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# Critical point dominance in quantum field models

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ABSTRACT. — We study the generating functional $\Gamma\{A\}$ for vertex functions in $\mathscr{P}(\varphi)_2$ models and prove the Callan-Symanzik equations for the weakly coupled $\lambda_0\varphi_2^4$ model. Under reasonable (but unproved) assumptions, we show that the dimensionless physical coupling constant $g$ in $\varphi_{2,3}^4$ models achieves its maximum value at the critical point. We discuss a more general picture of « critical point dominance ».

## 1. THE CALLAN-SYMANZIK EQUATIONS

The Euclidean $\mathscr{P}(\varphi)_2$ quantum field models are parameterized by $(m_0, g_0)$. Here $m_0$ is the bare mass and $g_0 = \lambda_0/m_0^2$ is the dimensionless bare charge. The bare charge $\lambda_0$ is the vector of coefficients of the interaction polynomial

$$\mathscr{P}(\varphi) = \sum_{j=1}^{2N} \lambda_0^{(j)} \frac{1}{j!} : \varphi^j : . \tag{1.1}$$

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We discuss the variation of the solution with respect to these bare parameters, and the transformation from bare parameters to a physical parameter space $(m, g)$. Here $m$ is the physical mass, inf (spectrum (M) $\sim \{ 0 \}$) where the mass operator $M = (H^2 - P^2)^{\frac{1}{2}}$. A physical coupling constant $g$ may be defined by the momentum space vertex function (for $l \geqq 3$) as

(1.2) $$g^{(l)} = - \Gamma^{(l)}(p)/m^2 \,|_{p=0}$$

where

$$\Gamma^{(l)}(p)\delta\Big(\sum_i p_i\Big) = \int e^{i\Sigma p_j x_j} \tilde{\Gamma}^{(l)}(x) dx .$$

For $j = 1$, we take $g^{(1)} = \langle \varphi \rangle$. The field strength renormalization constant Z is

(1.3) $$Z = (p^2 + m^2) \langle \Phi^{\sim}(p)\Phi(0) \rangle \,|_{p^2 = -m^2}$$

where $\langle \quad \rangle$ denotes expectation in the Euclidean ground state measure. For increased flexibility we consider theories without the field strength renormalization, i. e. with $Z \neq 1$. Then the Green's functions and vertex functions can be considered as functions of Z as well as $(m_0^2, g_0)$. We note that

(1.3a) $$Z^{l/2}\Gamma^{(l)}(p, m_0, g_0, Z) = \Gamma^{(l)}(p, m_0, g_0, 1) ,$$

or in terms of the generating functional

(1.3b) $$\Gamma \{ Z^{\frac{1}{2}}A, m_0, g_0, Z \} = \Gamma \{ A, m_0, g_0, 1 \} ,$$

and the field strength renormalization amounts to choosing the theory with $Z = 1$.

Under scaling, one can check that Z is invariant (dimensionless) and that

(1.4) $$\Gamma^{(l)}(\alpha p, \alpha m_0, g_0, Z)(\alpha m)^{-2} = \Gamma^{(l)}(p, m_0, g_0, Z)m^{-2} ,$$

see the appendix. Thus the definition of $g^{(j)}$ in (1.2) is scale invariant.

The Callan-Symanzik equations [*11*] describe a one parameter family of perturbations of the form $\mathscr{P}(\varphi)_2 + \sigma \int :\varphi^2: d\bar{x}$. After a new choice of bare mass $m_0(\sigma)$ (see the appendix), this family defines a smooth curve $(m_0(\sigma), g_0(\sigma), Z(\sigma))$. The Callan-Symanzik equations express the variation of the theory with respect to $\sigma$. One side of the equation is an expression for $(d/d\sigma)\Gamma \{ A, \sigma \}$, which can be recognized as a prescription for $\varphi^2$ vertex insertions into the lines of the vertex functions. The other side of the equation is the chain rule,

(1.5) $$\frac{d}{d\sigma}\Gamma \{ A, \sigma, Z \} = \frac{dm(\sigma)^2}{d\sigma}\frac{\partial\Gamma}{\partial m(\sigma)^2} + \frac{dg(\sigma)}{d\sigma}\frac{\partial\Gamma}{\partial g(\sigma)} + \frac{dZ(\sigma)}{d\sigma}\frac{\partial\Gamma}{\partial Z(\sigma)}$$

Here Γ is the generating function for the (amputated, one particle irreducible) vertex parts.

We now study (1.5) for weak coupling. In [5] we prove the existence of Γ, for weak coupling, and its analyticity in A and σ. Let $J_A(\sigma)$ be the solution of the equation

$$G_x \{ J, \sigma \} - G_x \{ 0, \sigma \} = A(x)$$

where $G\{ J, \sigma \}$ is the generating function for the Euclidean Green's functions and $G_x = \delta G/\delta J(x)$. From the Legendre transformation

$$\Gamma \{ A, \sigma \} = G \{ J_A, \sigma \} - A(J_A(\sigma)) - G^{(1)}(J_A(\sigma)),$$

we find

$$\frac{d}{d\sigma}\Gamma \{ A, \sigma \} = \frac{d}{d\sigma} G \{ J, \sigma \} |_{J = J_A(\sigma)},$$

for $A \in H_1$, $J \in H_{-1}$ and $\| A \|_1$ and $\| J \|_{-1}$ small. We use the notation

$$\mathscr{Z} \{ J, \sigma, Z_0 \} = \langle e^{Z_0^{-\frac{1}{2}}\Phi(J)} \rangle_\sigma$$
$$\mathscr{Z}_{:xx:} \{ J, \sigma, Z_0 \} = Z_0^{-1} \langle : \Phi(x)^2 : e^{Z_0^{-\frac{1}{2}}\Phi(J)} \rangle_\sigma$$

(with $m_0(\sigma)$-Wick ordering). The field strength renormalization then amounts to choosing $Z_0$ to be the Z of (1.3), but for the time being we regard $Z_0$ as an independent variable.

We have

$$G \{ J, \sigma, Z/Z_0 \} = \ln \mathscr{Z} \{ J, \sigma, Z_0 \}$$

and

$$\frac{d}{d\sigma}\Gamma = \frac{d}{d\sigma} G = \frac{d}{d\sigma} \ln \mathscr{Z} = \mathscr{Z}^{-1} \frac{d}{d\sigma} \mathscr{Z}.$$

In order to simplify subsequent formulas, we assume that $\mathscr{P}$ is even, so $G^{(1)}(J) = 0$, and then

$$(1.6) \quad \frac{d}{d\sigma}\Gamma \{ A, \sigma \}$$
$$= \mathscr{Z}^{-1} Z_0 \int [ - \mathscr{Z}_{:xx:} \{ J_A, \sigma \} + \mathscr{Z} \{ J_A, \sigma \} \mathscr{Z}_{:xx:} \{ 0, \sigma \} ] dx .$$

For J regular, we establish (1.6) by Dimock's asymptotic calculations [3]. First we keep $m_0$ fixed, and consider the interaction $\mathscr{P}(\varphi) + \sigma : \varphi^2 :$. Then the mass renormalization transformation (see appendix) is used to obtain (1.6) with $m_0(\sigma)$ Wick order. For the techniques to prove convergence of the integral in (1.6) with $J \in H_{-1}$, see [5, Chapter 2]. The key point is that since $H_{-1}$ is an $L_2$ norm, we must be careful to eliminate terms linear in J, or in localized portions $J_\alpha$ of J. Since $\mathscr{P}$ is even, these terms do not contribute to the cluster expansion of (1.6).

We note that the integrand

$$F(x) = - \langle : \Phi(x)^2 : e^{Z_0^{-\frac{1}{2}}\Phi(J)} \rangle_\sigma + \langle : \Phi(x)^2 : \rangle_\sigma \langle e^{Z_0^{-\frac{1}{2}}\Phi(J)} \rangle_\sigma ,$$

which clusters can be written

$$F(x) = \lim_{a \to 0} F(x, x + a)$$

where

$$(1.7) \quad F(x, y) = - \langle \Phi(x)\Phi(y) e^{Z_0^{-\frac{1}{2}}\Phi(J)} \rangle_\sigma + \langle \Phi(x)\Phi(y) \rangle_\sigma \langle e^{Z_0^{-\frac{1}{2}}\Phi(J)} \rangle_\sigma ,$$

so

$$\frac{1}{Z_0} \frac{d\mathscr{Z}\{ J, \sigma \}}{d\sigma} = \lim_{a \to 0} \int F(x, x + a) dx .$$

Recall that $G\{ J, \sigma \} = \ln \mathscr{Z}\{ J, \sigma \}$, the generating function for connected parts. Then regarding $Z_0$, $\sigma$ and $\lambda_0$ as independent variables,

$$(1.8) \quad \frac{d}{d\sigma}\Gamma\{ A, \sigma \} = \frac{d}{d\sigma} G\{ J, \sigma \} = \mathscr{Z}\{ J, \sigma \}^{-1} \frac{d}{d\sigma}\mathscr{Z}\{ J, \sigma \}$$
$$= - \lim_{a \to 0} Z_0 \int [G_{x,x+a}\{ J, \sigma \} + G_x\{ J, \sigma \}^2 - G_{x,x+a}\{ 0, \sigma \}] dx$$

We eliminate the J dependence in (1.8) by using $G_x\{ J, \sigma \} = A(x)$ and the fact that $-\Gamma_{xy}\{ A, \sigma \}$ is the kernel of the operator inverse to $G_{xy}\{ J, \sigma \}$. Thus by (1.8),

$$(1.9) \quad \frac{d}{d\sigma}\Gamma\{ A, \sigma \} = \lim_{a \to 0} Z_0 \int [\Gamma^{-1}_{x,x+a}\{ A, \sigma \} - \Gamma^{-1}_{x,x+a}\{ 0, \sigma \} - A(x)^2] dx$$

In order to equate (1.9) to the right side of (1.5), the only problem is to show that the indicated derivatives exist. We specialize further to the case of a pure $\lambda_0\varphi^4$ interaction. For small coupling, there are no multiple phases, and so $m$ and $g$ are functions of $m_0$ and $g_0$. The existence of derivatives $\partial\Gamma/\partial m_0^2$ and $\partial\Gamma/\partial g_0$ follows from the analyticity of $\Gamma\{ A, \sigma \}$ in the bare parameters [5], while $\partial\Gamma/\partial Z_0$ exists by (1.3*b*). The fact that $m_0(\sigma)^2$ is differentiable with a bounded inverse is the second transformation of the appendix. We regard Z as a function of $m_0$ and $g_0$, defined by (1.3), and we take $Z_0 = Z$. By scaling $\partial g/\partial m_0 = \partial Z/\partial m_0 = 0$ and $\partial m/\partial m_0 = 1$, while $\partial Z/\partial\lambda_0$ and $\partial m/\partial\lambda_0$ exist by [9]. Thus the derivatives $dZ(\sigma)/d\sigma$ and $dm(\sigma)/d\sigma$ taken at constant $\lambda_0$ exist also. We show below for weak coupling that $\partial g/\partial\lambda_0$ ($m_0$ constant) and $\partial g/\partial m_0$ ($\lambda_0$ constant) (i. e. $dg/d\sigma$) exist and are not zero. Furthermore, it follows that the Jacobian $\partial(m, g)/\partial(m_0, g_0)$ is not zero. The proof of the Callan-Symanzik equations for weak $\lambda_0\varphi_2^4$ models, namely

$$(1.10) \quad \frac{dm(\sigma)^2}{d\sigma}\frac{\partial\Gamma}{\partial m(\sigma)^2}\{ A, \sigma \} + \frac{dg(\sigma)}{d\sigma}\frac{\partial\Gamma}{\partial g(\sigma)}\{ A, \sigma \} + \frac{dZ(\sigma)}{d\sigma}\frac{\partial\Gamma}{\partial Z(\sigma)}\{ A, \sigma \}$$
$$= \lim_{a \to 0} Z_0 \int [\Gamma^{-1}_{x,x+a}\{ A, \sigma \} - \Gamma^{-1}_{x,x+a}\{ 0, \sigma \} - A(x)^2] dx$$

is reduced to the following result:

THEOREM 1.1. — For a pure $\varphi^4$ interaction with weak coupling, the coordinate transformation $(m, g) \leftrightarrow (m_0, g_0)$ is differentiable with a nonvanishing Jacobian, for real bare parameters and for $g$ defined by (1.4).

*Proof.* — We have already seen that it is sufficient to show $\partial g/\partial\lambda_0$ exists and is nonzero. With $m_0$ fixed, $\partial g/\partial\lambda_0$ is proportional to

$$(1.11) \qquad -\left.\frac{\partial\Gamma^{(4)}(p)}{\partial\lambda_0}\right|_{p=0} = -\frac{\partial}{\partial\lambda_0}\int \left. G_a^{(4)}(x)\right|_{x_1=0} dx_2dx_3dx_4\,,$$

where the amputated Green's function $G_a$ equals

$$G_a(x) = \int dx' G(x') \prod_{i=1}^{4} \tilde{\Gamma}^{(2)}(x_i - x_i').$$

Using Dimock's proofs [3] of the asymptotic formulas for the Euclidean Green's functions, (1.11) can be evaluated as

$$-\frac{\partial\Gamma^{(4)}(0)}{\partial\lambda_0} = 1 + 0(\lambda_0)$$

which is $\neq 0$ for small coupling.

## 2. CRITICAL POINT DOMINANCE

In this section we discuss the global nature of the transformation $(m, g) \leftrightarrow (m_0, g_0)$ for the $\varphi^4$ model. Unlike the mathematical proofs of Section 1, our discussion here is a heuristic picture, based on reasonable assumptions about the physical parameters $(m, g)$. In a multiple phase region we consider a pure phase. We conclude that the charge $g$ achieves its maximum value $g_c$ for the $\varphi_2^4$ or $\varphi_3^4$ model at the critical point ($m = 0$), which we presume exists. Here $g_c$ may be infinite, but corresponds to finite values in the bare parameter space $(m_0, g_0)$. We relate this behavior to $\beta(g)$, one of the coefficients in the Callan-Symanzik equation, and are led in the superrenormalizable cases ($d = 2, 3$) to:

THE PICTURE OF CRITICAL POINT DOMINANCE. — The curve $m(\sigma)^2$ has a unique zero at $\sigma = \sigma_c$, and increases monotonically to $+\infty$ on each side of this zero. The curve $g(\sigma)$ decreases monotonically to zero as $|\sigma - \sigma_c| \to \infty$, with a (possibly infinite) maximum $g_c$ at $\sigma = \sigma_c$, see Figure 1.

We now discuss our assumptions and conclusions. For $d = 2$, the cluster expansion deals with the $\sigma \to +\infty$ limit [6]. After scaling (to ensure that the mass remains bounded) the model converges as $\sigma \to \infty$ to a free theory. In this limit $\lambda \approx \lambda_0$, $m \approx m_0$, and $g \approx g_0$; the monotonicity of $g(\sigma)$ or $m(\sigma)$ follows by perturbation theory, see [3]. The more general result that $m(\sigma)$ is monotonic for $\sigma \geq \sigma_c$ was established in [7]. We remark that in the Ising model, a stronger result holds for the inverse correlation length $\xi^{-1}$. For $\text{Ising}_2$, $\xi^{-1}$ is monotonic in both the single phase region and the two phase region, see, e. g. [8], and $\xi^{-1}$ vanishes at the critical point, near which there is a linear power law behavior $\xi^{-1} = a\,|\,\sigma - \sigma_c\,|$.

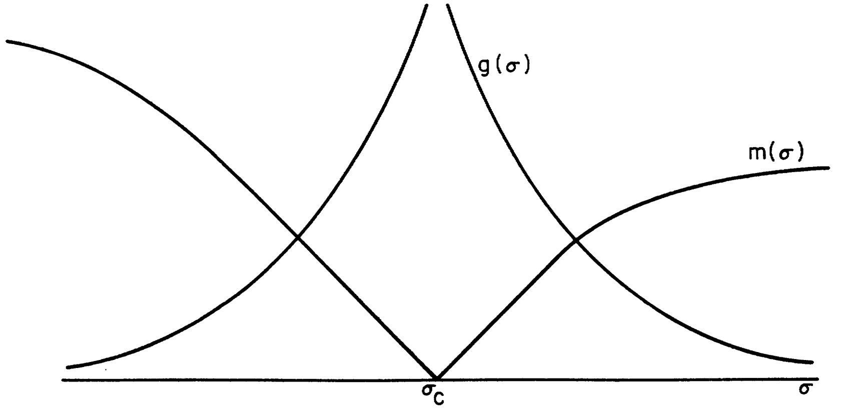


FIG. 1. — Critical Point Dominance.

For $d = 3$, the $\sigma$-perturbation produces no new ultraviolet divergences in a finite volume [4], and we expect the same behavior in an infinite volume. Thus for $d = 3$ the same picture should hold for $m(\sigma)$ as in $d = 2$. The dimensionless coupling constant in three dimensions is $g = -\Gamma^{(4)}(0)/m$.

This picture of monotonicity of $m(\sigma)$ is related to a choice of variables made by Symanzik in analyzing the characteristics of the Callan-Symanzik equations [11]. Symanzik chooses a dimensionless parameter $s$ to reparameterize the $m(\sigma)$ curve, $\sigma = \sigma(s)$, starting from a point $m(0)$, and such that

$$\frac{d}{ds}\, m(s)^2 = m(s)^2\,. \tag{2.1}$$

Since (2.1) can be integrated to yield $m(s)^2 = m(0)^2 e^s$, it is a valid assumption only on a branch of the $\sigma$ curve for which $m(\sigma)^2$ varies monotonically between 0 and $\infty$. Since $m(\sigma)^2$ is finite for $\sigma$ finite, and (assuming continuity) is bounded for $\sigma$ bounded, $m(\sigma)^2 \to \infty$ can occur at most for $\sigma \to \pm\infty$. In terms of the new parameter, an $s$-curve corresponds to a monotone branch $\sigma > \sigma_c$ or $\sigma < \sigma_c$ of a $\sigma$-curve.

Let us assume that on each $s$-curve the use of physical coordinates $m(s)$ and $g(s)$ is justified. We now argue that $g$ is monotone in $s$ on each side of the critical point. Let $\beta = dg/ds$. By scaling, $\beta$ is a function $\beta(g)$ of $g$ alone. (Here we use the fact that the $\sigma$-curves and $s$-curves are mapped into one another by scaling. Then the tangent lines

$$dg(s)/dm(s)^2 = (dg(s)/ds) \times (dm(s)^2/ds)^{-1}$$

transform as $m^{-2}$ and $\beta$ is scale invariant.) Thus the equation

$$\frac{dg}{ds} = \beta(g) \tag{2.2}$$

is a first order ordinary differential equation.

PROPOSITION. — Assume $g$ and $d\beta(g)/dg$ are bounded for $\sigma - \sigma_c \geqq \varepsilon > 0$, for all $\varepsilon > 0$. Then $g(s)$ is monotone in $s$ and $\beta(g) < 0$.

*Proof.* — By the assumed bound, $\beta(g)$ satisfies

$$|\beta(g_1) - \beta(g_2)| \leq M |g_1 - g_2|,$$

giving existence and uniqueness for the solution of the equation (2.2). Let $(\bar{s}, \infty)$ be the maximum interval on which $g(s)$ is strictly monotonic in $s$. Then $\beta(g(\bar{s})) = 0$ and by the uniqueness of the solution to (2.2), $g(s) = g(\bar{s}) =$ const. for $s \in [\bar{s}, \infty)$. However, the $d = 2$ cluster expansion [6] concerns the $\sigma \to \infty$ ($s \to \infty$) limit. In this limit $g \approx g_0$, $m \approx m_0$ and $g_0(s) \to 0$ as $s \to \infty$. By (2.1) and perturbation theory (which is asymptotic [3])

$$\beta(g) = -g + 0(g^2) \tag{2.3}$$

as $g \to 0$. We conclude for small $g$ that $\beta(g) < 0$, a contradiction to $\beta(g) = \beta(g(s)) = 0$, so $s = -\infty$, completing the proof for $d = 2$. For $d=3$, we expect a similar result with $\beta = -$ const $g + 0(g^2)$, but the required estimates have not yet been proved.

We now argue that for $\sigma < \sigma_c$, $g(s) \to 0$ as $s \to \infty$ ($\sigma \to -\infty$). In that case, the proposition also applies for $\sigma < \sigma_c$, i. e. in the two phase region. We appeal to the Goldstone picture of the ground state, in which the ground state is determined by the minimum of the Euclidean action function V, and where the curvature at the minimum approximates the mass. For the $\varphi^4$ model with $\sigma \ll 0$, the minimum of $V(\cdot)$ occurs at $\pm (6|\sigma|/\lambda_0)^{\frac{1}{2}} \equiv \pm \bar{x}$. At the minimum, the curvature if $V^{(2)}(\pm \bar{x}) = 2|\sigma|$, while $V^{(3)}(\pm \bar{x}) = \pm (6\lambda_0|\sigma|)^{\frac{1}{2}}$ and $V^{(4)}(\pm \bar{x}) = \lambda_0$. Hence the dimensionless couplings at the minimum $\pm \bar{x}$ are for $d = 2$,

$$g^{(j)} = V^{(j)}(\pm \bar{x})/V^{(2)}(\pm \bar{x}) = \begin{cases} \pm (3\lambda_0/2|\sigma|)^{\frac{1}{2}}, & j = 3 \\ \lambda_0/2|\sigma|, & j = 4 \end{cases} \tag{2.4}$$

We conclude that as $\sigma \to -\infty$, and after scaling so that the mass $(2|\sigma|)^{\frac{1}{2}}$ remains bounded, the model converges to a free field (or a sum of two free fields, depending on the boundary conditions). By (2.4), we also see that the

Goldstone picture suggests $g(s) \approx \lambda_0/m(s)^2$ in this limit, and by (2.1), $\beta(g) \approx -g$ in this limit. In other words (2.3) holds for $\sigma \to -\infty$ as well as for $\sigma \to +\infty$. A similar picture holds for the quartic coupling $g^{(4)}$ (but not $g^{(3)}$) in three dimensions.

With this picture of critical point dominance, the function $\beta(g)$ has a zero at $g = 0$, and a second zero at $g = g_c$ (possibly infinity). The coupling constant is bounded by $g_c$, and $\beta$ is represented in Figure 2. The zero at $g = 0$ governs the canonical (short distance, ultraviolet) behavior of the theory, while the presumed zero at $g = g_c$ governs the infrared (long distance) behavior.

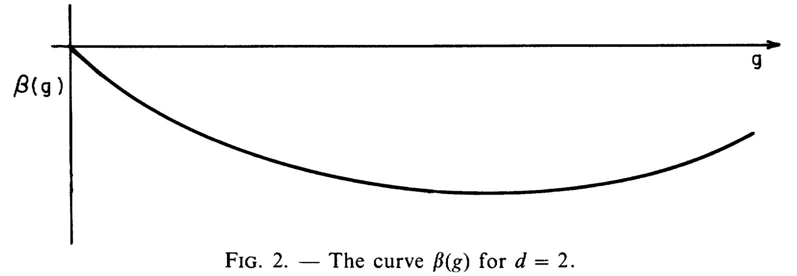


FIG. 2. — The curve $\beta(g)$ for $d = 2$.

This question remains open whether $g_c = \infty$, in which case the function $\beta(g) < 0$ for $g > 0$. In order for $g_c$ to be finite, it is necessary that $\lambda_c = 0$ and that $\lambda$ vanish as $\sigma \to \sigma_c$ in such a way that $\lambda/m^2 \to g_c$. If $g_c < \infty$, then our picture of critical point dominance says that $g_c$ is the maximum value of $g$, and in the two phase region $\beta(g)$ returns to zero along another curve $\beta < 0$, see Figure 3.

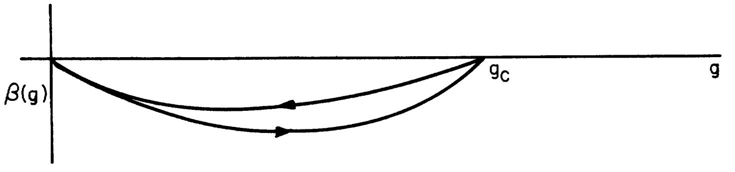


FIG. 3. — The trajectory of $\beta(g(\sigma))$, assuming $g_c < \infty$.

Critical theories ($g = g_c$) need not be scale invariant, but lie on the critical line of bare parameters $g_{0,c}$ (see the appendix). This line is transformed into itself by scale transformations. If a scale invariant $\varphi_2^4$ model exists, it is obtained by an infinite scaling of a critical (but not scale invariant) theory.

In four dimensions, as opposed to $d < 4$, the coupling constant $\lambda$ is dimensionless, i. e. $g = \lambda$. Hence the first order contribution to $g(g_0)$ does not contribute to $\beta(g)$ and explicit calculation shows that the second order contribution is positive. Hence for weak coupling,

$$\beta(g) = \alpha g^2 + 0(g^3), \qquad \alpha > 0.$$

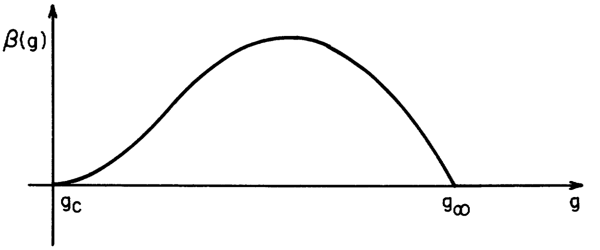


FIG. 4. — The curve $\beta(g)$ for $d = 4$.

The change in the sign of $\beta(g)$ from $d = 2$ or $d = 3$ reverses the roles of the zeros of $\beta(g)$. Thus the zero at $g = 0$ is the critical zero, $g_c = 0$, and governs the infrared (long distance) behavior of the theory. On the other hand, a zero at $g = g_\infty > 0$ would govern ultraviolet behavior (and is expected to be noncanonical). The arguments from the two dimensional case, applied to $\beta > 0$, yield characteristic curves drawn in Figure 5.

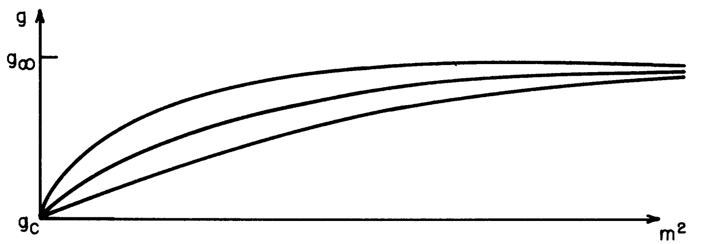


FIG. 5. — Presumed characteristics for $d = 4$.

As for $d < 4$, we expect in four dimensions that there are infinitely many critical theories, all interchanged by scaling. These theories presumably have canonical long distance behavior and infinite scaling of these theories in the long distance direction should produce a free theory of zero mass particles.

Finally we reconsider the behavior of

$$g = -Z^2 G^{(2)\sim}(0)^{-4} G^{(4)\sim}(0) m^{-4+d}$$

as we approach the critical point from the single phase region. By the definition of Z, $Z \leqq m^2 G^{(2)\sim}(0)$, so

$$g \leqq -G^{(2)\sim}(0)^{-2} G^{(4)\sim}(0) m^d.$$

Each of $G^{(2)\sim}$, $G^{(4)\sim}$ and $m^2$ is assumed to have behavior characterized by a critical exponent. We use the terminology of statistical mechanics, see [*10*]: the mass $m$ is the inverse correlation length $\xi^{-1}$, and $G^{(2)\sim}(0)$ is proportional to susceptibility $(\partial M/\partial H)_T = \chi_T$. If $\varepsilon = (T/T_c - 1)$, the $\varepsilon$-dependence in the neighborhood of the critical point is given by

$$m = \xi^{-1} \sim \varepsilon^{\nu},$$
$$G^{(2)\sim}(0) \sim \chi_T \sim \varepsilon^{-\gamma}.$$

In field theory, we use $\varepsilon = \left(\frac{m_0^2}{m_{0,c}^2} - 1\right)$. The four point correlation function $G^{(4)}$ has the gap exponent $\Delta_4$, which relates its critical behavior to the behavior of $G^{(2)}$,

$$- G^{(4)} \sim \varepsilon^{-2\Delta_4} G^{(2)} \sim \varepsilon^{-2\Delta_4 - \gamma}.$$

Thus

$$g \leqq 0(\varepsilon^{\gamma - 2\Delta_4 + d\nu}).$$

For the Ising model in the single phase region ($\varepsilon > 0$) the exponents are

$$d = 2: \quad \gamma = \frac{7}{4}, \quad \nu = 1, \quad \Delta_4 = \frac{15}{8},$$
$$d = 3: \quad \gamma = \frac{5}{4}, \quad \nu = \frac{5}{8}, \quad \Delta_4 = \frac{25}{16},$$

so

$$\gamma - 2\Delta_4 + d\nu = \begin{cases} 0, & d = 2 \\ 0, & d = 3 \end{cases}$$

Here we use exponents as in [*10*], and the $d = 3$ exponents are approximate. While there is no known relation between the Ising model exponents and the field theory exponents, Wilson and others have argued that they may coincide. These exponents do not distinguish for $d = 2, 3$ whether $g$ has a finite limit at the critical point. Furthermore, these exponents indicate that the dimensional coupling constant $\lambda_c = 0$.

The critical exponent $\eta$ describes the rate of decay of the pair correlation function at the critical point,

$$G(r) \sim r^{-(d-2+\eta)},$$

and $\eta \neq 0$ is the case of anomalous (noncanonical) dimensions. In the $I_2$ model, $\eta = \frac{1}{4}$. This anomalous dimension indicates that the propagator decays faster than it would in the presence of zero mass particles (i. e. $\eta = 0$). Thus we do not expect zero mass particles in the critical theory and we expect $Z_c = 0$. If $Z \sim \varepsilon^\zeta$ near the critical point, the bound $Z \leqq m^2 G^{(2)\sim}(0)$ shows

$$\zeta \geqq 2\nu - \gamma .$$

By the known inequality $(2 - \eta)\nu \geqq \gamma$, we have $\zeta \geqq \eta\nu$. Furthermore, $\zeta = \frac{1}{4}$ for $I_2$, using a recent calculation of Tracy and McCoy [*12*], and thus for $I_2$ the inequality is an identity. Thus anomalous dimensions, $\eta > 0$, indicate $Z_c = 0$.

Rigorous bounds on the two point function [*6*] exclude zero mass particles in the two point function, and assuming $m \to 0$ as the critical point is approached, these same bounds then prove $Z \to 0$. For $I_3$, $\eta \approx .04 \neq 0$, so again we expect $Z_c = 0$ and no zero mass particles in the $\varphi_3^4$ field theory. It is expected (up to logarithms) that the exponents are given by their canonical values in $I_4$.

# APPENDIX

## REVIEW OF SCALE AND MASS SHIFT TRANSFORMATIONS

We assume $d = 2$ in the appendix. The corresponding results for $d > 2$ are easily derived.

SCALE TRANSFORMATIONS. — On Fock space, the scale transformation is a unitary operator U defined on an $n$-particle state $f$ by

$$(U^*f)(\vec{k}_1, \ldots, \vec{k}_n) = \alpha^{n/2} f(\alpha\vec{k}_1, \ldots, \alpha\vec{k}_n),$$

for $\alpha > 0$. The free field $\varphi_{m_0}$ of mass $m_0$, and its Hamiltonian $H_{0,m_0}$, transform as

$$U\varphi_{m_0}(\vec{x})U^* = \varphi_{\alpha m_0}(\vec{x}/\alpha),$$
$$UH_{0,m_0}U^* = \alpha^{-1}H_{0,\alpha m_0}.$$

Also the no particle state $\Omega_0$ is invariant.

$$U\Omega_0 = \Omega_0,$$

and $Ua(\vec{k})U^* = \alpha^{\frac{1}{2}}\alpha(k\alpha)$.

Furthermore, on Wick monomials, scale transformations map

$$U : \varphi_{m_0}(x)^j : U^* = : \varphi_{\alpha m_0}(\vec{x}/\alpha)^j : .$$

With an ultraviolet cutoff $\kappa$ and box of length L,

$$UH_1(m_0, \lambda, \kappa, L)U^* = H_1(\alpha m_0, \alpha\lambda, \alpha\kappa, \alpha^{-1}L),$$

since the Wick ordering constant is invariant under scaling in two dimensions:

$$c(m_0, \kappa) = c(\alpha m_0, \alpha\kappa).$$

The vacuum energy for

$$H_{0,m_0} + H_1(m_0, \lambda_0, \kappa, L)$$

is the same as the vacuum energy for

$$\begin{aligned} U[H_{0,m_0} + H_1(m_0, \lambda_0, \kappa, L)]U^* &= \alpha^{-1}H_{0,\alpha m_0} + H_1(\alpha m_0, \alpha\lambda_0, \alpha\kappa, \alpha^{-1}L) \\ &= \alpha^{-1}[H_{0,\alpha m_0} + H_1(\alpha m_0, \alpha^2\lambda_0, \alpha\kappa, \alpha^{-1}L)]. \end{aligned}$$

Thus

$$E(m_0, \lambda_0, \kappa, L) = \alpha^{-1}E(\alpha m_0, \alpha^2\lambda_0, \alpha\kappa, \alpha^{-1}L)$$

and writing $H = H_0 + H_1 - E$,

$$UH(m_0, \lambda_0, \kappa, L)U^* = \alpha^{-1}H(\alpha m_0, \alpha^2\lambda_0, \alpha\kappa, \alpha^{-1}L).$$
$$U\Omega(m_0, \lambda_0, \kappa, L) = \Omega(\alpha m_0, \alpha^2\lambda_0, \alpha\kappa, \alpha^{-1}L).$$

Thus for a cutoff $\mathscr{P}(\varphi)_2$ field theory, scale transformations on Fock space provide a unitary map between Hamiltonians with different cutoffs, but the same $g_0$. Since the vacuum expectation values converge as $\kappa$, $L \to \infty$, we obtain in the limit an isomorphism of theories with constant $g_0$. We may define a unitary transformation

$$U : \mathscr{H}_1 \to \mathscr{H}_2$$

between the Hilbert spaces $\mathscr{H}_i$ of two such theories, where $U\Omega_1 = \Omega_2$,

$$UH_1U^* = \alpha^{-1}H_2,$$

and

$$U\varphi_1(x)U^* = \varphi_2(x/\alpha).$$

Theories described by different values of the parameter $\alpha$ are physically equivalent, and in the $(m_0^2, \lambda_0)$ plane correspond to various points on lines of constant $g_0$, see Figure 6.

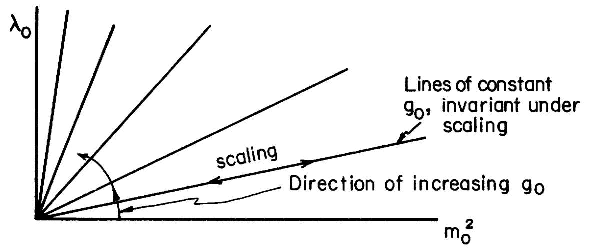


FIG. 6. — Scaling in the bare parameter space.

MASS SHIFT TRANSFORMATION. — The second transformation arises from a unitary operator on a Fock space over a finite volume (with e. g. periodic boundary conditions). In the infinite volume limit, we obtain an isomorphism of free fields of different mass, and of interacting fields as follows: we define in the Q-representation the scale (Bogoliubov) transformation,

$$U\varphi^{\sim}(\vec{k})U^* = \varphi^{\sim}(\vec{k})\alpha(\vec{k}),$$

where $\alpha(\vec{k}) = (\vec{k}^2 + m_0^2)^{\frac{1}{4}}(\vec{k}^2 + m_1^2)^{-\frac{1}{4}} = \mu_0^{\frac{1}{2}}\mu_1^{-\frac{1}{2}}$. Then by a standard calculation, see for example [1],

$$U\varphi_{m_0}(\vec{x})U^* = \varphi_{m_1}(\vec{x}),$$

$$U : \varphi_{m_0}(\vec{x})^j : U^* = \sum_{r=0}^{[j/2]} \frac{j!}{(j-2r)!r!2^r}(-\delta c)^r : \varphi_{m_1}(\vec{x})^{j-2r} : ,$$

with

(A1) $$\delta c = \frac{1}{4\pi}\int\left[\frac{1}{k^2+m_0^2} - \frac{1}{k+m_1^2}\right]dk = \frac{1}{2}\ln\left(\frac{m_1}{m_0}\right).$$

We note that $\delta c$ varies over $(-\infty, \infty)$ as $m_1/m_0$ varies over $(0, \infty)$. Also

$$UH_{0,m_0}U^* = H_{0,m_1} + \frac{1}{2}(m_0^2 - m_1^2)\int : \varphi_{m_1}^2 : d\vec{x} - EL$$

where E is the (finite) vacuum energy density

$$E = -\int\left[\frac{1}{2}(\mu_0 - \mu_1) + \frac{1}{4}\frac{m_0^2 - m_1^2}{\mu_1}\right]d\vec{k}.$$

Let $\mathscr{P}$ be an interaction polynomial for mass $m_0$,

$$\mathscr{P}(\varphi_{m_0}) = \sum_{j=1}^{2N} \lambda_0^{(j)} : \varphi_{m_0}^j : .$$

Define $\hat{\mathscr{P}}$ as the transformed polynomial,

$$\hat{\mathscr{P}}(\varphi_{m_1}) = U\mathscr{P}(\varphi_{m_0})U^* = \sum_{j=0}^{2N} \hat{\lambda}_0^{(j)} : \varphi_{m_1}^j : ,$$

and note $\lambda_0^{(j)} = \hat{\lambda}_0^{(j)}$ for $j = 2N, 2N-1$. Let $H_1$ and $\hat{H}_1$ be the respective interaction Hamiltonians. Then for $H(m_0, \lambda) = H_{0,m_0} + H_1(m_0, \lambda) - E(m_0, \lambda)$, where E is the vacuum energy, we have

(A2) $$UH(m_0, \lambda_0)U^* = H(m_1, \tilde{\lambda}_0).$$

Here

$$\tilde{\lambda}_0^{(j)} = \hat{\lambda}_0^{(j)}, \qquad j \neq 2, \tag{A3}$$

$$\tilde{\lambda}_0^{(2)} = \hat{\lambda}_0^{(2)} + \frac{1}{2}(m_0^2 - m_1^2).$$

We note in particular that $\lambda_0^{(2N)} > 0$ and

$$\tilde{\lambda}_0^{(2)} = \lambda_0^{(2)} + \frac{1}{2}(m_0^2 - m_1^2) + N(2N-1)!!\lambda_0^{(2N)}(-\delta c)^{N-1} + 0(|\delta c|^{N-2}). \tag{A4}$$

By (A1), $-\delta c \to +\infty$ as $m_1 \to 0$. Also

$$|\delta c| \leqq 0\left(\left|\ln \frac{m_1}{m_0}\right|\right),$$

so $\tilde{\lambda}_0^{(2)} \to -\infty$ as $m_1 \to \infty$. Thus for $m_0$ fixed, $\tilde{\lambda}_0^{(2)}$ ranges over $(\infty, -\infty)$ as $m_1$ varies over $(0, \infty)$.

*Remark 1.* — Given $H(m_0, \lambda_0)$, there exists a bare mass $m_1$ for which

$$UH(m_0, \lambda_0)U^* = H(m_1, \tilde{\lambda}_0)$$

with $\tilde{\lambda}_0^{(2)} = 0$.

For the $\lambda_0 : \varphi_{m_0}^4 : + \sigma : \varphi_{m_0}^2 :$ interaction, $\tilde{\lambda}_0^{(2)}$ is monotonic in $m_1$, so the zero is unique. (More generally, $\mathscr{P}(\varphi_{m_0}) + \sigma\varphi_{m_0}^2$ has a unique zero for $|\sigma|$ large.) Hence with $\lambda_0$ fixed, we obtain a one parameter family of theories, parameterized by $\sigma$, with the interaction $\lambda_0 : \varphi_{m_1(\sigma)}^4 :$. In our diagram of couplings, this $\sigma$ line is horizontal, with its height determined by $\lambda_0$, see Figure 7.

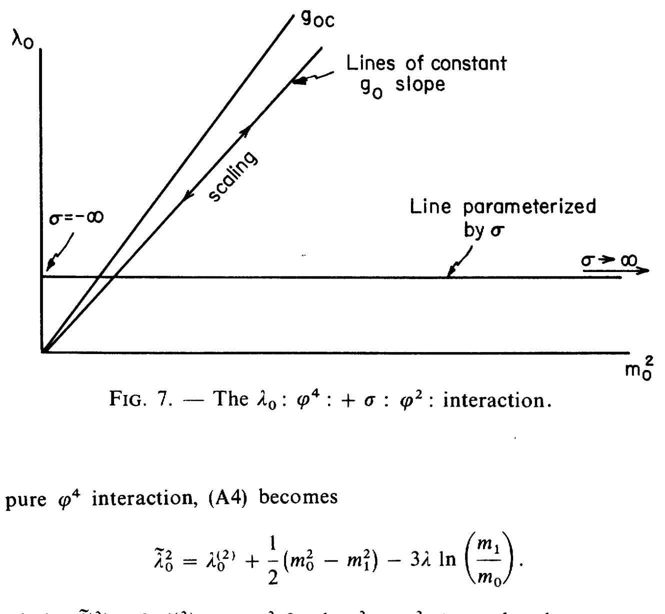


FIG. 7. — The $\lambda_0 : \varphi^4 : + \sigma : \varphi^2 :$ interaction.

For a pure $\varphi^4$ interaction, (A4) becomes

$$\tilde{\lambda}_0^2 = \lambda_0^{(2)} + \frac{1}{2}(m_0^2 - m_1^2) - 3\lambda \ln\left(\frac{m_1}{m_0}\right). \tag{A5}$$

With the choice $\tilde{\lambda}_0^{(2)} = 0$, $\lambda_0^{(2)} = \sigma$, $m_0^2$ fixed, $m_1^2 = m_1^2(\sigma)$, we then have

$$\frac{dm_1^2(\sigma)}{d\sigma} = \left(\frac{1}{2} + \frac{3}{2}\frac{\lambda}{m_1^2(\sigma)}\right)^{-1} \tag{A6}$$

*Remark 2.* — For the $\lambda\varphi_2^4$ interaction, $m_1^2(\sigma)$ is a differentiable function of $\sigma$, bounded away from zero for $\sigma$ bounded away from $-\infty$.

*Remark 3.* — The finite volume may be introduced by periodic boundary conditions. Within the region of convergence of the cluster expansion, the infinite volume theory is independent of the choice of boundary conditions.

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