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Section A :
Physique théorique.

An existence theorem for a massive spin one particle in an external tensor field

by

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ABSTRACT. — The system of equations describing a massive vector meson in a suitable external tensor field in four dimensional space-time is shown to be essentially equivalent to a symmetric hyperbolic system, to which standard existence theorems can be applied. The results cover the case of sufficiently weak external field and include the situation of non-causal propagation of signals.

1. INTRODUCTION

In recent years the relativistic equations describing particles of any spin interacting with external fields have raised an ever increasing interest [*1*], [*2*]. For higher spin new phenomena like the possibility that these equations can lose the property of propagating signals or can propagate at a speed greater than the speed of light were brought to attention [*3*], [*4*].

Here we wish to discuss a specific example of an equation in which some of the preceding features are present. In fact, strictly speaking, the arguments expressed in ref. [*3*] and [*4*] and in subsequent papers were not complete insofar that the existence of solutions to the equations was assumed but not proved. The appearance of these phenomena was later confirmed by the investigation of some simple cases of constant external fields [*5*], [*6*] and by a rigorous analysis performed for the theory of a spin one particle interacting with an external tensor field in two dimensional space-time [*5*].

The purpose of this paper is to establish an existence theorem for a massive vector meson in a suitable external tensor field in four dimensional space-time. This result covers also a situation of noncausal behaviour and extends to the physical space-time the results of ref. [*5*]. The methods of the proof are quite different. In ref. [*5*] the fact was used that the roots of the characteristic determinant of some equations related to the initial ones are simple, while here the difficulty that in four dimensions the roots are multiple will be overcomed by an energy estimate [*7*]. The initial system of equations will be essentially reduced to a symmetric hyperbolic system to which standard theorems can be applied. This kind of procedure is susceptible of being used in the case of other interactions of vector mesons and presumably in some other higher spin problems. It appears quite natural from the physical point of view because the presence of positive forms conserved or almost conserved helps the stability of the physical systems.

Other existence theorems for a spin 3/2 particle in external electromagnetic field were announced by Bellissard and Seiler [*8*]. They propose to use the method of Leray and Ohya [*9*] who established an existence theorem for hyperbolic systems with multiple characteristics in some Gevrey spaces. Here instead, because of the energy estimates, the natural function spaces for the solutions are the much easier to handle Sobolev spaces.

2. THE CAUCHY INITIAL VALUE PROBLEM

Let us consider a real vector field $V^{\mu}(x)$ in four dimensions coupled to an external symmetric real tensor field $T^{\mu\nu}$ through the following equations (1) :

$$E^{\nu} = 0 \tag{2.1}$$

where:

$$E^{\nu} \equiv \partial_{\mu}(\partial^{\mu}V^{\nu} - \partial^{\nu}V^{\mu}) + m^2(V^{\nu} + T^{\nu\rho}V_{\rho}). \tag{2.2}$$

Here m^2 is taken to be a strictly positive constant.

The system of equations (2.1) appears as a second order system of partial differential equations (PDE). We cannot analyze it directly because its characteristic determinant is identically zero due to the presence of constraints. To bypass this difficulty we will study an equivalent problem in which the constraints are clearly separated from the equations of motion. This reduction to a new system is described in the following and is different from the one performed in ref. [*4*].

(1) Our conventions are $g^{\mu\nu} = \mathrm{diag}\ (1, -1, -1, -1)$, $\varepsilon^{0123} = 1$, latin indices run from 1 to 3, greek indices run from 0 to 3.

Let us define the following expressions:

$$\begin{aligned} N^i &\equiv \partial_\mu G^{\mu i} + m^2(V^i + T^{i\rho}V_\rho) \\ F^i &\equiv \partial_\mu H^{\mu i} \\ L &\equiv \partial_\mu V^\mu + \partial_\mu T^{\mu\nu}V_\nu \\ S^i &\equiv \partial^0 V^i - \partial^i V^0 - G^{0i} \end{aligned} \tag{2.3}$$

where $G^{\mu\nu}(x)$ is an antisymmetric real tensor field and:

$$H^{\mu\nu} \equiv \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}G_{\rho\sigma}. \tag{2.4}$$

It is immediate to check that if we take:

$$G^{\mu\nu} = \partial^\mu V^\nu - \partial^\nu V^\mu \tag{2.5}$$

and if the field V^μ is a solution of the system (2.1), then V^μ satisfy also the following equations:

$$\begin{aligned} N^i &= 0 \\ F^i &= 0 \\ L &= 0 \\ S^i &= 0. \end{aligned} \tag{2.6}$$

Conversely we are going to show that if the $G^{\mu\nu}$ and the V^μ satisfy eqs (2.6) and if at a specific time, say $t = 0$, the relations

$$G_{ij} - (\partial_i V_j - \partial_j V_i) = 0 \tag{2.7}$$

and

$$\partial_i G^{i0} + m^2(V^0 + T^{0\rho}V_\rho) = 0 \tag{2.8}$$

hold, then V^μ is a solution of the system eqs (2.1). The proof is a consequence of the identities

$$\partial_0 \{ G_{ij} - (\partial_i V_j - \partial_j V_i) \} \equiv - \varepsilon_{0ijk}F^k - \partial_i S_j + \partial_j S_i \tag{2.9}$$

and

$$\partial_0 \{ \partial_i G^{i0} + m^2(V^0 + T^{0\rho}V_\rho) \} \equiv m^2 L - \partial_j N^j \tag{2.10}$$

which are easily derivable from the definitions (2.3). Therefore the expressions on the left hand side of eqs (2.7) and (2.8) are prime integrals of the system of PDE (2.6). Since, by hypothesis, they are taken to be zero at a specific time they will continue to be zero at all times. The assertion now follows immediately because of the identities:

$$E^0 \equiv \partial_i G^{i0} + m^2(V^0 + T^{0\rho}V_\rho) - \partial_i S^i \tag{2.11}$$

and

$$E^i \equiv N^i + \partial_\mu \{ (\partial^\mu V^i - \partial^i V^\mu) - G^{\mu i} \}. \tag{2.12}$$

In the preceding equivalence theorem we did not state in which space of functions we were working. The content of the existence theorem for the system (2.6), to be proved in the next paragraph, will fix these points.

Moreover we wanted to stress more the spirit than the technical details behind the idea of going to a problem different but, in some sense, equivalent to the initial one.

From now on we will analyze the system of PDE (2.6). To see on which surfaces the Cauchy initial value problem can be defined we write the characteristic matrix M of the system (2.6):

$$M=\begin{vmatrix} n_0 & 0 & 0 & -n_2 & 0 & n_3 & 0 & 0 & 0 & 0 \\ 0 & n_0 & 0 & n_1 & -n_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & n_0 & 0 & n_2 & -n_1 & 0 & 0 & 0 & 0 \\ -n_2 & n_1 & 0 & n_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -n_3 & n_2 & 0 & n_0 & 0 & 0 & 0 & 0 & 0 \\ n_3 & 0 & -n_1 & 0 & 0 & n_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & n_0+n_\mu T^\mu_0 & n_1+n_\mu T^\mu_1 & n_2+n_\mu T^\mu_2 & n_3+n_\mu T^\mu_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & n_1 & n_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & n_2 & 0 & n_0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & n_3 & 0 & 0 & n_0 \end{vmatrix} \tag{2.13}$$

The value of the determinant of the first six rows and six columns of M is $n_0^2(n^2)^2$ and the value of the determinant of the last four rows and four columns is $n_0^2(n^2 + n\cdot T\cdot n)$. The determinant of M is zero if the n^μ satisfy either the equation

$$n_0(n^2) = 0 \tag{2.14}$$

or the equation

$$n^2 + n\cdot T\cdot n = 0. \tag{2.15}$$

Given $\vec{n}$ the solutions for n^0 of eq. (2.15) are:

$$n^0 = \frac{-n_iT^{0i} \pm [(T^{0i}n_i)^2 + (1 + T^{00})(\delta_{ij} - T_{ij})n_in_j]^{\frac{1}{2}}}{1 + T^{00}}.$$

Therefore, as long as $1 + T^{00} \neq 0$ and as long as $(T^{0i}n_i)^2 + (1 + T^{00})(\delta_{ij} - T_{ij})n_in_j$ is $\geqslant 0$, the system (2.6) is hyperbolic because the solutions n^0 to the equation $|M| = 0$ are all real [*7*], [*10*]. It is not strictly hyperbolic because some of the roots are multiple.

In the next paragraph we will show that, under suitable assumptions, the system (2.6) is equivalent to a symmetric hyperbolic system.

3. EXISTENCE OF SOLUTIONS

A careful examination of the characteristic matrix (2.13) suggests a way of going to a new system whose matrix coefficients of the first derivatives are symmetric. More precisely we have the following:

THEOREM 1. — If matrix $(1 + T^{00})(\delta_{ij} - T_{ij})$ is strictly positive definite, then the system of PDE (2.6) is equivalent to a symmetric hyperbolic system, whose space-like surfaces have normal n^{μ} which satisfy the condition:

$$| n^0 | > \max \left(| \vec{n} |, \frac{- n_i T^{0i} \pm [(T^{0i}n_i)^2 + (1 + T^{00})(\delta_{ij} - T_{ij})n_i n_j]^{\frac{1}{2}}}{1 + T^{00}} \right)$$

Proof. — Let us define the matrix $B = (b_{ij})$ by:

$$b_{ij} = \delta_{ij} - T_{ij} \tag{3.1}$$

and let us put

$$\alpha = (I + T^{00})^{-1}. \tag{3.2}$$

The expression α in eq. (3.2) is well defined by hypothesis. It is then obvious that the system (2.6) is equivalent to the following new system:

$$\begin{aligned} N^i &= 0 \\ F^i &= 0 \\ \alpha L - \alpha T_{0i} S^i &= 0 \\ \alpha b_{ij} S^j &= 0. \end{aligned} \tag{3.3}$$

The characteristic matrix $C(n^0, \vec{n})$ of the system (3.3) differs from the matrix (2.13) only in the elements of the last four rows and four columns. We will write for brevity only these ones which concern the last four equations of the system (3.3):

$$\begin{vmatrix} n_0 + 2\alpha T^{0i} n_i & \alpha b_{1j} n_j & \alpha b_{2j} n_j & \alpha b_{3j} n_j \\ \alpha b_{1j} n_j & \alpha b_{11} n_0 & \alpha b_{12} n_0 & \alpha b_{13} n_0 \\ \alpha b_{2j} n_j & \alpha b_{21} n_0 & \alpha b_{22} n_0 & \alpha b_{23} n_0 \\ \alpha b_{3j} n_j & \alpha b_{31} n_0 & \alpha b_{32} n_0 & \alpha b_{33} n_0 \end{vmatrix} \tag{3.4}$$

The other elements can be read directly from the matrix (2.13). It is clear that $C(n^0, \vec{n})$ is symmetric for all n^{μ} and that $C(n^0, \vec{0})$ is positive for $n^0 > 0$. This establishes that the system is symmetric hyperbolic. The space-like surfaces, on which the initial value problem is well posed, are defined by the condition that $C(n^0, \vec{n}) > 0$, where n^{μ} has to be interpreted as the normal to the surface and n^0 is taken to be positive. It is easy to verify that the part of the matrix $C(n^0, \vec{n})$ formed by the first six rows and columns is positive if and only if $n^0 > | \vec{n} |$. A simple computation, as can be seen from eq. (3.4), yields, for the determinant of the last four rows and columns, the following expression:

$$(n^0)^2 \, [\det(\alpha B)] [n_0^2 + 2\alpha T^{0i} n_i - \alpha b_{ij} n_i n_j]. \tag{3.5}$$

In order for the matrix (3.4) to be positive, its determinant must be positive, which implies, for $n^0 > 0$, the inequality:

$$n^0 > - \alpha T^{0i} n_i + [(\alpha T^{0i} n_i)^2 + \alpha b_{ij} n_i n_j]^{\frac{1}{2}}. \tag{3.6}$$

This inequality is also sufficient for the positivity of the matrix (3.4). In fact, because of the restriction (3.6), the matrix (3.4) is greater than the matrix:

$$n_0^{-1} \begin{vmatrix} \alpha b_{ij}n_in_j & \alpha b_{1j}n_jn_0 & \alpha b_{2j}n_jn_0 & \alpha b_{3j}n_jn_0 \\ \alpha b_{1j}n_in_0 & & & \\ \alpha b_{2j}n_jn_0 & & n_0^2\alpha \mathrm{B} & \\ \alpha b_{3j}n_jn_0 & & & \end{vmatrix}, \tag{3.7}$$

which is positive because $\alpha\mathrm{B}$ is positive. It is then the biggest number between $|\vec{n}|$ and the expression on the RHS of eq. (3.6) which decides the positivity of $\mathrm{C}(n^0, \vec{n})$.

Turning now to the existence of solutions we have the following.

THEOREM 2. — Let the matrix $(1 + \mathrm{T}^{00})(\delta_{ij} - \mathrm{T}_{ij})$ be strictly positive. Let all components of the external tensor field be infinitely differentiable functions of space-time. Then the system of PDE (3.3) has, for infinitely differentiable initial data, a unique infinitely differentiable solution.

Proof. — The proof is now obvious by theorem 1 and by a standard theorem on hyperbolic symmetric first order systems of PDE (See ref. [*7*], p. 669, and ref. [*10*], p. 89).

More detailed results could be stated by looking for solutions in suitable Sobolev spaces.

Concerning the propagation of signals for the system (3.3) by standard results we know that the maximal speed of propagation v is given by the expression:

$$v = \max\left(1, -\alpha\widehat{n}_i\mathrm{T}^{0i} + [(\widehat{n}_i\mathrm{T}^{0i}\alpha)^2 + \alpha b_{ij}\widehat{n}_i\widehat{n}_j]^{\frac{1}{2}}\right). \tag{3.8}$$

Under suitable conditions for $\mathrm{T}^{\mu 0}$, still satisfying the assumptions of Theorem 2, we can easily reproduce a situation in which v is greater than one.

The existence theorem extends now to the system (2.6) and by the equivalence theorem to the initial system (2.1) provided we impose, at a fixed time, the conditions (2.7) and (2.8). The maximal speed of the signals remains the same even after we require eqs (2.7) and (2.8) to be satisfied, because the expressions at the LHS of eqs (2.7) and (2.8) are constants of motion for each point of space. They do not propagate at all and therefore they cannot change the maximal velocity of propagation of disturbances of the system (2.6).

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