deviation, in the case of an arbitrary parametrization. According to Proposition 3.3 from [5] the corresponding action is of the form

$$W^{(2)}[\xi^\alpha; r^\beta, w^\gamma] = \int_{\tau_0}^{\tau} \left[L^{(1)}\left(\xi^\alpha, u^\alpha, w^\alpha \frac{Dw^\alpha}{d\tau}\right) + L^{(2)}\left(\xi^\alpha, u^\alpha, r^\alpha, \frac{Dr^\alpha}{d\tau}\right)\right] d\tau, \tag{3.11}$$

where

$$\begin{aligned} L^{(1)} &:= \frac{u^\alpha}{\sqrt{u_\lambda u^\lambda}} \frac{Dw^\alpha}{d\tau}; \\ L^{(2)} &= \frac{h_{\alpha\beta}}{\sqrt{u_\lambda u^\lambda}} \frac{Dr^\alpha}{d\tau}\frac{Dr^\beta}{d\tau} - \frac{1}{\sqrt{u_\lambda u^\lambda}} R_{\alpha\beta\gamma\delta} r^\alpha u^\beta r^\gamma u^\delta. \end{aligned} \tag{3.12}$$

If one makes use of (1.4), (2.2), (2.15′), (3.2) and of

$$\Delta \frac{Dw^\alpha}{d\tau} = \frac{D}{d\tau} \Delta w^\alpha + R^\alpha{}_{\beta\gamma\delta} w^\beta \delta x^\gamma u^\delta,$$

one obtains the following expression for the complete variation of $W^{(2)}$:

$$W^{(2)} = - \int_{\tau_0}^{\tau_1} d\tau \left(\frac{j_\alpha}{\sqrt{u_\rho u^\rho}} \delta x^\alpha + 2h_\alpha \Delta r^\alpha + \Delta w^\alpha \frac{D}{d\tau} \frac{u_\alpha}{\sqrt{u_\rho u^\rho}} \right) + \tilde{\tau}_\mu \delta x^\mu \Big|_{\tau_0}^{\tau_1} + 2 \frac{h_{\alpha\beta}}{\sqrt{u_\rho u^\rho}} \frac{Dr^\beta}{d\tau} \Delta r^\alpha \Big|_{\tau_0}^{\tau_1} + \frac{u_\alpha}{\sqrt{u_\rho u^\rho}} \Delta w^\alpha \Big|_{\tau_0}^{\tau_1},$$

where h_α is the l. h. side of (2.4), j_α is an expression which differs from the l. h. side of (3.7) by terms vanishing modulo the geodesic and first geodesic deviation equations, and

$$\tilde{\tau}_\alpha = \frac{h_{\alpha\beta}}{\sqrt{u_\lambda u^\lambda}} \frac{Dw^\beta}{d\tau} + \frac{2}{\sqrt{u_\rho u^\rho}} R_{\alpha\beta\gamma\delta} r^\beta r^\gamma u^\delta - \frac{2h_{\alpha\beta}}{(u_\lambda u^\lambda)^{3/2}} \frac{Dr^\beta}{d\tau} \left(u_\mu \frac{Dr^\mu}{d\tau} \right) - \frac{u_\alpha}{(u_\rho u^\rho)^{3/2}} \left(h_{\mu\nu} \frac{Dr^\mu}{d\tau} \frac{Dr^\nu}{d\tau} - R_{\mu\nu\rho\sigma} r^\mu u^\nu r^\rho u^\sigma \right). \tag{3.13}$$

Taking into account the equality (2.15) and similar equality for Δw^α we obtain

THEOREM 3.1. — The variation of the functional (3.11) $\delta W^{(2)} = 0$, for δx^α, δr^α and δw^α vanishing at τ_0 and τ_1 and being otherwise arbitrary, if and only if $x^\alpha = \xi^\alpha(\tau)$ fulfil the equations of geodesics (1.6), $r^\alpha = r^\alpha(\tau)$ those of the first geodesic deviation (2.4) and $w^\alpha = w^\alpha(\tau)$ of the second geodesic deviation (3.7).

In the approach based on the unified action principle with the action (3.11) the dynamical properties of the first and of the second geodesic deviation vectors are, therefore, not being separated from those of the geodesic motion.

To pass to the cannonical formalism we calculate in the standard way the generalized momenta

$$p_\mu := \frac{\partial(L^{(1)} + L^{(2)})}{\partial \dot{w}^\mu} = \frac{u_\mu}{\sqrt{u_\rho u^\rho}};$$

$$\pi_\mu^{(2)} := \frac{\partial(L^{(1)} + L^{(2)})}{\partial \dot{r}^\mu} = \frac{2h_{\mu\nu}}{\sqrt{u_\lambda u^\lambda}} \frac{Dr^\beta}{d\tau};$$

$$\tau_\mu := \frac{\partial(L^{(1)} + L^{(2)})}{\partial u^\mu} = \tilde{\tau}_\mu + \frac{2h_{\alpha\beta}}{\sqrt{u_\lambda u^\lambda}} \Gamma^\alpha_{\mu\lambda} r^\lambda \frac{Dr^\beta}{d\tau} + \frac{\Gamma^\nu_{\mu\rho} u_\nu w^\rho}{\sqrt{u_\lambda u^\lambda}};$$

where $\tilde{\tau}_\mu$ is given by (3.13). The following primary constraint conditions are satisfied by them:

$$\begin{aligned} &g^{\alpha\beta} p_\alpha p_\beta = 1; \\ &g^{\alpha\beta} \pi_\alpha^{(2)} p_\beta = 0; \\ &g^{\alpha\beta}(\tau_\alpha - \Gamma^\sigma_{\alpha\rho} \pi_\sigma^{(2)} r^\rho - \Gamma^\sigma_{\alpha\rho} p_\sigma w^\rho) p_\beta + \frac{1}{4} g^{\alpha\beta} \pi_\alpha^{(2)} \pi_\beta^{(2)} + R^\alpha{}_\beta{}^\gamma{}_\sigma p_\alpha r^\beta p_\gamma r^\delta = 0. \end{aligned} \tag{3.14}$$

The Hamiltonian is again equal to zero. If one defines the principal function $S^{(2)}$ as the value of the action (3.11) at the end point on a true trajectory, one gets

$$\frac{\partial S^{(2)}}{\partial x^\alpha} = \tau_\alpha, \qquad \frac{\partial S^{(2)}}{\partial r^\alpha} = \pi^{(2)}_\alpha, \qquad \frac{\partial S^{(2)}}{\partial w^\alpha} = p_\alpha.$$

Thus, the Hamilton-Jacobi equations are given by the system

$$\begin{gathered}
\frac{\partial S^{(2)}}{\partial \tau} = 0\,; \\
g^{\alpha\beta}\frac{\partial S^{(2)}}{\partial w^\alpha}\frac{\partial S^{(2)}}{\partial w^\beta} = 1\,; \\
g^{\alpha\beta}\frac{\partial S^{(2)}}{\partial r^\alpha}\frac{\partial S^{(2)}}{\partial w^\beta} = 0\,; \\
g^{\alpha\beta}\left(\frac{\partial S^{(2)}}{\partial x^\alpha} - \Gamma^\sigma_{\alpha\rho}\frac{\partial S^{(2)}}{\partial r^\sigma} r^\rho - \Gamma^\sigma_{\alpha\rho}\frac{\partial S^{(2)}}{\partial w^\sigma} w^\rho\right)\frac{\partial S^{(2)}}{\partial w^\beta} \\
= -\frac{1}{4}g^{\alpha\beta}\frac{\partial S^{(2)}}{\partial r^\alpha}\frac{\partial S^{(2)}}{\partial r^\beta} - R^\alpha{}_\beta{}^\gamma{}_\delta\frac{\partial S^{(2)}}{\partial w^\alpha} r^\beta \frac{\partial S^{(2)}}{\partial r^\gamma} r^\delta.
\end{gathered} \tag{3.15}$$

The easiest way to verify the consistency of the system (3.15) consists in showing that it admits the separation of variables. Assuming $S^{(2)}$ to be of the form

$$S^{(2)}(x^\mu, r^\nu, w^\rho) = w^\mu \frac{\partial U}{\partial x^\mu} + r^\mu r^\nu \frac{\partial^2 U}{\partial x^\mu \partial x^\nu} - \Gamma^\lambda_{\mu\nu} r^\mu r^\nu \frac{\partial U}{\partial x^\lambda} + r^\mu \frac{\partial V}{\partial x^\mu} + W, \tag{3.16}$$

where U, V, W are functions of x^α only, one can verify (cf. Appendix B) that Eqs. (3.15) are satisfied, provided

$$\begin{gathered}
g^{\alpha\beta}\frac{\partial U}{\partial x^\alpha}\frac{\partial U}{\partial x^\beta} = 1\,; \\
g^{\alpha\beta}\frac{\partial V}{\partial x^\alpha}\frac{\partial U}{\partial x^\beta} = 0\,; \\
g^{\alpha\beta}\frac{\partial W}{\partial x^\alpha}\frac{\partial U}{\partial x^\beta} = -\frac{1}{4}g^{\alpha\beta}\frac{\partial V}{\partial x^\alpha}\frac{\partial V}{\partial x^\beta}.
\end{gathered} \tag{3.17}$$

After the first of these equations, the usual geodesic Hamilton-Jacobi equation, is solved, there is no difficulty in integrating successively the next two of them.

In the case of the natural geodesic deviation all these considerations can be performed on the level of the « dynamic » variational principle. To (2.22) and (2.24) corresponds then the following accessoric action

$$J^{(2)} = \int_{\tau_0}^{\tau_1} \mathscr{L}^{(d)}_2 d\tau \tag{3.18}$$

(cf. (3.5)). It leads to the second natural geodesic deviation equations

$$\frac{D^2w^\alpha}{d\tau^2} + R^\alpha{}_{\beta\gamma\delta}u^\beta w^\gamma u^\delta = (R^\alpha{}_{\beta\gamma\delta;\varepsilon} + R^\alpha{}_{\varepsilon\gamma\delta;\beta})u^\beta u^\gamma r^\delta r^\varepsilon + 4R^\alpha{}_{\beta\gamma\delta}\frac{Dr^\beta}{d\tau}u^\gamma r^\delta. \quad (3.19)$$

The action (3.18) is invariant under the transformation $w^\mu \mapsto w^\mu + \psi u^\mu$, with ψ being an arbitrary constant. This invariance is connected with the existence of the first integral of (3.19):

$$\frac{Dr^\alpha}{d\tau}\frac{Dr_\alpha}{d\tau} + u_\alpha\frac{Dw^\alpha}{d\tau} - R_{\mu\nu\rho\sigma}r^\mu u^\nu r^\rho u^\sigma = \text{const.} \quad (3.20)$$

As has been indicated in [5], in the case of the natural geodesic deviation the constant here must be taken to be equal to zero. The relation (3.20) then (*i. e.* for const. = 0) plays the role of the constraint condition selecting the natural geodesic deviation. The action (3.18), as in (2.24), is not any more invariant under the translation of the parameter τ.

The cannonical formalism connected with (3.18) is a nondegenerate one and its Hamilton-Jacobi equation can be brought into the same form to which one can reduce the system (3.10) after the elimination of the « non-dynamical » variables.

Also within the framework of the « dynamic » approach the most complete description of the problem is yielded by the unified action principle. The corresponding action has the form

$$I^{(2)} = \frac{1}{2}\int_{\tau_0}^{\tau_1}\left(u_\alpha\frac{Dw^\alpha}{d\tau} + \frac{Dr^\alpha}{d\tau}\frac{Dr_\alpha}{d\tau} - R_{\mu\nu\rho\sigma}r^\mu u^\nu r^\rho u^\sigma\right)d\tau \quad (3.21)$$

and it leads, when varied with respect to w^α, r^α and ξ^α, simultaneously to (2.21), (2.25) and (3.19).

The action (3.21) is invariant under the following transformations

$$\begin{aligned} \tau &\mapsto \tilde{\tau} = \tau + \varepsilon & (\varepsilon = \text{const.}) \\ r &\mapsto \tilde{r}^\alpha = r^\alpha + \kappa u^\alpha & (\kappa = \text{const.}) \\ w &\mapsto \tilde{w}^\alpha = w^\alpha + 2\frac{Dr^\alpha}{d\tau}\kappa + \psi u^\alpha & (\psi = \text{const.}) \end{aligned} \quad (3.22)$$

The first of them implies the conservation law (3.20) and the quantity ξ equal to the l. h. side of (3.20) can be interpreted as the relative energy of two neighbouring freely falling test bodies (in the approximation considered when higher deviations are neglected) (Another motivation of such an interpretation follows from the fact that (2.27) can be, due to introducing of new degrees of freedom represented by w^α, brought into the form $\frac{d\xi}{d\tau} = 0$; cf. [*10*]). In the case, however, when the proper times along the world lines of both particles are measured by two identical standard ideal clocks comoving with them, the total energy ε must, for reasons already pointed out in I, be equal to zero. The formula (3.20) can then be written in the form

$$\frac{1}{2}\frac{Dr^\alpha}{ds}\frac{Dr_\alpha}{ds}\tilde{\rho}^2 + \frac{1}{2}u_\alpha\frac{Dw^\alpha}{d\tau}\tilde{\rho}^2 = \frac{1}{2}R_{\alpha\beta\gamma\delta}r^\alpha u^\beta r^\gamma u^\delta\tilde{\rho}^2$$

($\tilde{\rho}$ is characterizing the initial distance; cf. I) and still can be interpreted as the equality of the relative kinetic and the relative potential energy of two neighbouring particles (in the approximation considered).

It is not difficult to verify that the second transformation in (3.22) implies: $u_\alpha\frac{Dr^\alpha}{d\tau} = \text{const.}$; and the third: $g_{\alpha\beta}u^\alpha u^\beta = \text{const.}$

The Hamiltonian formalism corresponding to (3.21) is a nondegenerate one. Its single Hamilton-Jacobi equation can be reduced (in the case of the natural parametrization) to Eqs. (3.17) in a way similar to that shown at the end of section 2.

APPENDIX A

From (2.19) one gets

$$\frac{\partial S^{(1)}}{\partial r^\alpha} = \frac{\partial U}{\partial x^\alpha}; \tag{A.1}$$

$$\frac{\partial S^{(1)}}{\partial x^\alpha} = r^\mu \frac{\partial}{\partial x^\alpha}\frac{\partial U}{\partial x^\mu} + \frac{\partial V}{\partial x^\alpha} = r^\mu \nabla_\alpha \frac{\partial U}{\partial x^\mu} + \Gamma_{\alpha}{}^{\sigma}{}_{\mu} r^\mu \frac{\partial U}{\partial x^\sigma} + \frac{\partial V}{\partial x^\alpha}; \tag{A.2}$$

where ∇_α stands for covariant differentiation.

Because of (A.1) the first of Eqs. (2.18) turns over into the first of Eqs. (2.20). Making in (A.2) use of

$$\nabla_\alpha \frac{\partial U}{\partial x^\mu} = \nabla_\mu \frac{\partial U}{\partial x^\alpha}, \tag{A.3}$$

one can bring the second of Eqs. (2.18) to the form

$$\frac{1}{2} r^\mu \nabla_\mu \left(g^{\alpha\beta} \frac{\partial U}{\partial x^\alpha}\frac{\partial U}{\partial x^\beta} \right) + g^{\alpha\beta} \frac{\partial U}{\partial x^\alpha}\frac{\partial V}{\partial x^\beta} = 0$$

which, for U being a solution of the geodesic Hamilton-Jacobi equation, is equivalent to the second equation in (2.20).

APPENDIX B

Eq. (3.16) can also be written as

$$S^{(2)}(x^\mu, r^\nu, w^\rho) = w^\mu \frac{\partial U}{\partial x^\mu} + r^\mu r^\nu \nabla_\mu \frac{\partial U}{\partial x^\nu} + r^\mu \frac{\partial V}{\partial x^\mu} + W. \tag{B.1}$$

From here

$$\frac{\partial S^{(2)}}{\partial w^\alpha} = \frac{\partial U}{\partial x^\alpha}; \tag{B.2}$$

$$\frac{\partial S^{(2)}}{\partial r^\alpha} = 2r^\mu \nabla_\alpha \frac{\partial U}{\partial x^\mu} + \frac{\partial V}{\partial x^\alpha}; \tag{B.3}$$

$$\begin{aligned}\frac{\partial S^{(2)}}{\partial x^\alpha} &= w^\mu \frac{\partial U}{\partial x^\mu} + \Gamma^\sigma_{\alpha\mu} w^\mu \frac{\partial U}{\partial x^\sigma} + r^\mu r^\nu \nabla_\alpha \nabla_\mu \frac{\partial U}{\partial x^\nu} \\ &\quad + 2r^\mu r^\nu \Gamma^\sigma_{\alpha\mu} \nabla_\sigma \frac{\partial U}{\partial x^\nu} + r^\mu \nabla_\alpha \frac{\partial V}{\partial x^\mu} + r^\mu \Gamma^\sigma_{\alpha\mu} \frac{\partial V}{\partial x^\sigma} + \frac{\partial W}{\partial x^\alpha}.\end{aligned} \tag{B.4}$$

The second of Eqs. (3.15), due to (B.2), implies for U the first of Eqs. (3.17). By means of (B.3), after taking into account (A.3), one can show, similarly as in Appendix A, that the third of Eqs. (3.15) turns over into the second of (3.17).

For the third term in (B.4), due to the Ricci identity and to (A.3), we can write

$$r^\mu r^\nu \nabla_\alpha \nabla_\mu \frac{\partial U}{\partial x^\nu} = r^\mu r^\nu \nabla_\mu \nabla_\nu \frac{\partial U}{\partial x^\alpha} - R^\sigma{}_{\nu\alpha\mu} \frac{\partial U}{\partial x^\sigma}.$$

Inserting then (B.2), (B.3) and (B.4) into the l. h. side of the fourth of Eqs. (3.15) one can bring it to the form

$$\begin{aligned} &g^{\alpha\beta}\left(\frac{\partial S^{(2)}}{\partial x^\alpha} - \Gamma^\sigma_{\alpha\rho} \frac{\partial S^{(2)}}{\partial r^\sigma} r^\rho - \Gamma^\sigma_{\alpha\rho} \frac{\partial S^{(2)}}{\partial w^\sigma} w^\rho\right) \frac{\partial S^{(2)}}{\partial w^\beta} \\ &\quad = \frac{1}{2} w^\mu \nabla_\mu\left(g^{\alpha\beta} \frac{\partial U}{\partial x^\alpha} \frac{\partial U}{\partial x^\beta}\right) + \frac{1}{2} r^\mu r^\nu \nabla_\mu \nabla_\nu\left(g^{\alpha\beta} \frac{\partial U}{\partial x^\alpha} \frac{\partial U}{\partial x^\beta}\right) \\ &\quad - g^{\alpha\beta} r^\mu r^\nu \left(\nabla_\mu \frac{\partial U}{\partial x^\alpha}\right)\left(\nabla_\nu \frac{\partial U}{\partial x^\beta}\right) - R^\alpha{}_\beta{}^\gamma{}_\delta \frac{\partial U}{\partial x^\alpha} r^\beta \frac{\partial U}{\partial x^\gamma} r^\delta \\ &\quad + g^{\alpha\beta} r^\mu \left(\nabla_\mu \frac{\partial V}{\partial x^\alpha}\right) \frac{\partial U}{\partial x^\beta} + g^{\alpha\beta} \frac{\partial W}{\partial x^\alpha} \frac{\partial U}{\partial x^\beta}. \end{aligned}$$

Substituting next (B.2) and (B.3) into the r. h. side of the fourth of Eqs. (3.15) we obtain the following equation

$$\begin{aligned} &\frac{1}{2} w^\mu \nabla_\mu\left(g^{\alpha\beta} \frac{\partial U}{\partial x} \frac{\partial U}{\partial x^\beta}\right) + \frac{1}{2} r^\mu r^\nu \nabla_\mu \nabla_\nu\left(g^{\alpha\beta} \frac{\partial U}{\partial x^\alpha} \frac{\partial U}{\partial x^\beta}\right) \\ &\quad + r^\mu \nabla_\mu\left(g^{\alpha\beta} \frac{\partial V}{\partial x^\alpha} \frac{\partial U}{\partial x^\beta}\right) + g^{\alpha\beta} \frac{\partial W}{\partial x^\alpha} \frac{\partial U}{\partial x^\beta} + \frac{1}{4} g^{\alpha\beta} \frac{\partial V}{\partial x^\alpha} \frac{\partial V}{\partial x^\beta} = 0 \end{aligned}$$

which due to the first two of Eqs. (3.17) implies the last of them.

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