

ANNALES DE L'I. H. P., SECTION A

J. T. LEWIS

L. C. THOMAS

On the existence of a class of stationary quantum stochastic processes

Annales de l'I. H. P., section A, tome 22, n° 3 (1975), p. 241-248

http://www.numdam.org/item?id=AIHPA_1975__22_3_241_0

© Gauthier-Villars, 1975, tous droits réservés.

L'accès aux archives de la revue « Annales de l'I. H. P., section A » implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>

On the existence of a class of stationary quantum stochastic processes

by

J. T. LEWIS

School of Theoretical Physics, Dublin Institute for Advanced Studies

and

L. C. THOMAS

Department of Pure Mathematics, University College of Swansea

ABSTRACT. — Given a classical stationary stochastic process we construct a corresponding quantum stochastic process. As an example we use the Ornstein-Uhlenbeck process to construct the quantum process whose existence was suggested by the work of Ford, Kac and Mazur.

RÉSUMÉ. — Donné un processus aléatoire stationnaire classique, on construit un processus aléatoire quantique correspondant. Par exemple on se sert du processus Ornstein-Uhlenbeck pour construire le processus quantique dont l'existence a été proposée par Ford, Kac et Mazur.

1. INTRODUCTION

The work of Ford, Kac and Mazur [1] suggests the existence of a quantum-mechanical analogue of the Ornstein-Uhlenbeck stochastic process [2]. In this paper we prove a theorem which establishes the existence of a class of such quantum stochastic processes.

Recall that a classical stochastic process $\{X_t : -\infty < t < \infty\}$ is a family of random variables on a probability space Σ with a probability

measure $\mathbb{P}$ given on a field $\mathcal{B}$ of subsets of Σ ; the expectation $\mathbb{E}Y$ of a random variable Y on Σ is given by

$$\mathbb{E}Y = \int_{\Sigma} Y(\sigma) \mathbb{P}(d\sigma). \quad (1)$$

We shall be interested in processes for which for all t

$$\mathbb{E}X_t = 0 \quad , \quad \mathbb{E}X_t^2 < \infty \quad (2)$$

so that $t \rightarrow X_t$ can be regarded as a curve in the real Hilbert space $L^2(\Sigma, \mathbb{P})$ and such a process will be called stationary if the correlation $\mathbb{E}X_t X_{t+\tau}$ depends only on the time-difference τ . The history $\mathfrak{H}^X$ of the process $t \mapsto X_t$ is the closed subspace of $L^2(\Sigma, \mathbb{P})$ spanned by the X_t :

$$\mathfrak{H}^X = V \{ X_t : -\infty < t < \infty \}. \quad (3)$$

It follows that for a stationary process $t \mapsto X_t$ there exists a group $\{ U_t : -\infty < t < \infty \}$ of unitary operators on its history $\mathfrak{H}^X$ such that for all t, τ

$$X_{t+\tau} = U_{\tau} X_t. \quad (4)$$

Thus a classical process involves three objects: an underlying space Σ , a probability measure $\mathbb{P}$ and a family $\{ X_t : -\infty < t < \infty \}$ of random variables. If in addition the process is stationary, its time-development is given by a family of operators which preserve expectations. This suggests the following tentative definition of a quantum stochastic process:

A quantum stochastic process is a family of self-adjoint operators $\{ Q_t : -\infty < t < \infty \}$ on a Hilbert space $\mathfrak{H}$ with a state vector Ω ($\|\Omega\| = 1$). The expectations are given by

$$\mathbb{E}Q_t = \langle \Omega, Q_t \Omega \rangle, \quad (5)$$

$$\mathbb{E}Q_t \circ Q_s = \frac{1}{2} \langle \Omega, (Q_t Q_s + Q_s Q_t) \Omega \rangle. \quad (6)$$

The symmetrized expectation (6) is taken because in general Q_t does not commute with Q_s and so $Q_t Q_s$ is not self-adjoint. For a discussion of quantum correlations see [1] and [3].

A quantum stochastic process $t \mapsto Q_t$ is said to be stationary if there exists a one-parameter group $\{ V_t : -\infty < t < \infty \}$ of unitary operators on $\mathfrak{H}$ such that for all t, τ

$$Q_{t+\tau} = V_{\tau} Q_t V_{\tau}^{-1} \quad (7)$$

and

$$V_{\tau} \Omega = \Omega. \quad (8)$$

It then follows from (5) and (6) that for all t, τ

$$\mathbb{E}Q_t = \mathbb{E}Q_0 \quad (9)$$

and

$$\mathbb{E}Q_t \circ Q_{t+\tau} = \mathbb{E}Q_0 \circ Q_{\tau} \quad (10)$$

so that the correlation depends only on the time-difference τ .

2. DESCRIPTION OF A CLASS OF QUANTUM PROCESSES

It is well-known (for example Doob [4]) that if $\tau \mapsto \gamma(\tau)$ is the correlation of a stationary stochastic process $t \mapsto X_t$ so that

$$\gamma(\tau) = \mathbb{E}X_t X_{t+\tau} \quad (11)$$

then γ is a function of positive type, and that conversely if γ is a function of positive type there exists a stochastic process $(X_t, \Sigma, \mathbb{P})$ such that (11) holds. Given a correlation function γ we define for each $\mu > 0$ a quantum modification γ_μ such that for all τ

$$\lim_{\mu \rightarrow 0+} \gamma_\mu(\tau) = \gamma(\tau), \quad (12)$$

and we prove the existence of a stationary quantum stochastic process $(Q_t, \mathfrak{H}, \Omega)$ such that

$$\gamma_\mu(\tau) = \mathbb{E}Q_t \circ Q_{t+\tau}. \quad (13)$$

One motivation for this is that in the case in which γ is the correlation function of the Ornstein-Uhlenbeck x -process, the quantum modification $\gamma_\hbar$ is the correlation function obtained by Ford, Kac and Mazur. The Ornstein-Uhlenbeck process arises as the stationary solution of a Langevin equation which describes the motion of an oscillator of frequency ω_0 and frictional constant f coupled to a heat-bath at inverse temperature β . Then

$$\gamma(\tau) = \frac{e^{-f|\tau|/2}}{\beta\omega_0^2} \left(\cos \tilde{\omega}\tau + \frac{f}{2\omega_0} \sin \tilde{\omega}|\tau| \right). \quad (14)$$

Since the process X_t is stationary the oscillator is in thermal equilibrium with the heat-bath and so we can regard the inverse temperature β as describing the stationary state of the process. Indeed the joint probability distribution of the position and momentum of the oscillator is just the Maxwell-Boltzmann equilibrium distribution at inverse temperature β . A quantum-analogue of the Maxwell-Boltzmann condition is the Kubo-Martin-Schwinger boundary condition; we use it as formulated by Haag, Hugenholtz and Winnink [5]. The pair (A, B) satisfy the KMS-boundary condition at inverse temperature β if for every pair (A, B) of bounded self-adjoint operators on $\mathfrak{H}$ there exists a function $z \mapsto F_{AB}(z)$ analytic and uniformly bounded on the strip $-\beta\mu \leq \text{Im}z \leq 0$ such that for all t

$$\begin{aligned} F_{AB}(t) &= \langle \Omega, A_t B \Omega \rangle \\ F_{AB}(t - i\beta\mu) &= \langle \Omega, B A_t \Omega \rangle \end{aligned} \quad (15)$$

where

$$A_t = V_t A V_{-t} \quad (16)$$

Suppose $\tau \mapsto \gamma(\tau)$ is a positive definite function given by

$$\gamma(\tau) = \gamma(0) \int_0^\infty \cos \omega \tau dG(\omega). \quad (17)$$

Then for $\mu > 0$ the quantum modification at inverse temperature β is defined to be

$$\gamma_\mu(\tau) = \gamma(0) \int_0^\infty b_\mu(\omega) \cos \omega \tau dG(\omega) \quad (18)$$

where

$$b_\mu(\omega) = \frac{\beta \mu \omega}{2} \coth \frac{\beta \mu \omega}{2}. \quad (19)$$

Notice that, because of the inequality $x \coth x < 1 + x$ for $x > 0$, the function γ_μ is bounded provided $\int_0^\infty \lambda dG(\lambda) < \infty$.

THEOREM. — *Let $(X_t, \Sigma, \mathbb{P})$ be a real stationary stochastic process with correlation function*

$$\gamma(\tau) = \gamma(0) \int_0^\infty \cos \omega \tau dG(\omega)$$

such that $\Delta G(0) = 0$ and $\int_0^\infty \omega dG(\omega) > \infty$.

Then there exists a stationary quantum stochastic process $(Q_t, \mathfrak{H}, \Omega)$ with $Q_t = V_t Q_0 V_t^{-1}$ such that

$$(i) \quad \langle \Omega, \exp \left\{ i \sum_{j=1}^n k_j Q_{t_j} \right\} \Omega \rangle = \exp \left\{ -\frac{1}{2} \sum_{j,l=1}^n k_j k_l \gamma_\mu(t_j - t_l) \right\} \quad (20)$$

where γ_μ is the quantum modification (18) of γ at inverse temperature β .

$$(ii) \quad \exp(iQ_s) \exp(iQ_t) = \exp\{i\mu\beta\gamma'(t-s)\} \exp(iQ_t) \exp(iQ_s). \quad (21)$$

(iii) The pair (V_t, Ω) satisfies the KMS boundary condition (15) at inverse temperature β .

Remark. — The properties (i) and (ii) are precise formulations of the more familiar looking

$$(i)' \quad \mathbb{E} Q_s \circ Q_t = \gamma_\mu(t-s), \quad (22)$$

$$(ii)' \quad [Q_s, Q_t] = i\mu\beta\gamma'(s-t)1. \quad (23)$$

The theorem is proved by constructing a process with the claimed properties.

3. THE CONSTRUCTION OF THE QUANTUM PROCESS

The construction proceeds in three parts. First we prove a lemma about classical processes which states that a real process can be complexified in

such a way that time-translation acts as a unitary group with positive generator. Next we use the Araki-Woods construction [6], [7] to construct a quantum process and finally we verify the properties claimed for the process.

LEMMA. — *Let $(X_t, \Sigma, \mathbb{P})$ be the real process of the theorem. Then there exists a complex process $(\xi_t, \Sigma, \mathbb{P})$ and a one-parameter unitary group*

$$U_t = \int_{\mathbb{R}} e^{it\lambda} E(d\lambda) \quad (24)$$

on the complex Hilbert space $\mathfrak{H}^\xi$ such that

$$(i) \quad X_t = \sqrt{2} \Re e \xi_t, \quad (25)$$

$$(ii) \quad \xi_t = U_t \xi_0, \quad (26)$$

$$(iii) \quad U_t = \exp(iC^2 t), \quad (27)$$

with $C \geq 0$,

$$(iv) \quad \|E(d\lambda)\xi_0\|^2 = 2\gamma(0)dG(\lambda). \quad (28)$$

Proof. — By Bôchner's theorem there exists a distribution function F such that

$$\gamma(\tau) = \gamma(0) \int_{-\infty}^{\infty} e^{it\omega} dF(\omega). \quad (29)$$

For $\omega \geq 0$ define $G(\omega)$ by

$$G(\omega) = \begin{cases} F(\omega) - F(-\omega - 0) & , \quad \omega > 0, \\ 0 & , \quad \omega = 0 \end{cases} \quad (30)$$

Then

$$\gamma(\tau) = \gamma(0) \int_0^\infty \cos \omega \tau dG(\omega) \quad (31)$$

since γ is real. Further, X_t has a representation

$$X_t = \int_{-\infty}^{\infty} e^{it\lambda} \zeta(d\lambda) \quad (32)$$

as the Fourier transform of a stochastic process ξ_λ with orthogonal increments such that

$$\mathbb{E} |\zeta(d\lambda)|^2 = \gamma(0)dF(\lambda). \quad (33)$$

Let $\tilde{X}_t$ be the Hilbert transform of X_t :

$$\tilde{X}_t = \int_{-\infty}^{\infty} e^{it\lambda} g(\lambda) \zeta(d\lambda) \quad (34)$$

where

$$g(\lambda) = \begin{cases} i & , \quad \lambda < 0, \\ 0 & , \quad \lambda = 0, \\ -i & , \quad \lambda > 0. \end{cases} \quad (35)$$

Then we have

$$\mathbb{E}\tilde{X}_t\tilde{X}_{t+\tau} = \gamma(\tau), \quad (36)$$

$$\mathbb{E}X_t\tilde{X}_{t+\tau} = \tilde{\gamma}(\tau) = \gamma(0) \int_0^\infty \sin \lambda \tau dG(\lambda). \quad (37)$$

Put

$$\xi_t = \frac{1}{2^{1/2}} (X_t + i\tilde{X}_t) \quad (38)$$

so that

$$\mathbb{E}\bar{\xi}_t\xi_{t+\tau} = \gamma(\tau) + i\bar{\gamma}(\tau). \quad (39)$$

Then there exists a unitary operator

$$U_t = \int_{-\infty}^\infty e^{it\lambda} E(d\lambda) \quad (40)$$

on $\mathfrak{H}^\xi$ such that

$$\xi_t = U_t \xi_0 \quad (41)$$

and so

$$\int_{-\infty}^\infty e^{it\lambda} \|E(d\lambda)\xi_0\|^2 = \gamma(0) \int_0^\infty e^{it\lambda} dG(\lambda). \quad (42)$$

Thus the spectrum of the generator of U_t is positive and

$$\|E(d\lambda)\xi_0\|^2 = \gamma(0)dG(\lambda). \quad (43)$$

Put

$$C = \int_0^\infty \lambda^{1/2} E(d\lambda) \quad (44)$$

then

$$U_t = e^{itC^2}, \quad C \geq 0. \quad (45)$$

For the Araki-Woods construction we follow Chaiken [7]. A Weyl system over a complex vector space $\mathcal{V}$ is a map W from $\mathcal{V}$ to the unitary operators $\mathcal{U}(\mathfrak{H})$ on a Hilbert space $\mathfrak{H}$ together with a vector Ω in $\mathfrak{H}$ such that

$$\bigvee \{ W(v)\Omega : v \in \mathcal{V} \} = \mathfrak{H},$$

which satisfies

$$(i) \quad W(v_1)W(v_2) = \exp \left\{ -\frac{i}{2} \mathcal{J}m \langle v_1, v_2 \rangle \right\} W(v_1 + v_2), \quad (46)$$

(ii) for all $v \in \mathcal{V}$ the map $\lambda \mapsto W(\lambda v)$ is continuous from $\mathbb{C}$ to $\mathcal{U}(\mathfrak{H})$ in the strong operator topology.

It follows from (ii) by Stone's theorem that for each $v \in \mathcal{V}$ there exists a self-adjoint operator $R(v)$ such that

$$W(v) = \exp \{ iR(v) \}. \quad (47)$$

A Weyl system is determined up to equivalence by its generating functional

$$\mu(v) = \langle \Omega, W(v)\Omega \rangle. \quad (48)$$

THEOREM (Chaiken [7]). — *Let $\mathcal{V}$ be a complex Hilbert space. Let T be a self-adjoint operator on $\mathcal{V}$ with domain $\mathcal{D}(T)$ and such that $Y \geq 1$. Then there exists a Weyl system (W, ξ, Ω) over $\mathcal{D}(T)$ with generating functional*

$$\mu(v) = \exp \left(-\frac{1}{4} \|Tv\|^2 \right). \quad (49)$$

Proof. — Let $A = \frac{1}{2}(T^2 - 1)$ and let J be a conjugation on $\mathcal{V}$ which commutes with A . Let $\mathcal{M}$ be the closure of the range of $A^{1/2}$; let $(W_1, \mathcal{I}(\mathcal{V}), \Omega_1)$ be the Fock-Cook Weyl system over $\mathcal{M}$ and let $(W_2, \mathcal{I}(\mathcal{M}), \Omega_2)$ be the Fock-Cook Weyl system over $\mathcal{M}$. For $v \in \mathcal{D}(T)$ put

$$W(v) = W_1((1 + A)^{1/2}v) \otimes W_2(A^{1/2}Jv) \quad (50)$$

on

$$\xi = \mathcal{I}(\mathcal{V}) \otimes \mathcal{I}(\mathcal{M})$$

with

$$\Omega = \Omega_1 \otimes \Omega_2.$$

Then direct calculation shows that

$$\langle \Omega, W(v)\Omega \rangle = \exp \left(-\frac{1}{4} \|Tv\|^2 \right). \quad (51)$$

The methods of Araki-Woods [6] show that Ω is cyclic for W and that W is a factor representation; it is irreducible if and only if $A = 0$.

With the notation of the lemma take, in the Araki-Woods construction, $\mathcal{V} = \xi^\xi$, $v_t = (\mu\beta)^{1/2}CU_{t\zeta_0}$, and let T be the unique positive operator such that

$$T^2 = \coth \frac{\mu\beta}{2} C^2. \quad (52)$$

Then $T^2 > 1$ since $\coth x < 1$ for $x > 0$, and $C\zeta_0 \in \mathcal{D}(T)$ since

$$\begin{aligned} \|TC\zeta_0\|^2 &= \int_0^\infty \lambda \coth \frac{\mu\beta\lambda}{2} \|E(d\lambda)\zeta_0\|^2 \\ &= \gamma(0) \int_0^\infty \lambda \coth \frac{\mu\beta\lambda}{2} dG(\lambda) = \frac{2}{\mu\beta} \gamma_\mu(0) < \infty. \end{aligned} \quad (53)$$

A similar calculation gives

$$\operatorname{Re} \langle Tv_s, Tv_t \rangle = 2\gamma_\rho(t - s), \quad (54)$$

and

$$\operatorname{Im} \langle v_s, v_t \rangle = -\mu\beta\gamma'(t - s). \quad (55)$$

Putting

$$Q_t = R(v_t) \quad (56)$$

we check that (20) and (21) follow from (46) and (49).

Let $(W, \mathcal{I}(\mathcal{K}), \Omega)$ be the Fock-Cook Weyl system over a Hilbert space $\mathcal{K}$;

then to each self-adjoint operator B on $\mathcal{H}$ there corresponds a self-adjoint operator $\Gamma(B)$ on $\mathcal{I}(\mathcal{H})$ such that

$$W(e^{iBt}v) = e^{i\Gamma(B)t}W(v)e^{-i\Gamma(B)t} \quad (57)$$

for all t and all v in $\mathcal{H}$. It is straightforward to check that

$$V_t \exp(i\lambda Q_s) V_t^{-1} = \exp(i\lambda Q_{s+t}) \quad (58)$$

and

$$V_t \Omega = \Omega \quad (59)$$

where

$$V_t = e^{i\Gamma(C^2)} \otimes e^{-i\Gamma(C^2)} \quad (60)$$

so that the conditions (7) and (8) are satisfied. It remains to check that (V_t, Ω) satisfy the KMS boundary condition at inverse temperature β . It is enough to consider the function defined for $v, w \in \mathcal{D}(T)$ by

$$\begin{aligned} & F_{vw}(t + iy) \\ &= \exp \left[-\frac{1}{4} \left\{ \|T(U_t e^{-yC^2}v + w)\|^2 + \langle U_t e^{-yC^2}v, w \rangle - \langle w, U_t e^{-yC^2}v \rangle \right\} \right] \end{aligned} \quad (61)$$

which is analytic and bounded in the strip $-\beta\mu \leq y \leq 0$.

Using (52), we find that

$$T^2(e^{\mu\beta C^2} - 1) = (e^{\mu\beta C^2} + 1) \quad (62)$$

which together with (46) and (49) gives

$$F_{vw}(t) = \langle \Omega, W(U_t v) W(w) \Omega \rangle, \quad (63)$$

$$F_{vw}(t - i\mu\beta) = \langle \Omega, W(w) W(U_t v) \Omega \rangle, \quad (64)$$

so that the KMS boundary condition is satisfied and the proof of the theorem is complete.

REFERENCES

- [1] G. W. FORD, M. KAC and P. MAZUR, *J. Math. Phys.*, t. **6**, 1965, p. 504.
- [2] G. E. UHLENBECK and L. S. ORNSTEIN, *Phys. Rev.*, t. **36**, 1930, p. 823.
- [3] G. W. FORD and J. T. LEWIS, *in course of preparation*.
- [4] J. L. DOOB, *Stochastic Processes*, Wiley and Sons, New York, 1953.
- [5] R. HAAG, N. HUGENHOLTZ and M. WINNINK, *Commun. Math. Phys.*, t. **5**, 1967, p. 215.
- [6] H. ARAKI and E. J. WOODS, *J. Math. Phys.*, t. **4**, 1963, p. 637.
- [7] J. M. CHAIKEN, *Commun. Math. Phys.*, t. **8**, 1968, p. 164.

(Manuscrit reçu le 2 septembre 1974)