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Unitary equivalence of local algebras in the quasifree representation

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ABSTRACT. — We consider the von Neumann algebras $\mathscr{A}(\mathrm{B}, m)$ generated in the free representation by bose fields of mass $m \geqslant 0$ with test functions supported in a bounded region $\mathrm{B} \subseteq \mathbb{R}^d$. We show that in $d = 2, 3$ space dimensions $\mathscr{A}(\mathrm{B}, m)$ is unitarily equivalent to $\mathscr{A}(\mathrm{B}, 0)$, by proving a more general theorem about equivalence of local algebras.

1. INTRODUCTION

This note deals with a problem arising in the algebraic approach to quantum field theory. We compare different (relativistic) *local* algebras (of observables) generated by neutral bose fields with different two point functions in the *quasi-free representation.* We derive criteria for such algebras to be isomorphic or to be unitarily equivalent.

There is an extensive literature on quasi-free representations (see e. g. [*1*] [*2*]), in which necessary and sufficient conditions have been given for such representations to give rise to isomorphic algebras. However, these criteria seem unfortunately *not to apply to local* algebras, due to an initial setup which does not incorporate our case at hand.

Another line of ideas, initiated by Glimm and Jaffe in [*3*] and discussed in [*4*] [*5*] and [*6*], uses the existence of a local *non-relativistic* number operator as the starting point for the proof of isomorphism of von Neumann

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algebras. We apply this idea to the quasi-free case at hand and we show that if the non-diagonal part of the Bogoliubov transformation connecting two representations is « locally » a Hilbert-Schmidt (H. S.) operator then the two corresponding local algebras are isomorphic (Theorem 4.3). In particular, the local algebras $\mathscr{A}(B, m_1)$ and $\mathscr{A}(B, m_2)$, $m_1 \geqslant 0$, $m_2 \geqslant 0$ of free scalar bose fields of mass m_1 and m_2 respectively are unitarily equivalent in $d = 2$ and 3 space dimensions if $m_1 . m_2 = 0$ and in $d = 1, 2, 3$ dimensions otherwise (Theorems 5.1, 5.2). The case $m_1 . m_2 = 0$ is an extension of results in [5].

This note is motivated by and has applications in an analysis of scattering states in relativistic field theories without a mass gap [8].

2. NOTATION. STATEMENT OF THE PROBLEM

Let $\mathbb{R}^d$ be the configuration space and let $\mathscr{H} = L^2(\mathbb{R}^d)$ be the Hilbert space of one particle wave functions. As usual, the Fock space $\mathscr{F}$ over $\mathscr{H}$ is

$$\mathscr{F} = \bigoplus_{m=0}^{\infty} L^2(\mathbb{R}^d)^{\otimes_s m} \qquad \text{where} \qquad L^2(\mathbb{R}^d)^{\otimes_s 0} = \mathbb{C},$$

and $\otimes_s$ is the symmetric tensor product. The vector $\{ 1, 0, 0, \dots \}$ is called the vacuum and is denoted by Ω_0.

On $\mathscr{F}$, the creation and annihilation operators $A^*(f)$ and $A(g)$ obey for $f, g \in \mathscr{H}$ the commutation relations (CCR)

$$[A(f), A^*(g)] = (f, g)_2 = \int_{\mathbb{R}^d} dx \bar{f}(x) g(x), \tag{2.1}$$

$$[A(f), A(g)] = [A^*(f), A^*(g)] = 0$$

and

$$A(f)\Omega_0 = 0 \qquad \text{for all} \qquad f \in \mathscr{H}. \tag{2.2}$$

Let μ be a real positive selfadjoint operator on $\mathscr{H} = L^2(\mathbb{R}^d)$ such that for some dense $D_\mu \subseteq L^2_{\text{real}}(\mathbb{R}^d)$ we have

$$\mathscr{D}(\mu^{1/2}) \supseteq \mathscr{D}_\mu = D_\mu + iD_\mu, \qquad \mathscr{D}(\mu^{-1/2}) \supseteq \mathscr{D}_\mu .$$

We define the field φ_μ « with two point function μ^{-1} » and its conjugate momentum π_μ by

$$\begin{aligned} \varphi_\mu(f) &= 2^{-1/2}(A^*(\mu^{-1/2}f) + A(\mu^{-1/2}f)), \\ \pi_\mu(f) &= i2^{-1/2}(A^*(\mu^{+1/2}f) + A(\mu^{+1/2}f)), \end{aligned} \tag{2.3}$$

for $f \in D_\mu$. They satisfy the CCR in Weyl form,

$$e^{i\varphi_\mu(f)}e^{i\pi_\mu(g)} = e^{-i(f,g)_2}e^{i\pi_\mu(g)}e^{i\varphi_\mu(f)}, \qquad f, g \in D_\mu . \tag{2.4}$$

For any bounded open region $B \subseteq \mathbb{R}^d$ we define $\mathring{\mathscr{A}}_\mu(B)$ as the complex *-algebra generated by

$$\{ e^{i\varphi_\mu(f)}, e^{i\pi_\mu(f)}, \quad f \in D_\mu, \text{ supp } f \subset B \}, \tag{2.5}$$

and we let $\mathscr{A}_\mu(B)$ be the von Neumann algebra generated by $\mathring{\mathscr{A}}_\mu(B)$ with respect to the weak operator topology on $\mathscr{F}$.

We shall deal with problems where two different two point functions are considered. Call them μ_1^{-1} and μ_2^{-1}. We shall always assume that for some dense $D \subseteq L^2_{\text{real}}(\mathbb{R}^d)$, one has with $\mathscr{D} = D + iD$,

$$\mathscr{D}(\mu_1^{+1/2}) \supseteq \mathscr{D} \cup \mu_2^{-1/2}\mathscr{D}, \qquad \mathscr{D}(\mu_2^{+1/2}) \supseteq \mathscr{D} \cup \mu_1^{-1/2}\mathscr{D},$$

and

$$\mathscr{D}(\mu_1^{-1/2}) \supseteq \mathscr{D} \cup \mu_2^{+1/2}\mathscr{D}, \qquad \mathscr{D}(\mu_2^{-1/2}) \supseteq \mathscr{D} \cup \mu_1^{+1/2}\mathscr{D}. \tag{2.6}$$

With D replacing D_μ in equ. (2.5), we define the algebras $\mathring{\mathscr{A}}_{\mu_i}(B)$ and $\mathscr{A}_{\mu_i}(B)$, $i = 1, 2$. We want to give sufficient conditions for $\mathscr{A}_{\mu_1}(B)$, and $\mathscr{A}_{\mu_2}(B)$ to be isomorphic or unitarily equivalent.

3. BOGOLIUBOV TRANSFORMATIONS

This section reviews well-known formulae and conditions involving Bogoliubov transformations [*1*]. We need them for our derivation of a formula for the existence of a local number operator in Section 4. We also show why the usual isomorphism criteria [*1*] [*2*] do not apply.

A Bogoliubov transformation β is a pair of real linear maps $\beta = (\beta_+, \beta_-)$ on $L^2(\mathbb{R}^d)$ satisfying

$$\beta_+^*\beta_+ - \beta_-^*\beta_- = \mathbb{1}, \qquad \beta_+^*\beta_- = \beta_-^*\beta_+ . \tag{3.1}$$

The restriction to real β_+ and β_- is for convenience only and will have the effect of producing only transformations which map fields and conjugate momenta onto themselves. Given β, we define for $f \in \mathscr{D}(\beta_\pm)$,

$$A_\beta^*(f) = A^*(\beta_+ f) + A(\beta_- f),$$

and

$$A_\beta(f) = A^*(\beta_- f) + A(\beta_+ f), \tag{3.2}$$

so that by (3.1), A_β^* and A_β satisfy the CCR (2.1).

If μ_1 and μ_2 satisfy (2.6), we set

$$\beta_\pm = \frac{1}{2}(\mu_2^{-1/2}\mu_1^{1/2} \pm \mu_2^{1/2}\mu_1^{-1/2}). \tag{3.3}$$

Then the map $(A^*, A) \to (A_\beta^*, A_\beta)$ defines through equation (2.3) a map $(\varphi_{\mu_1}(f), \pi_{\mu_1}(g)) \to (\varphi_{\mu_2}(f), \pi_{\mu_2}(g))$ and hence a natural invertible homomorphism τ_β from the algebra $\mathring{\mathscr{A}}_{\mu_1}(\mathbb{R}^d)$ onto $\mathring{\mathscr{A}}_{\mu_2}(\mathbb{R}^d)$. It is well known that for irreducible representations τ_β is unitarily implementable (spatial) on $\mathscr{F}$ if and only if β_- is a Hilbert-Schmidt operator [*1*]. If τ_β is spatial then obviously for every open $B \subseteq \mathbb{R}^d$ the algebras $\mathscr{A}_{\mu_1}(B)$ and $\mathscr{A}_{\mu_2}(B)$ are unitarily equivalent. This paper deals with situations in which (the global) τ_β *is not spatial*, but where, for *bounded* B, the algebras $\mathscr{A}_{\mu_1}(B)$ and $\mathscr{A}_{\mu_2}(B)$ will turn out to be isomorphic or unitarily equivalent. We wish to use the fact that β_- is « locally H. S. » in a sense to be made precise below. Since μ_1 does in general *not* necessarily map $D_{\mu_1} \cap L^2_{\text{real}}(B)$ into $L^2_{\text{real}}(B)$, it seems that the general criteria of [*1*] [*2*] do not apply, because such a condition is always implied by the general setup of the problem in these papers.

4. LOCAL PARTICLE NUMBER OPERATORS

We want to apply the basic philosophy used in [*3*]: if a representation of the CCR is given by a state ω and if local number operators from another representation can be defined on ω, then the two representations are isomorphic. This motivates our analysis of local particle number operators which now follows.

For a Borel set $\Delta \subseteq \mathbb{R}^d$ we define the local number operator $N(\Delta)$ on $\mathscr{F}$ by

$$N(\Delta) = \int_\Delta dx\, A^*(x)\, A(x)\,. \tag{4.1}$$

This is a selfadjoint operator on $\mathscr{F}$ whose spectrum is $\{ 0, 1, 2, \dots \}$. For simplicity we restrict ourselves to the case where Δ is a d-dimensional hypercube of unit volume.

Let β be a Bogoliubov transformation and set

$$\omega_\beta(A) = \omega_0(\tau_\beta(A)) \qquad \text{for} \qquad A \in \mathring{\mathscr{A}}_1(\mathbb{R}^d)\,,$$

where $\omega_0(A) = (\Omega_0, A\Omega_0)$ and $\mathring{\mathscr{A}}_1 = \mathring{\mathscr{A}}_{\mu=1}$.

The GNS space associated with the algebra $\mathring{\mathscr{A}}_1(\mathbb{R}^d)$ and the state ω_β is denoted by $\mathscr{F}_\beta$ and its cyclic vector is denoted by Ω_β. By construction,

$$\begin{aligned}(\Omega_\beta, \exp i(\varphi_1(f) &+ \pi_1(g))\Omega_\beta) \\ &= (\Omega_0, \exp i(\varphi_1((\beta_+ + \beta_-)f) + \pi_1((\beta_+ - \beta_-)g))\Omega_0)\,, \end{aligned} \tag{4.2}$$

for all f and g in $\mathscr{D}(\beta_\pm)$.

We assume for convenience that the operator β_- has a kernel $\beta_-(x, y)$ which is a measurable function on $\mathbb{R}^d \times \mathbb{R}^d$. Then we have

THEOREM 4.1. — *If* $\mathscr{D}(\beta_+) \cap L^2(\Delta)$ *is dense in* $L^2(\Delta)$ *and*

$$\int_\Delta dx \int_{\mathbb{R}^d} dy\, |\, \beta_-(x, y)\,|^2 < \infty\,, \tag{4.3}$$

then there exists a positive, s. a. *(local) number operator for the algebra* $\Pi_{\mathscr{F}_\beta}(\mathring{\mathscr{A}}_1(\Delta))''$ *on* $\mathscr{F}_\beta$ *(which is formally given by the operator* $N(\Delta)$ *defined in* (4.1)*).*

COROLLARY 4.2. — *If, in addition, ω_β is a pure state then*

$$\Pi_{\mathscr{F}_\beta}(\mathring{\mathscr{A}}_1(\Delta))'' \cong \mathscr{A}_1(\Delta) . \tag{4.4}$$

In particular, if (4.3) holds for $\Delta = \mathbb{R}^d$, then $\mathscr{F}_\beta = \mathscr{F}$,

$$\Pi_{\mathscr{F}_\beta}(\mathring{\mathscr{A}}_1(\mathbb{R}^d))'' = \mathscr{A}_1(\mathbb{R}^d),$$

and the isomorphism τ_β is unitarily implementable on $\mathscr{F}$ (Shale's theorem).

Proof. — Let P_Δ be the orthogonal projection onto $L^2(\Delta)$. It follows from hypothesis (4.3) that $\beta_- P_\Delta$ is a bounded operator and

$$\| \beta_- P_\Delta \|^2 \leqslant \int_\Delta dx \int_{\mathbb{R}^d} dy \, | \beta_-(x, y) |^2 < \infty .$$

Let h be a normalized vector in $L^2(\Delta) \cap \mathscr{D}(\beta_+)$. By equations (4.2), (3.2),

$$\begin{aligned}\omega_\beta(A^*(h)A(h)) &= \omega_0(A^*_\beta(h)A_\beta(h)) \\ &= (\beta_- h, \beta_- h)_2 \leqslant \| \beta_- P_\Delta \|^2 \, \| h \|_2^2 < \infty . \end{aligned} \tag{4.5}$$

Since τ_β is a homomorphism, $A^*_\beta(h)A_\beta(h)$ is a number operator for one degree of freedom for the algebra spanned by $\{ \exp i(\varphi_1((\beta_+ + \beta_-)h)), \exp i(\pi_1((\beta_+ - \beta_-)h)) \}$.

Let $\{ h_n \}_{n=0}^\infty$ be an arbitrary complete orthonormal system contained in $\mathscr{D}(\beta_+) \cap L^2(\Delta)$. Then, by (4.5),

$$\begin{aligned}\omega_\beta(N(\Delta)) &:= \lim_{M \to \infty} \omega_\beta\left(\sum_{n=0}^{M} A^*(h_n)A(h_n) \right) \\ &= \lim_{M \to \infty} \sum_{n=0}^{M} (\beta_- h_n, \beta_- h_n)_2 \end{aligned} \tag{4.6}$$

exists and is equal to

$$\int_\Delta dx \int_{\mathbb{R}^d} dy \, | \beta_-(x, y) |^2$$

and hence independent of the basis $\{ h_n \}_{n=0}^\infty$. Therefore $\omega_\beta(N(\Delta))$ is defined for net convergence.

Let $\mathscr{L}$ be the subspace of $\mathscr{F}_\beta$ spanned by $\exp i(\varphi_1(f) + \pi_1(g))\Omega_\beta$, f and g in $\mathscr{D}(\beta_\pm)$. The subspace $\mathscr{L}$ is dense in $\mathscr{F}_\beta$. We use the canonical commutation relations and equation (4.6) and conclude that for all θ in $\mathscr{L}$,

$$\lim_{M \to \infty} \sum_{n=0}^{M} (A(h_n)\theta, A(h_n)\theta) = (\theta, N(\Delta)\theta)$$

exists and is independent of the basis $\{ h_n \}_{n=0}^\infty$.

Therefore $N(\Delta)$ is a densely defined, positive quadratic form. Hence

Theorem 1 of Chaiken [2], applies and proves Theorem 4.1. Theorem 2 of [2] proves Corollary 4.2.

Q. E. D.

The relation (4.4) is only an isomorphism between so called *Newton-Wigner* (type I_∞) local algebras ($\mu = \mathbb{1}$). In this paper we are interested in proving that local algebras with *different*, non trivial *two point functions* are isomorphic. Theorem 4.1 does not apply any more. Nevertheless, the existence of local number operators $\{ N(\Delta) \mid \Delta \subset \mathbb{R}^d \}$ is the starting point of the powerful isomorphism theorem of Glimm and Jaffe [3] for relativistic local algebras. Its application yields our main theorem.

Given μ_1 and μ_2, we need the following conditions:

C1) μ_1 and μ_2 satisfy (2.6).

C2) μ_1 satisfies: let ξ be a C^∞ function which equals one on a hypercube Δ of unit volume and let $\Delta' \supset \operatorname{supp} \xi$ be a cube. If Δ_a is a hypercube of unit volume centered at $\mathscr{A}$ then for some $\alpha > 0$

$$\| \chi_{\Delta_a} \mu_1^{\pm 1/2} \xi \mu_2^{\pm 1/2} \|_{\text{H.S.}} \leqslant O(1) (e^{-\alpha \operatorname{dist}(\Delta_a, \Delta)})$$

whenever $\Delta_a \cap \Delta' = \emptyset$.

C3) Set $\beta_\pm = \frac{1}{2}(\mu_2^{-1/2}\mu_1^{1/2} \pm \mu_2^{1/2}\mu_1^{-1/2})$. Then

$$\int_{\Delta_a} dx \int_{\mathbb{R}^d} dy \, | \beta_-(x, y) |^2 < C$$

uniformly in $\mathscr{A} \in \mathbb{R}^d$.

THEOREM 4.3. — *Suppose μ_1 and μ_2 satisfy* C1, C2, C3. *If the algebras $\mathscr{A}_{\mu_1}(B)$ and $\mathscr{A}_{\mu_2}(B)$ are factors, then they are isomorphic for all bounded open regions* B.

Proof. — By definition, $\mathscr{A}_{\mu_i}(B) = \pi_{\omega_0}(\mathring{\mathscr{A}}_{\mu_i}(B))''$, $i = 1, 2$. Also, for β as in C3, $\pi_{\omega_\beta}(\mathring{\mathscr{A}}_{\mu_1}(B)) = \pi_{\omega_0}(\mathring{\mathscr{A}}_{\mu_2}(B))$ and we have to show that

$$\pi_{\omega_0}(\mathring{\mathscr{A}}_{\mu_2}(B))'' = \pi_{\omega_\beta}(\mathring{\mathscr{A}}_{\mu_1}(B))'' \cong \pi_{\omega_0}(\mathring{\mathscr{A}}_{\mu_1}(B))'' . \qquad (4.6)$$

We construct a sequence of states approximating ω_β. Cover $\mathbb{R}^d$ with unit volume hypercubes. Let P_M be the projection onto fewer than $M(a + 1)^{d+1}$ particles in each hypercube at a distance a of the origin whenever $a < e^M$ and onto zero particles elsewhere (cf. equ. (4.7), (4.44) in [3]). The states $\omega_M(.) = \omega_\beta(P_M . P_M)$ are normal on $\mathscr{A}_{\mu_1}(B)$. By C1 and C3 we can apply Theorem 4.1 to conclude that $\omega_\beta(N(\Delta_a))$ and hence $\omega_M(N(\Delta_a))$ is uniformly bounded in a and M. From C2 and [3], we first conclude that

$$\omega_\beta(A) = \lim_{M \to \infty} \omega_M(A)$$

for all $A \in \mathring{\mathscr{A}}_{\mu_1}(B)$, (equ. (4.14) and equ. (4.47) of [3]). In fact by Theorem 4.1 of [3], the above limit is a *norm limit* on $\mathscr{A}_{\mu_1}(B)$, (but not a norm limit, e. g.

in the dual of $\mathscr{A}_1(\mathbb{R}^d)$). Some modifications of the original proof [3] are necessary so that it will apply in our case and these modifications have been given by Rosen in Appendix B of the preprint version of [5]. Since $\omega_\beta = n - \lim_{M\to\infty} \omega_M$, on $\mathscr{A}_{\mu_1}(B)$, we conclude that ω_β is ultraweakly continuous on $\mathscr{A}_{\mu_1}(B)$ and this proves (4.6), i. e. $\tau_\beta : \mathscr{A}_{\mu_1}(B) \to \mathscr{A}_{\mu_2}(B)$ extends to an isomorphism of factors (see e. g. [3]).

The following theorem deals with a situation which is typical for field theory applications.

THEOREM 4.4 [7]. — *Suppose*

(1) $\mathscr{A}_{\mu_1}(B)$ *is a separable factor for each* B.

(2) *If the closure of* B *is contained in the interior of* C *then* $\mathscr{A}_{\mu_1}(B)' \cap \mathscr{A}_{\mu_1}(C)$ *contains a factor of type* I_∞.

Then every isomorphism between $\mathscr{A}_{\mu_1}(B)$ *and* $\mathscr{A}_{\mu_2}(B)$ *is unitarily implementable.*

We note that the hypotheses of Theorem 4.4 follow from a mild regularity assumption on μ_1:

LEMMA 4.5. — *Let* $B \subseteq \mathbb{R}^d$ *be bounded open with piecewise smooth boundaries. Let* $\{ h_n \}$ *and* $\{ h'_n \}$ *be bases of* $\mathscr{S}(B)$ *and* $\mathscr{S}(\sim B)$, *respectively. Assume that* $L_2(\mathbb{R}^d)$ *is spanned by*

$$\{ \mu^{+1/2} h_n \} \cup \{ \mu^{+1/2} h'_n \} \quad \text{and by} \quad \{ \mu^{-1/2} h_n \} \cup \{ \mu^{-1/2} h'_n \}.$$

Then the hypotheses (1) *and* (2) *of Theorem* 4.4 *hold.*

The proof is easy (see [7] for similar results).

5. APPLICATION

We consider the case where μ_1 is the Fourier transform (F. T.) of multiplication by $(k.k + m^2)^{1/2}$ on $L^2(\mathbb{R}^d, dk)$, $m > 0$ and μ_2 is the F. T. of multiplication by $|k|$. These μ's are conventionally associated to free bose fields of mass m and mass zero respectively.

THEOREM 5.1. — *With the above μ's, in $d = 2$ or $d = 3$ space dimensions the factors* $\mathscr{A}_{\mu_1}(B)$ *and* $\mathscr{A}_{\mu_2}(B)$ *are unitarily equivalent for bounded open* B *with piecewise smooth boundaries.*

Proof. — We first apply Theorem 4.3 to show isomorphism of $\mathscr{A}_{\mu_1}(B)$ and $\mathscr{A}_{\mu_2}(B)$. C1 is satisfied with $D = \mathscr{S}(\mathbb{R}^d)_{\text{real}}$. C2 is shown e. g. in [5], equ. (6.4). We show that C3 holds. Since $\mu_i^{\pm 1/2}(x, y) = \mu_i^{\pm 1/2}(x - y)$, $i = 1, 2$, it is advantageous to go into the F. T. representation: β_- has a kernel whose F. T. is

$$\tilde{\beta}_-(k) = \frac{1}{2}\left(\frac{(k^2 + m^2)^{1/4}}{|k|^{1/2}} - \frac{|k|^{1/2}}{(k^2 + m^2)^{1/4}}\right).$$

Hence

$$0 \leqslant \tilde{\beta}_-(k) = \frac{1}{2}\frac{(k^2 + m^2)^{1/2} - |k|}{|k|^{1/2}(k^2 + m^2)^{1/4}} \leqslant 0(1)(|k| + 1)^{-3/2}.|k|^{-1/2}$$

Therefore

$$\int_{\mathbb{R}^d} dx \, |\beta_-(x - y)|^2 = \int_{\mathbb{R}^d} dk \, |\tilde{\beta}_-(k)|^2 \leqslant 0(1) \int_{\mathbb{R}^d} dk(|k| + 1)^{-3} |k|^{-1} < \infty$$

if $d = 2$ or $d = 3$.

One can verify that Lemma 4.5 applies for $\mu = \mu_1$ in $d \geq 2$ dimensions (See [9] [10] for results of this type). Hence the von Neumann algebra $\mathscr{A}_{\mu_1}(\mathrm{B})$ is a factor [7]. It is easy to show that $\mathscr{A}_{\mu_1}(\mathrm{B})$ is separable [7]. Hypothesis (2) of Theorem 4.4 is obviously satisfied. Thus we have verified the hypotheses of Theorems 4.3 and 4.4.

Q. E. D.

THEOREM 5.2. — *If* $\tilde{\mu}_i(k) = (k^2 + m_i^2)^{1/2}$, $m_i > 0$, $i = 1, 2$, *then the factors* $\mathscr{A}_{\mu_1}(\mathrm{B})$ *and* $\mathscr{A}_{\mu_2}(\mathrm{B})$ *are unitarily equivalent for* B *as in Theorem* 5.1 *and* $d = 1, 2, 3$.

Proof. — In $d = 2$, 3 dimensions the proof is identical to the one of Theorem 5.1 except for the verification of C3. Condition C3 follows from the inequality

$$\int_{\mathbb{R}^d} dx \, |\beta_-(x - y)|^2 \leqslant 0(1) \int_{\mathbb{R}^d} dk(|k| + 1)^{-4},$$

and the R. H. S. is finite for $d = 1, 2, 3$.

In $d = 1$ dimension conditions C1, C2, C3 are verified as before. However, we cannot apply lemma 4.5 to verify that $\mathscr{A}_{\mu_1}(\mathrm{B})$ is a factor. But Glimm and Jaffe [3] [7] (see also [5]) have shown that $\mathscr{A}_{\mu_1}(\mathrm{B})$ is a separable factor. Again, hypothesis (2) of Theorem 4.4 is obviously true.

Q. E. D.

Remark. — Theorem 5.2 has been shown earlier in [5].

COROLLARY. — Let s be a positive number and let $\mathrm{B}_s = \left\{ x \,|\, \frac{1}{3} x \in \mathrm{B} \right\}$.
Let μ be the Fourier transform of $(k^2 + m^2)^{1/2}$, $m > 0$ and $d = 1, 2, 3$. Then $\mathscr{A}_\mu(\mathrm{B})$ and $\mathscr{A}_\mu(\mathrm{B}_s)$ are unitarily equivalent.

Proof. — This follows from theorem 5.2 by a dilation argument. We thank Prof. M. Guenin for calling our attention to this problem.

REFERENCES

[1] D. SHALE, *Trans. A. M. S.*, t. **103**, 1962, p. 149.
J. MANUCEAU et A. VERBEURE, *Comm. Math. Phys.*, t. **8**, 1968, p. 315.
A. VAN DAELE et A. VERBEURE, *Comm. Math. Phys.*, t. **20**, 1971, p. 268.

A. VAN DAELE, *Comm. Math. Phys.*, t. **21**, 1971, p. 171.
H. ARAKI et M. SHIRAISHI, *Publ. RIMS*, t. **7**, 1972, p. 185.
[*2*] J. CHAIKEN, *Ann. Phys.*, t. **42**, 1967, p. 23.
[*3*] J. GLIMM et A. JAFFE, *Acta Math.*, t. **125**, 1970, p. 203.
[*4*] O. BRATTELI, Preprint, Courant Institute.
[*5*] L. ROSEN, *J. Math. Phys.*, t. **13**, 1972, p. 918.
[*6*] J.-P. ECKMANN, *Comm. Math. Phys.*, t. **25**, 1972, p. 1.
[*7*] J. GLIMM et A. JAFFE, In *Mathematics of Contemporary Physics*, Academic Press, London, New York, 1972.
[*8*] J. FROHLICH, *To appear*.
[*9*] H. ARAKI, *J. Math. Phys.*, t. **4**, 1963, p. 1343.
[*10*] K. OSTERWALDER, *Comm. Math. Phys.*, t. **29**, 1973, p. 1.

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