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## On the linearized Einstein-Cartan theory

by

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**ABSTRACT.** — This paper is concerned with the linear approximation of the Einstein-Cartan theory, this being considered the approximation of weak metric field and weak torsion. The linearized field equations are applied to the simple case of the Weysenhoff fluid. It is shown that a static sphere of the Weysenhoff fluid may serve as a source of the Kerr metric (in the linear approximation). Finally, junction conditions across the surface of spin discontinuity are shortly discussed.

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### 1. INTRODUCTION

During the past few years there has been a renewal of interest in the Einstein-Cartan theory, which is a slight modification of the classical, Einsteinian theory of gravitation. A complete review of this subject can be found in the papers by Hehl [5] and by Trautman [8]. In this theory, gravitational field is described by two tensor fields, namely the metric  $g_{ij}$  and the torsion  $Q^i_{jk} = \Gamma^i_{kj} - \Gamma^i_{jk}$  ( $\Gamma^i_{jk}$  are the coefficients of the linear, metric connection  $\Gamma$  with respect to a holonomic frame). The Einstein tensor of

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the connection  $\Gamma$  is proportional to the canonical energy-momentum tensor of matter  $t_j^i$ , similarly as in the classical theory of gravitation,

$$(1) \quad R_j^i - \frac{1}{2} \delta_j^i R = \frac{8\pi G}{c^4} t_j^i.$$

The torsion tensor of  $\Gamma$  is determined by the spin tensor of matter  $s_{jk}^i$ ,

$$(2) \quad Q_{jk}^i = \frac{8\pi G}{c^3} \left( s_{jk}^i + \frac{1}{2} \delta_j^i s_k - \frac{1}{2} \delta_k^i s_j \right), \quad s_i := s_{ii}^i.$$

Taking into account the metric condition imposed on  $\Gamma$  and equation (2) one may express the connection coefficients by the Christoffel symbols and the spin tensor,

$$(3) \quad \Gamma_{jk}^i = \{ \}_{jk}^i - \frac{4\pi G}{c^3} (s_{jk}^i + s_{jk}^i + s_{kj}^i - \delta_k^i s_j + g_{jk} s^i).$$

If we substitute (3) into equation (1) we obtain the single-written equation

$$(4) \quad \tilde{R}_{ij} - \frac{1}{2} g_{ij} \tilde{R} = \frac{8\pi G}{c^4} t_{ij} + \frac{4\pi G}{c^3} \tilde{\nabla}_k (s_{ij}^k + s_{ij}^k + s_{ji}^k) - \left( \frac{4\pi G}{c^3} \right)^2 (2s_{ijk} s^k + 2s_{ikm} s^{mk}_j + s_{ikm} s_j^{km} + g_{ij} \left( s_k s^k - s_{klm} s^{mlk} - \frac{1}{2} s_{klm} s^{klm} \right))$$

instead of the system of equations (1) and (2). The symbol  $\tilde{\phantom{x}}$  denotes objects related to the Riemannian connection associated with the metric tensor  $g_{ij}$ . A similar formula has been obtained by Hehl [5].

Since, due to equation (4), the torsion was eliminated, one may use this equation to compare the Einstein-Cartan theory with the classical theory of gravitation. One sees that metric of space-time depends not only on energy-momentum distribution but also on spin distribution.

Let us now apply equation (4) to estimate the influence of spin in the case of the Weyssenhoff fluid [4] [9], its matter tensors being defined by

$$t_i^k = u^k h_i - p \delta_i^k, \\ s_{ij}^k = u^k s_{ij}, \quad u^k s_{kj} = 0.$$

In these formulae the vector field  $u^i$  is the velocity vector of the fluid,  $h^i$  is the vector of its enthalpy density,  $p$  is its pressure, while  $s_{ij}$  is the tensor of spin density in the matter rest-frame. One can write the vector of enthalpy density with the help of the Bianchi identities in the form

$$h_i = (e + p)u_i + cu^j u^k \tilde{\nabla}_k s_{ij},$$

where  $e = u^i u^j t_{ij}$  is energy density in the matter rest-frame.

In the present case equation (4) takes on the form

$$(5) \quad \tilde{R}_{ij} \frac{1}{2} - g_{ij} \tilde{R} \\ = \frac{8\pi G}{c^4} \left( \left( e + p - \frac{4\pi G}{c^2} s^2 \right) u_i u_j - \left( p - \frac{2\pi G}{c^2} s^2 \right) g_{ij} - c(g^{kl} + u^k u^l) \tilde{\nabla}_k s_{l(i} u_{j)} \right),$$

where  $s^2 := \frac{1}{2} s_{ij} s^{ij}$ . One sees that the square term in spin appearing in the above equation contribute to the effective energy density and pressure,

$$e_{\text{eff}} = e - \frac{2\pi G}{c^2} s^2, \\ p_{\text{eff}} = p - \frac{2\pi G}{c^2} s^2.$$

The square term in spin behaves as an effective repulsive force. The repulsion can become important if the quantity  $\frac{2\pi G}{c^2} s^2$  is of the same order as the energy density. Let us assume that the energy density  $e$  and the spin density  $s$  are proportional to the concentration of spin alligned nucleons  $n$ :  $e \cong nmc^2$ ,  $s \cong \frac{1}{2} n\hbar$  ( $m$  is the mass of nucleon). In this case, both the terms appearing in the expression for the effective energy  $e_{\text{eff}}$  are equal, when  $n \cong \frac{2mc^4}{\pi G \hbar^2} \cong 10^{79} \text{ cm}^{-3}$ . Such great concentration may be present in collapsing stars or at the early stages of the Universe evolution only. Due to the square terms in spin there is the possibility to avert the singularities of the Friedmann type in homogeneous cosmological models [7].

In those models the linear term in the spin derivatives occuring on the right-hand side of equation (5) vanishes. It may be essential for sufficiently large gradient of spin density. We estimate its effect by replacing the derivative of spin density  $\tilde{\nabla}_s$  by  $s/R$ , where  $R$  denotes the typical size of inhomogeneity. To have this term comparable with the energy term,  $R$  must be of the order  $\hbar/mc \cong 10^{-13} \text{ cm}$ , if the ratio of the energy density to the spin density is  $mc^2/\hbar$ . Thus spin may have an essential influence on the space-time geometry in the regions of the size of elementary particles.

## 2. THE LINEAR APPROXIMATION

Let us now study the linear approximation of the Einstein-Cartan theory. In this theory there are various possible approximations, namely the

approximations of: (i) weak metric field, (ii) weak torsion, (iii) weak metric field and weak torsion. Consider the weak metric field approximation. In this case exists a coordinate system in which  $g_{ij} = \eta_{ij} + \lambda h_{ij} + O(\lambda^2)$ , where  $\eta_{ij} = \text{diag}(+1, -1, -1, -1)$  is the Minkowskian metric matrix and  $\lambda h_{ij}$  is a small perturbation. In the classical theory of gravitation the expansion of metric with respect to the parameter  $\lambda$  is equivalent to its expansion with respect to the gravitational constant  $G$ . In the Einstein-Cartan theory the situation is slightly different, the expansion with respect to the gravitational constant corresponds to the simultaneous expansion of the metric tensor  $g_{ij}$  and the torsion tensor  $Q^i{}_{jk}$ . In turn, (iii) is the approximation of weak sources.

In the first-order approximation of weak sources equation (4) becomes

$$(6) \quad -\square \psi_{ij} + \partial_k(\partial_i \psi_j^k + \partial_j \psi_i^k - \eta_{ij} \partial_i \psi^{kl}) \\ = \frac{16\pi G}{c^4} t_{ij} + \frac{8\pi G}{c^3} \partial_k(s^{kij} + s_{ij}^k + s_{ji}^k), \quad \lambda = 1.$$

We have introduced the notation  $\psi_j^i = h_j^i - \frac{1}{2} \delta_j^i h_k^k$  and  $\square = \eta^{ij} \partial_i \partial_j$ . This equation has the simplest form in a harmonic coordinate system, which is defined by the de Donder condition  $\partial_i(\sqrt{-g} g^{ij}) = 0$  [2], taking on (in the linear approximation) the form

$$(7) \quad \partial_i \psi^{ij} = 0.$$

Let us note that the generalization of the Fock form of the harmonic condition [3] to the Einstein-Cartan theory reads

$$g^{jk} \Gamma_{jk}^i = -\frac{4\pi G}{c^3} s^i.$$

By virtue of (7), equation (6) becomes

$$(8) \quad -\square \psi_{ij} = \frac{16\pi G}{c^4} T_{ij},$$

where

$$T_{ij} = t_{ij} + \frac{c}{2} \partial_k(s^{kij} + s_{ij}^k + s_{ji}^k)$$

is the symmetric energy-momentum tensor associated by the method of Belinfante [4] to the canonical one  $t_j^i$ . On the other hand,  $T_{ij}$  is the linear approximation of the symmetric energy-momentum tensor deduced from Hehl's variational principle [5].

Due to the harmonic condition (7), equation (8) implies the differential conservation laws for the canonical energy-momentum tensor  $t_j^i$  and the symmetric energy-momentum tensor  $T_j^i$

$$\partial_i t_j^i = 0, \quad \partial_i T_j^i = 0,$$

and for the angular momenta

$$\partial_k \left( \frac{1}{c} x_i t_j^k - \frac{1}{c} x_j t_i^k + s^k_{ij} \right) = 0, \quad \partial_k \left( \frac{1}{c} x_i \Gamma_j^k - \frac{1}{c} x_j \Gamma_i^k \right) = 0.$$

The divergence of the symmetric energy-momentum tensor  $t_{(ij)}$  obtained from Trautman's variational principle [8] is not equal to zero,

$$\partial_i t^{(i)}_{j)} = \frac{c}{2} \partial_i \partial_k s^{ik}_{j}.$$

### 3. SPHERE OF WEYSSENHOFF FLUID

Let us consider the linear approximation for the gravitational field of a static, finite body made of the Weyssenhoff fluid. In the comoving system of coordinates equation (6) takes the form

$$\begin{aligned} \Delta \psi_{00} &= \frac{16\pi G}{c^4} e, \\ \Delta \psi_{\alpha\beta} &= \frac{16\pi G}{c^4} p \delta_{\alpha\beta}, \\ \Delta \psi_{0\alpha} &= \frac{8\pi G}{c^3} \partial_\beta s_\alpha^\beta. \end{aligned}$$

(The Greek indices run from 1 to 3). One sees that the spin term and the energy-momentum term contribute to the different components of the metric tensor. Since we are interested in spin's effect on the gravitational field, we shall consider the third of the above equations only. This equation may be rewritten in the form

$$(9) \quad \Delta \vec{h} = - \frac{8\pi G}{c^3} \text{rot } \vec{s},$$

where we have introduced the notation  $\vec{h} = (h_{0\alpha}) = (\psi_{0\alpha})$  and  $\vec{s} = \left( \frac{1}{2} \varepsilon^{\alpha\beta\gamma} s_{\beta\gamma} \right)$ .

The de Donder condition (7) reduces to

$$(10) \quad \text{div } \vec{h} = 0.$$

Equations (9) and (10) are similar to the equations of magnetostatics. If  $\vec{h}$  tends to zero at infinity, the solution of this system of equations is

$$(11) \quad \vec{h}(\vec{r}) = \frac{2G}{c^3} \text{rot } \int \frac{\vec{s}(\vec{r}')}{|\vec{r} - \vec{r}'|} \delta V'.$$

When  $r = |\vec{r}|$  is much greater than the size of the body,

$$\vec{h}(\vec{r}) = \frac{2G}{c^3} \frac{\vec{r} \times \vec{S}}{r^3} + O\left(\frac{1}{r^3}\right),$$

where  $\vec{S} = \int \vec{s}(\vec{r}) dV$  is the total spin of the body. Choosing the z-axis in the direction of  $\vec{S}$ , one may write-down the approximate solution in the spherical system of coordinates  $(r, \theta, \varphi)$  as

$$h_{01} \cong h_{02} \cong 0, \quad h_{03} \cong \frac{2GS}{c^3} \frac{1}{r} \sin^2 \theta, \quad (r \rightarrow \infty),$$

where  $S = |\vec{S}|$ . Thus, the metric tensor of a static body constituted of the Weyssenhoff dust ( $p = 0$ ) turns out to be a linearization of the Kerr solution [6],

$$ds^2 = c^2 \left(1 - \frac{2MG}{rc^2}\right) dt^2 - \left(1 + \frac{2MG}{rc^2}\right) dr^2 - r^2(d\theta^2 + \sin^2 \theta d\varphi^2) + \frac{4GS}{c^2} \frac{1}{r} \sin^2 \theta d\varphi dt,$$

where  $M$  is the mass of the body.

In the particular case of a homogeneous sphere (of the radius  $R$ ) with a constant spin density vector  $\vec{s}$ , one can calculate the integral (11) exactly. In the spherical coordinate system used above the metric tensor is found to be:

$$h_{01} = h_{02} = 0, \\ h_{03} = \frac{2GS}{c^3} \sin^2 \theta \times \begin{cases} R^{-3} r^2 & \text{for } r < R \\ r^{-1} & \text{for } r > R. \end{cases}$$

Thus, we have obtained the solution of the linearized Einstein-Cartan equations describing the gravitational field of a static sphere of Weyssenhoff dust, which exterior part is a linearization of the Kerr metric. One may suppose that in the exact Einstein-Cartan theory it will be possible to accord the exterior Kerr solution with an interior solution for a body made of spinning (Weyssenhoff's) matter. This supposition seems to be plausible due to the situation in the linearized theory and to the value of the gyromagnetic ration in the Kerr-Newman solution.

Finally, it is worthwhile to discuss the junction conditions which must be satisfied at the surface of discontinuity of the spin density. Replacing the differential equations (9) and (10) by the integral equation (11) we have avoided this problem so far. Similarly to the Einstein theory we require that the metric should be continuous. Moreover, we look for the conditions on the first derivatives of the metric. To this purpose let us introduce  $\vec{\Gamma} := \text{rot } \vec{h}$ , which satisfies

$$(12) \quad \text{div } \vec{\Gamma} = 0$$

and (due to equation (10))

$$(13) \quad \text{rot} \left( \vec{\Gamma} - \frac{8\pi\mathbf{G}}{c^3} \vec{s} \right) = 0.$$

The vector  $\vec{\Gamma}$  is an analogue of the magnetic induction in magnetostatics. From equations (12) and (13) there follows the continuity of  $\vec{n} \cdot \vec{\Gamma}$  and of  $\vec{n} \times \left( \vec{\Gamma} - \frac{8\pi\mathbf{G}}{c^3} \vec{s} \right)$  across the surface of discontinuity of the spin density, where  $\vec{n}$  is the unit vector normal to the surface. In our case, the boundary conditions of the other components of the metric tensor are the same as in the classical theory of gravitation. It is necessary to underline that not every derivative of any metric tensor component must be continuous in contrast to the classical theory of gravitation. The authors are going to discuss the problem of junction conditions in the exact Einstein-Cartan theory in more details.

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