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Section A :
Physique théorique.

On the validity of Huygens' principle for second order partial differential equations with four independent variables. Part I : derivation of necessary conditions (*)

by

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ABSTRACT. — Five necessary conditions are obtained in tensorial form for the validity of Huygens' principle for second order linear partial differential equations of normal hyperbolic type in four independent variables. They are derived from Hadamard's necessary and sufficient condition by expanding the diffusion kernel in a Taylor series in normal coordinates. The transformation laws for the elementary solutions and diffusion kernel under the trivial transformations are given and the invariance of the necessary conditions under these transformations is investigated. The conditions are employed to determine the self-adjoint Huygens' differential equations on symmetric spaces.

RÉSUMÉ. — Nous donnons, sous forme tensorielle, cinq conditions nécessaires de validité du principe de Huygens pour les équations aux dérivées partielles linéaires hyperboliques du second ordre à quatre variables indépendantes. Ces conditions résultent de la condition nécessaire et suffisante de Hadamard et sont obtenues en développant en série le noyau de diffusion dans un système de coordonnées normales. On donne les règles de transformation des noyaux élémentaires et du noyau de diffusion pour les transformations triviales et on étudie l'invariance des condi-

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tions nécessaires pour ces mêmes transformations. Ces conditions sont utilisées pour déterminer, sur les espaces symétriques, les équations auto-adjointes satisfaisant le principe de Huygens.

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1. INTRODUCTION

In this paper we consider second order linear partial differential equations of normal hyperbolic type for an unknown function $u(x^1, \ldots, x^n)$ of n independent variables. Such an equation may be written in coordinate invariant form as follows:

$$F[u] \equiv g^{ab}u_{;ab} + A^a u_{,a} + Cu = 0, \tag{1.1}$$

where g^{ab} are the contravariant components of the metric tensor of a pseudo-Riemannian space V_n of signature $(+ \; - \; \ldots \; -)$ and $;a$ denotes the covariant derivative with respect to the pseudo-Riemannian connection. The coefficients g^{ab}, A^a and C as well as V_n are assumed to be of class C^∞.

Cauchy's problem for the equation (1.1) is the problem of determining a solution u of (1.1) which, on some fixed initial manifold S (1), assumes prescribed values and prescribed values for the normal derivative. These values are called Cauchy data. Cauchy's problem for (1.1) was first solved by Hadamard [*14*] who introduced the concept of a fundamental solution. Alternate solutions have been given by Mathisson [*21*], Sobolev [*28*], Bruhat [*2*], and Douglis [*7*]. Hadamard's theory is local in the sense that it is restricted to geodesic simply convex neighbourhoods of V_n. The global theory of Cauchy's problem has been developed by Leray [*18*]. The considerations in the present paper will be purely local.

Of particular importance in Cauchy's problem is the domain of dependence of the solution. In this respect the equation (1.1) is said to satisfy *Huygens' principle* (or be a *Huygens' differential equation*) if, for any Cauchy problem, the value of the solution at any point x_0 depends only on the data on the intersection $C^-(x_0) \cap S$ of the past characteristic conoid with the initial manifold S. The best known Huygens' differential equations are the wave equations

$$\frac{\partial^2 u}{\partial x^{1 2}} - \sum_{\alpha=2}^{n} \frac{\partial^2 u}{\partial x^{\alpha 2}} = 0, \tag{1.2}$$

where $n \geq 4$ is even (see for example Courant and Hilbert [*6*], p. 690).

Hadamard in his lectures on Cauchy's problem [*14*] posed the problem, as yet unsolved, of determining all Huygens' differential equations. He

(1) Assumed to satisfy $g^{ab}f_{,a}f_{,b} > 0$, where $f(x^1, \ldots, x^n) = 0$ is the equation of S.

showed that for such an equation it is necessary that $n \geq 4$ be even. Furthermore he established that a necessary and sufficient condition for the validity of Huygens' principle is the vanishing of the coefficient of the logarithmic term in the fundamental solution. Hadamard wondered if every differential equation might be transformed into the wave equation (1.2) by one or a combination of the following transformations, called *trivial transformations*, which preserve the Huygens' character of the differential equation (2):

(*a*) a transformation of coordinates,

(*b*) the multiplication of both sides of (1.1) by a non-vanishing factor $e^{-2\phi(x)}$ (this transformation induces a conformal transformation of the metric: $\tilde{g}_{ab} = e^{2\phi} g_{ab}$),

(*c*) replacing the unknown function by λu where $\lambda(x)$ is a non-vanishing function.

This is often referred to as « Hadamard's conjecture » in the literature.

The conjecture has been proved in the case $n = 4$, g^{ab} constant, A^a and C variable by Mathisson [*22*], Hadamard [*15*] and Asgeirsson [*1*]. However, it has been disproved in general by Stellmacher ([*29*] [*30*]) who provided counter examples for $n = 6, 8, \ldots$ and by Günther [*13*] who produced a family of counter examples in the physically interesting case $n = 4$. These examples arise from the metric

$$ds^2 = 2dx^0 dx^3 - a_{\alpha\beta} dx^\alpha dx^\beta \qquad (\alpha, \beta = 1, 2), \tag{1.3}$$

where $a_{\alpha\beta}$ are functions only of x^0 and the symmetric matrix $(a_{\alpha\beta})$ is positive definite. The above metric may be interpreted physically as a solution of the Einstein-Maxwell equations for exact plane waves in the General Theory of Relativity.

Günther [*11*] has derived after lengthy calculations the following four necessary conditions for the validity of Huygens' principle:

$$C = \frac{1}{2} A^a{}_{;a} + \frac{1}{4} A_a A^a + \frac{1}{6} R, \tag{1.4}$$

$$H^k{}_{a;k} = 0, \tag{1.5}$$

$$S_{abk;}{}^k - \frac{1}{2} C^k{}_{ab}{}^l L_{kl} = -5\left(H_{ak} H_b{}^k - \frac{1}{4} H_{kl} H^{kl}\right), \tag{1.6}$$

$$3S_{(ab|k|} H^k{}_{c)} + C^k{}_{(ab}{}^l H_{c)k;l} = K_{(a} g_{bc)}, \tag{1.7}$$

where R_{abcd} is the Riemannian curvature tensor, $R_{bc} = g^{ad} R_{abcd}$ is the Ricci tensor, $R = g^{bc} R_{bc}$ is the curvature scalar,

$$H_{ab} = A_{[a,b]}, \tag{1.8}$$

(2) Two equations related by trivial transformations are said to be *equivalent*. An equation equivalent to the wave equation is said to be *trivial*.

$$L_{ab} = - R_{ab} + \frac{1}{6} g_{ab} R \,, \tag{1.9}$$

$$S_{abc} = L_{a[b;c]} \,, \tag{1.10}$$

and

$$C_{abcd} = R_{abcd} - 2g_{[a[d}L_{b]c]} \tag{1.11}$$

is the Weyl conformal curvature tensor. These conditions are invariant under the trivial transformations. However, as Günther [*11*] has pointed out, they are not sufficient to characterize the Huygens' differential equations. For example when $R_{ab} = 0$ the equation (1.1) is equivalent to $g^{ab}u_{;ab} \equiv \Box u = 0$ with no further conditions on the g_{ab}.

The present author [*23*] has derived the following additional necessary condition for the case $n = 4$ and $R_{ab} = 0$:

$$\mathrm{TS}(R_{kcdl;m}R^{k}{}_{ef}{}^{l}{}_{;}{}^{m}) = 0 \,, \tag{1.12}$$

where TS( ) is the operator which forms the trace-free symmetric tensor from a given tensor. This condition implies [*23*] that the only Huygens' differential equation in four independent variables with $R_{ab} = 0$ are those arising from the metric (1.3).

Recently Wünsch [*32*] has considered differential equations which are infinitesimally close to the wave equation for $n = 4$. He derives necessary and sufficient conditions for such an equation to satisfy Huygens' principle to second order. From these conditions he singles out the one with four free indices. He then constructs a tensor which he requires to be conformally invariant and to reduce in the second order approximation to the expression with four indices in the aforementioned condition. Arguing that the full necessary condition must be unique he gives a further necessary condition for the self-adjoint equation

$$\Box u + Cu = 0 \tag{1.13}$$

to satisfy Huygens' principle. This condition may be written

$$\begin{aligned}\mathrm{TS}(3C_{kcdl;m}C^{k}{}_{ef}{}^{l}{}_{;}{}^{m} + 8C^{k}{}_{cd}{}^{l}{}_{;e}S_{klf} + 40S_{cd}{}^{k}S_{efk} - 8C^{k}{}_{cd}{}^{l}S_{kle;f} \\ - 24C^{k}{}_{cd}{}^{l}S_{efk;l} + 4C^{k}{}_{cd}{}^{l}C_{l}{}^{m}{}_{ek}L_{fm} + 12C^{k}{}_{cd}{}^{l}C^{m}{}_{efl}L_{km}) = 0 \,. \end{aligned} \tag{1.14}$$

Wünsch's derivation of (1.14) depends on the conformal invariance of the tensor on the left hand side. However, it seems to the author that a complete proof of this important property has not been given.

In this paper we derive the complete fifth necessary condition for the validity of Huygens' principle for the general equation (1.1), verifying Wünsch's result in the case $A^a = 0$. The invariance of the condition under the trivial transformations is proved in § 6. The derivation of the necessary conditions is based on Hadamard's necessary and sufficient condition extended to C^∞ equations and is a generalization of the method employed by the author in [*23*] to treat the case $R_{ab} = 0$. This method is based on the Taylor expansion of the diffusion kernel in normal coordinates about some fixed point x_0, using an appropriate choice of the trivial transformations to simplify the calculations. It seems that we are able to shorten the derivation of the conditions (1.4) to (1.7) found by Günther.

It is not known if the first five necessary conditions characterize the Huygens' differential equations. However, in a symmetric space they characterize the self-adjoint Huygens' differential equations. Helgason [*16*], p. 68 observes that if V_n is symmetric the evidence available seems to indicate that Hadamard's conjecture might hold for the equation $\Box u = 0$. We show, however, the Hadamard's conjecture is not true in this case.

2. THE NECESSARY AND SUFFICIENT CONDITION

In the modern version of Hadamard's theory for the equation (1.1) (see for example Friedlander [*10*]) the fundamental solution is replaced by the scalar distributions $E^{\pm}_{x_0}(x)$, where x_0 is a fixed point of V_4 and x is a variable point in a simply convex set Ω containing x_0. These distributions, called elementary solutions, satisfy the equation

$$G[E^{\pm}_{x_0}(x)] = \delta_{x_0}(x), \tag{2.1}$$

where

$$G[v] \equiv g^{ab}v_{;ab} - (A^a v)_{;a} + Cv \tag{2.2}$$

is the differential operator adjoint to $F[u]$ and $\delta_{x_0}(x)$ is the Dirac delta distribution. It has been shown [*19*] that these elementary solutions exist and are unique for C^∞ equations. Furthermore [*10*] for $n = 4$ they decompose as follows:

$$E^{\pm}_{x_0}(x) = V(x_0, x)\delta^{\pm}(\Gamma(x_0, x)) + \mathscr{V}^{\pm}(x_0, x)\Delta^{\pm}(x_0, x). \tag{2.3}$$

In the above V is a C^∞ function on Ω defined by

$$V(x_0, x) = \frac{1}{2\pi}\exp\left\{ -\frac{1}{4}\int_0^{s(x)} (g^{ab}\Gamma_{;ab} - 8 - A^a\Gamma_{,a})\frac{dt}{t} \right\}, \tag{2.4}$$

where the integration is along the geodesic joining x_0 to x, $\Gamma(x_0, x)$ denotes the square of the geodesic distance from x_0 to x and s is an affine parameter. Let $D^{\pm}(x_0)$ denote the interiors of the future and past pointing characteristic conoids $C^{\pm}(x_0)$. Then $\mathscr{V}^{\pm}$ are C^∞ functions on $D^{\pm}(x_0)$ defined as follows:

$$G[\mathscr{V}^{\pm}] = 0 \qquad \text{when} \qquad x \in D^{\pm}(x_0) \tag{2.5}$$

and

$$\mathscr{V}^{\pm}(x_0, x) = \frac{V(x_0, x)}{s}\int_0^{s(x)} \frac{G[V]}{V}dt \qquad \text{when} \qquad x \in C^{\pm}(x_0). \tag{2.6}$$

The functions $\mathscr{V}^{\pm}$ are thus solutions of a characteristic initial value problem [*10*]. The distributions $\delta^{\pm}(\Gamma(x_0, x))$ are defined as

$$\delta^{\pm}(\Gamma(x_0, x)) = \begin{cases} \delta(\Gamma(x_0, x)) & , \quad x \in C^{\pm}(x_0) \\ 0 & , \quad x \in C^{\mp}(x_0), \end{cases}$$

where $\delta(\ )$ represents the one dimensional Dirac delta distribution. The $\Delta^{\pm}(x_0, x)$ denote the characteristic functions on $D^{\pm}(x_0)$.

Let S be a non-compact space-like 3-manifold in the convex set Ω. Then it may be shown on taking account of (2.3) and the results of Lichnerowicz [19] that a weak solution ([3]) of Cauchy's problem for (1.1) in the future of S is

$$u(x_0) = \int_S *[(uV_{,a} - Vu_{,a} + u\mathscr{V}^-\Gamma_{,a} - uVA_a)\delta^-(\Gamma(x_0, x)) + uV\Gamma_{,a}\delta'^-(\Gamma(x_0, x)) + (u\mathscr{V}^-{}_{,a} - \mathscr{V}^-u_{,a} - u\mathscr{V}^-A_a)\Delta^-(x_0, x)]dx^a, \quad (2.7)$$

where $*$ is Hodge's operator. It is clear from the above formula that Huygens' principle will be valid if $\mathscr{V}^-(x_0, x) = 0$ for any x_0 and $x \in D^-(x_0)$ since $u(x_0)$ will depend only on the Cauchy data on the intersection $C^-(x_0) \cap S$. Conversely if Huygens' principle is true one can show that $\mathscr{V}^-(x_0, x) = 0$. Thus a necessary and sufficient condition for the validity of Huygens' principle for the retarded Cauchy problem is

$$\mathscr{V}^-(x_0, x) = 0 \qquad \forall x_0 \text{ and } \forall x \in D^-(x_0). \quad (2.8)$$

Similarly it may be shown that

$$\mathscr{V}^+(x_0, x) = 0 \qquad \forall x_0 \text{ and } \forall x \in D^+(x_0) \quad (2.9)$$

is a necessary and sufficient condition for the validity of Huygens' principle for the advanced Cauchy problem. A necessary and sufficient condition for the validity of Huygens' principle for both the advanced and retarded Cauchy's problem is

$$\mathscr{V}^{\pm}(x_0, x) = 0 \qquad \forall x_0 \text{ and } \forall x \in D^{\pm}(x_0). \quad (2.10)$$

In view of (2.10) it is clear that the validity of Huygens' principle is equivalent to the elementary solution $E^{\pm}_{x_0}(x)$ having support on the characteristic semi-conoids $C^{\pm}(x_0)$. *The condition* (2.10) *is the generalization to* C^∞ *equations of Hadamard's necessary and sufficient condition* [14].

It can be shown [23] *that the condition* (2.10) *is equivalent to*

$$[G[V]] = 0, \quad (2.11)$$

where [ ] denotes the restriction of the enclosed function to

$$C(x_0) = C^+(x_0) \cup C^-(x_0).$$

The function [G[V]] is called the *diffusion kernel*. This form of the necessary and sufficient condition is more useful than (2.10) both for the derivation of necessary conditions [15] and for showing that an equation satisfies

([3]) If the data is C^∞, then (2.7) is a genuine solution.

Huygens' principle [13]. This is due to the fact that one has an explicit form for V :

$$V = \frac{1}{2\pi}\rho^{-1/2} \exp\left\{\frac{1}{4}\int_0^s A^a\Gamma_{,a}\frac{dt}{t}\right\}, \tag{2.12}$$

where

$$\rho = 8(g(x)g(x_0))^{1/2}\left[\det \frac{\partial^2\Gamma}{\partial x^a \partial x_0{}^b}\right]^{-1} \tag{2.13}$$

is the discriminant function and $g(x) = \det(g_{ab}(x))$.

It should be pointed out that the condition (2.11) is also valid when the coefficients of the differential equation (1.1) are merely sufficiently differentiable (see Chevalier [4] and Douglis [8]).

3. THE TRIVIAL TRANSFORMATIONS

We now turn our attention to the transformations (*a*), (*b*) and (*c*) defined in the introduction. Appropriate choices of these transformations will simplify our expansion of the diffusion kernel. Excluding consideration of (*a*) for the moment, we consider the effect only of (*b*) and the transformation Hadamard calls (*bc*) defined as follows:

(*bc*) Replacement of the function u in (1.1) by λu ($\lambda(x) \neq 0$) and simultaneous multiplation of the equation by λ^{-1}.

This transformation leaves invariant the pseudo-Riemannian metric.

The transformations (*b*) and (*bc*) transform the differential operator $F[u]$ into a similar operator $\bar{F}[u]$ with different coefficients $\tilde{g}^{ab}$, $\bar{A}^a$ and $\bar{C}$ and a different (conformally related) pseudo-Riemannian metric:

$$\bar{F}[u] \equiv \tilde{g}^{ab}u_{\tilde{;}ab} + \bar{A}^a u_{,a} + \bar{C}u = \lambda^{-1}e^{-2\phi}F[\lambda u]. \tag{3.1}$$

One has the following relations between the coefficients of $\bar{F}[u]$ and $F[u]$:

$$\tilde{g}^{ab} = e^{-2\phi}g^{ab} \qquad \text{or} \qquad \tilde{g}_{ab} = e^{2\phi}g_{ab}, \tag{3.2a}$$

$$\bar{A}_a = A_a + 2(\log\lambda)_{,a} - (n-2)\phi_{,a}\ , \quad \bar{A}^a = \tilde{g}^{ab}\bar{A}_b\ , \quad A_a = g_{ab}A^b, \tag{3.2b}$$

$$\bar{C} = e^{-2\phi}(C + \lambda^{-1}\Box\lambda + A^a(\log\lambda)_{,a}), \tag{3.2c}$$

It has been shown by Cotton [5] (4) that the necessary and sufficient conditions for (1.1) to be equivalent to the wave equation (1.2) are

$$C_{abcd} = 0, \tag{3.3a}$$

$$H_{ab} \equiv A_{[a,b]} = 0, \tag{3.3b}$$

$$\mathscr{C} \equiv C - \frac{1}{2}A^a{}_{;a} - \frac{1}{4}A_aA^a - \frac{n-2}{4(n-1)}R = 0. \tag{3.3c}$$

(4) See also Günther [12].

These conditions are invariant under the trivial transformations, since, on account of (3.2), $C^a{}_{bcd}$, H_{ab} and $\mathscr{C}$ transform as follows:

$$\tilde{C}^a{}_{bcd} = C^a{}_{bcd}\,, \tag{3.4a}$$

$$\bar{H}_{ab} = H_{ab}\,, \tag{3.4b}$$

$$\bar{\mathscr{C}} = e^{-2\phi}\mathscr{C}. \tag{3.5}$$

We shall now derive the relation between the elementary solutions for equivalent operators. We first note that the transformations (*b*) and (*bc*) of F[*u*], (3.1), induces the following transformation for the adjoint operator G[*v*]:

$$\bar{G}[v] = \lambda e^{-n\phi}G[\lambda^{-1}e^{(n-2)\phi}v]\,. \tag{3.6}$$

If $\bar{E}^{\pm}_{x_0}(x)$ are the elementary solutions of $\bar{G}[v]$, then

$$\bar{G}[\bar{E}^{\pm}_{x_0}(x)] = \tilde{\delta}_{x_0}(x)\,, \tag{3.7}$$

where $\tilde{\delta}_{x_0}(x) = e^{-n\phi(x)}\delta_{x_0}(x)$ is the Dirac delta distribution on $\tilde{V}_n$. Combining (3.6) and (3.7) we have

$$G[\lambda_0\lambda^{-1}e^{(n-2)\phi}\bar{E}^{\pm}_{x_0}(x)] = \delta_{x_0}(x)\,. \tag{3.8}$$

Thus by uniqueness of the elementary solutions we have

$$\bar{E}^{\pm}_{x_0}(x) = \lambda\lambda_0^{-1}e^{(2-n)\phi}E^{\pm}_{x_0}(x)\,, \tag{3.9}$$

where $\lambda_0 = \lambda(x_0)$, *in particular when* $n = 4$

$$\bar{E}^{\pm}_{x_0}(x) = \lambda\lambda_0^{-1}e^{-2\phi}E^{\pm}_{x_0}(x)\,. \tag{3.10}$$

Equation (3.10) enables us to derive the transformation laws for [V], $\mathscr{V}^{\pm}$ and [G[V]]. Using the decomposition of $E^{\pm}_{x_0}(x)$ given in (2.3) we have

$$\bar{V}\delta^{\pm}(\tilde{\Gamma}(x_0, x)) + \bar{\mathscr{V}}^{\pm}\Delta^{\pm}(x_0, x) = \lambda\lambda_0^{-1}e^{-2\phi}(V\delta^{\pm}(\Gamma(x_0, x)) + \mathscr{V}^{\pm}\Delta^{\pm}(x_0, x))\,. \tag{3.11}$$

We must first find the relation between $\delta^{\pm}(\tilde{\Gamma})$ and $\delta^{\pm}(\Gamma)$ or equivalently between $\delta(\tilde{\Gamma})$ and $\delta(\Gamma)$. Since $\Gamma = 0$ if and only if $\tilde{\Gamma} = 0$ we set

$$\tilde{\Gamma} = a_1\Gamma + a_2\Gamma^2 + \dots\,, \tag{3.12}$$

where the a_i are functions of x_0 and x to be determined. From the fact that Γ and $\tilde{\Gamma}$ satisfy respectively the equations

$$g^{ab}\Gamma_{,a}\Gamma_{,b} = 4\Gamma \quad , \quad \tilde{g}^{ab}\tilde{\Gamma}_{,a}\tilde{\Gamma}_{,b} = 4\tilde{\Gamma}\,, \tag{3.13}$$

we find, on substituting for $\tilde{\Gamma}$ from (3.12) in the second equation in (3.13) and equating coefficients of equal powers of Γ, the following differential equation for a_1:

$$s\frac{da_1}{ds} + a_1 = e^{2\phi}\,. \tag{3.14}$$

This equation has the regular solution at $s = 0$

$$a_1 = \frac{1}{s}\int_0^{s(x)} e^{2\phi}dt\,, \tag{3.15}$$

where one integrates along the null geodesic x_0x with respect to an affine parameter. Since

$$\delta(\Gamma) = \left(\frac{d\tilde{\Gamma}}{d\Gamma}\right)_{\Gamma=0}\delta(\tilde{\Gamma})\,,$$

one has in view of (3.12) and (3.15)

$$\delta(\tilde{\Gamma}) = a_1^{-1}\delta(\Gamma)\,. \tag{3.16}$$

Thus we may conclude from (3.11) *that*

$$[\bar{V}] = \lambda_0^{-1}a_1[\lambda e^{-2\phi}V] \tag{3.17}$$

and

$$\bar{\mathcal{V}}^{\pm} = \lambda_0^{-1}\lambda e^{-2\phi}\mathcal{V}^{\pm}\,. \tag{3.18}$$

From (3.17) and (3.18) the transformation law for the diffusion kernel may be deduced. Differentiating (2.6) yields

$$\frac{d}{ds}\left[\frac{s\mathcal{V}^{\pm}}{V}\right] = -\left[\frac{G[V]}{4V}\right]. \tag{3.19}$$

For the transformed operator one has equivalently

$$\frac{d}{d\tilde{s}}\left[\frac{\tilde{s}\bar{\mathcal{V}}^{\pm}}{\bar{V}}\right] = -\left[\frac{\bar{G}[\bar{V}]}{4\bar{V}}\right], \tag{3.20}$$

where $\tilde{s}$ is an affine parameter along the generators of $\tilde{C}(x_0) = C(x_0)$ ([5]) related to s along a fixed null geodesic by

$$\tilde{s} = \int_0^s e^{2\phi}dt\,. \tag{3.20}$$

In view of this and the transformation laws (3.17) and (3.18) equation (3.20) becomes

$$e^{-2\phi}\frac{d}{ds}\left[\frac{s\mathcal{V}^{\pm}}{V}\right] = -\left[\frac{\bar{G}[V]}{4\lambda_0^{-1}\lambda e^{-2\phi}V}\right].$$

Noting (3.19) we finally get

$$[\bar{G}[\bar{V}]] = \lambda_0^{-1}a_1[\lambda e^{-4\phi}G[V]]\,, \tag{3.21}$$

which is the transformation law for the diffusion kernel.

We may immediately deduce from (3.21) that the property of being a

([5]) The null conoids are identical since null geodesics are preserved under conformal transformations.

Huygens' operator is invariant under trivial transformations since $[\bar{G}[\bar{V}]] = 0$ if and only if $[G[V]] = 0$.

We consider now only the transformation (*bc*) which has the property of preserving the pseudo-Riemannian metric. In this case one can show using (2.4) and (3.2*b*) that

$$\bar{V} = \lambda_0^{-1}\lambda V . \tag{3.22}$$

In contrast to (3.17) the above relation holds at every point in some normal neighbourhood of x_0, not just on $C(x_0)$.

We are now in measure to specify how the trivial transformations will be chosen. Consider first the conformal transformation (*b*). We note that under a conformal transformation the tensor L_{ab}, defined in (1.9), transforms as follows:

$$\tilde{L}_{ab} = L_{ab} - 2\phi_{a;b} + 2\phi_a\phi_b - g_{ab}\phi_k\phi^k , \tag{3.23}$$

where $\phi_a = \phi_{,a}$. Let x_0 be any point of V_4. Then following Günther [11] (6) we can choose the derivatives of ϕ at x_0 such that

$$\overset{\circ}{\tilde{L}}_{ab} = 0 , \tag{3.24a}$$

$$\overset{\circ}{\tilde{L}}_{(ab;c)} = 0 , \tag{3.24b}$$

$$\overset{\circ}{\tilde{L}}_{(ab;cd)} = 0 , \tag{3.24c}$$

$$\vdots$$

where $\overset{\circ}{\tilde{L}}_{ab} = \tilde{L}_{ab}(x_0)$ and so on. We assume from here on that this transformation has been carried out and omit the tildes. Consequently at x_0 one has

$$\begin{gathered}
\overset{\circ}{R}_{ab} = 0 \quad , \quad \overset{\circ}{R} = 0 \quad , \quad \overset{\circ}{R}_{abcd} = \overset{\circ}{C}_{abcd} , \\
\overset{\circ}{R}_{ab;cd} = \overset{\circ}{R}_{ab;dc} \quad , \quad \overset{\circ}{L}_{ab;cd} = \overset{\circ}{L}_{ab;dc} , \\
\overset{\circ}{R}_{,a} = 0 \quad , \quad \overset{\circ}{L}_{ab;c} = -\overset{\circ}{R}_{ab;c} , \\
\Box\overset{\circ}{R} = 0 \quad , \quad \Box\overset{\circ}{R}_{ab} = -\frac{5}{3}\overset{\circ}{R}_{;ab} = -\Box\overset{\circ}{L}_{ab} , \\
\overset{\circ}{L}_{ab;c} = \frac{4}{3}\overset{\circ}{S}_{(ab)c} \quad , \quad 3\overset{\circ}{L}_{ab;cd} = 5\overset{\circ}{S}_{(ab)(c;d)} - \overset{\circ}{S}_{(cd)(a;b)} .
\end{gathered} \tag{3.25}$$

We now specify the choice of the transformation (*bc*). Following Hadamard [15] we set for the same point x_0 as above

$$\lambda(x) = \exp\left\{ -\frac{1}{4}\int_0^{s(x)} A^a\Gamma_{,a}\frac{ds}{s} \right\} . \tag{3.26}$$

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(6) See also Szekeres [31].

Consequently $\lambda_0 = \lambda(x_0) = 1$ and (2.12) and (3.22) imply

$$\bar{V} = \frac{1}{2\pi}\rho^{-1/2} . \tag{3.27}$$

It is equivalent to state that $\bar{A}^a$ given by (3.2*b*) (with $\phi = 0$) satisfies

$$\int_0^{s(x)} \bar{A}^a \Gamma_{,a} \frac{dt}{t} = 0$$

for all $x \in \Omega$ which implies in turn that

$$\bar{A}^a \Gamma_{,a} = 0 . \tag{3.28}$$

It should be emphasized that the transformation (3.26) depends in general on the choice of the point x_0. From here on it is assumed that the transformation (3.26) has been made, implying that $\bar{V}$ has the form (3.27). With this understanding the bars are dropped from the transformed quantities.

Finally it remains to choose the transformation (*a*) namely the system of coordinates in which to carry out the calculations. We choose a system of normal coordinates (x^a) about the point x_0 admissable in the convex set Ω. These coordinates are defined by the condition [*26*]

$$g_{ab}x^b = \overset{\circ}{g}_{ab}x^b . \tag{3.29}$$

In normal coordinates V takes the simple form [*26*] ([7])

$$V \overset{*}{=} \frac{1}{2\pi}\left(\frac{\overset{\circ}{g}}{g}\right)^{1/4} . \tag{3.30}$$

It is easy to show from (3.30) that

$$V_{,a} \overset{*}{=} -\frac{1}{4} V g^{bc} g_{bc,a} \tag{3.31}$$

and

$$\Box V \overset{*}{=} -\frac{1}{4} V\gamma , \tag{3.32}$$

where

$$\gamma \overset{*}{=} (g^{ab}g^{cd}g_{cd,a})_{,b} + \frac{1}{4} g^{ab}g_{ab,c}g^{cd}g^{ef}g_{ef,d} . \tag{3.33}$$

Consequently, if one defines

$$\sigma = -\frac{4G[V]}{V} , \tag{3.34}$$

one has

$$\sigma \overset{*}{=} \gamma + A^a g^{bc} g_{bc,a} + 4A^a{}_{,a} - 4C . \tag{3.35}$$

From (3.30) we see that $V(x_0, x) \neq 0$ for $x \in \Omega$. Thus $[\sigma] = 0$ if and only

([7]) $\overset{*}{=}$ signifies equality only in a system of normal coordinates.

if [G[V]] = 0 and we conclude that Huygens' principle is valid if and only if

$$[\sigma[x_0, x)] = 0 \qquad \forall x_0 \in V_4 \,. \tag{3.36}$$

In view of (3.17), (3.21) and (3.34) the quantity $[\sigma]$ transforms as follows:

$$[\bar{\sigma}] = [e^{-2\phi}\sigma] \,. \tag{3.37}$$

4. THE TAYLOR EXPANSION OF σ

Our objective now is to obtain a covariant Taylor expansion for G[V] or rather σ about an arbitrarily chosen point x_0. We shall determine the expansion in terms of a system of normal coordinates (x^a) with origin x_0. This expansion may be obtained from (3.33) and (3.35) once the expansions of g_{ab}, g^{ab}, A^a and C are known. In [*23*] the methods of Herglotz [*17*] and Günther [*11*] are used to obtain the following covariant expansions for g_{ab} and g^{ab} to sixth order and fourth order respectively:

$$\begin{aligned} g_{ab} \overset{*}{=} \mathring{g}_{ab} &+ \frac{1}{3}\mathring{R}_{acdb}x^{cd} + \frac{1}{6}\mathring{R}_{acdb;e}x^{cde} + \frac{1}{20}\left(\mathring{R}_{acdb;ef} + \frac{8}{9}\mathring{R}_{acdk}\mathring{R}^{k}{}_{efb}\right)x^{cdef} \\ &+ \frac{1}{90}(\mathring{R}_{acdb;efg} + 2(\mathring{R}_{acdk}\mathring{R}^{k}{}_{efb;g} + \mathring{R}_{acdk;e}\mathring{R}^{k}{}_{fgb}))x^{cdefg} \\ &+ \frac{1}{504}\left(\mathring{R}_{acdb;efgh} + \frac{17}{5}(\mathring{R}_{acdk}\mathring{R}^{k}{}_{efb;gh} + \mathring{R}_{acdk;ef}\mathring{R}^{k}{}_{ghb})\right. \\ &\left. + \frac{8}{9}\mathring{R}_{acdk}\mathring{R}^{k}{}_{efl}\mathring{R}^{l}{}_{ghb} + \frac{11}{2}\mathring{R}_{acdk;e}\mathring{R}^{k}{}_{fgb;h}\right)x^{cdefgh}, \end{aligned} \tag{4.1}$$

$$\begin{aligned} g^{ab} \overset{*}{=} \mathring{g}^{ab} &- \frac{1}{3}\mathring{R}^{a}{}_{cd}{}^{b}x^{cd} - \frac{1}{6}\mathring{R}^{a}{}_{cd}{}^{b}{}_{;e}x^{cde} \\ &- \frac{1}{20}\left(\mathring{R}^{a}{}_{cd}{}^{b}{}_{;ef} - \frac{4}{3}\mathring{R}^{a}{}_{cdk}\mathring{R}^{k}{}_{ef}{}^{b}\right)x^{cdef}, \end{aligned} \tag{4.2}$$

where $x^{cd} = x^c x^d$ and so on. Assuming that the conformal transformation (3.24) has been made, one obtains from (3.33), (4.1) and (4.2) the expansion of γ to fourth order:

$$\begin{aligned} \gamma &\overset{*}{=} \frac{3}{5}\left(\mathring{R}_{cd;ef} - \frac{2}{9}\mathring{C}_{kcdl}\mathring{C}^{k}{}_{ef}{}^{l}\right)\mathring{g}^{(cd}x^{ef)} \\ &+ \frac{4}{9}(\mathring{R}_{cd;efg} - \mathring{R}_{kcdl}\mathring{R}^{k}{}_{ef}{}^{l}{}_{;g})\mathring{g}^{(cd}x^{efg)} \\ &+ \frac{1}{1512}(90\mathring{R}_{cd;efgh} - 144\mathring{R}_{kcdl}\mathring{R}^{k}{}_{ef}{}^{l}{}_{;gh} + 32\mathring{C}^{p}{}_{cdk}\mathring{C}^{k}{}_{efl}\mathring{C}^{l}{}_{ghp} \\ &- 135\mathring{R}_{kcdl;e}\mathring{R}^{k}{}_{fg}{}^{l}{}_{;h})\mathring{g}^{(cd}x^{efgh)} - \frac{1}{5}\left(\mathring{R}_{cd;ef} - \frac{2}{9}\mathring{C}_{kcdl}\mathring{C}^{k}{}_{ef}{}^{l}\right)\mathring{C}^{(c}{}_{gh}{}^{d}x^{ef)}x^{gh}. \end{aligned} \tag{4.3}$$

We shall consider the construction of a covariant expansion for A_a. Expanding in normal coordinates about x_0 one has to pth order

$$A_a = \overset{\circ}{A}_a + \overset{\circ}{A}_{a,a_1} x^{a_1} + \frac{1}{2!} \overset{\circ}{A}_{a,a_1 a_2} x^{a_1 a_2} + \ldots + \frac{1}{p!} \overset{\circ}{A}_{a,a_1 \ldots a_p} x^{a_1 \ldots a_p} + \ldots . \quad (4.4)$$

We are assuming that the transformation (bc) has been made as a specified in (3.26). Consequently (3.28) is true. Now in normal coordinates [26]

$$g^{ab}\Gamma_{,b} \overset{*}{=} 2x^a . \quad (4.5)$$

Thus (3.28) becomes

$$A_a x^a \overset{*}{=} 0 \quad (4.6)$$

which is the same as that obtained by Hadamard [15] in the flat space case. Combining (4.4).and (4.6) yields

$$0 \overset{*}{=} \overset{\circ}{A}_a x^a + \overset{\circ}{A}_{a,a_1} x^{aa_1} + \frac{1}{2!} \overset{\circ}{A}_{a,a_1 a_2} x^{aa_1 a_2} + \ldots + \frac{1}{p!} \overset{\circ}{A}_{a,a_1 \ldots a_p} x^{aa_1 \ldots a_p} + \ldots .$$

Since this must hold for all $x \in \Omega$, one has at x_0 (8)

$$\begin{aligned} \overset{\circ}{A}_a &= 0 , \\ \overset{\circ}{A}_{(a,a_1)} &= 0 , \\ \overset{\circ}{A}_{(a,a_1 a_2)} &= 0 , \\ &\vdots \\ \overset{\circ}{A}_{(a,a_1 \ldots a_p)} &= 0 . \end{aligned} \quad (4.7)$$

It is a consequence of the conditions (4.7) and the symmetry of $\overset{\circ}{A}_{a,a_1 \ldots a_p}$ in the indices $a_1 \ldots a_p$ that

$$\overset{\circ}{A}_{a,a_1 \ldots a_p} = \frac{2p}{p+1} \overset{\circ}{H}_{a(a_1,a_2 \ldots a_{p-1})} . \quad (4.8)$$

Thus from (4.4) and (4.8) one has to fifth order

$$A_a \overset{*}{=} \overset{\circ}{H}_{ab} x^b + \frac{2}{3} \overset{\circ}{H}_{ab,c} x^{bc} + \frac{1}{4} \overset{\circ}{H}_{ab,cd} x^{bcd} + \frac{1}{15} \overset{\circ}{H}_{ab,cde} x^{bcde} + \frac{1}{72} \overset{\circ}{H}_{ab,cdef} x^{bcdef} . \quad (4.9)$$

It remains to replace the partial derivatives in (4.9) by covariant derivatives. We shall achieve this by expanding $H_{ab}x^b$ covariantly. Let

$$\overset{p}{\nabla} H = \left(H_{ab;a_1 \ldots a_p} x^{ba_1 \ldots a_p} \right) . \quad (4.10)$$

(8) These are the relations obtained by Mathisson [22] and Günther [11] from (3.2b) by a suitable choice of the derivatives of log λ at x_0. However, there true origin seems to a consequence of the choice (3.26).

Then the following recurrsion formula is valid in normal coordinates:

$$\overset{p}{\nabla}\mathrm{H} \overset{*}{=} \mathrm{X}(\overset{p-1}{\nabla}\mathrm{H}) - p\overset{p-1}{\nabla}\mathrm{H} - \frac{1}{2}\mathrm{X}(\mathrm{G})\mathrm{G}^{-1}\overset{p-1}{\nabla}\mathrm{H}, \qquad p = 1, 2, 3, \ldots \tag{4.11}$$

where $\mathrm{X}(f) = f_{,a}x^a$, $\mathrm{G} = (g_{ab})$ and $\mathrm{G}^{-1} = (g^{ab})$. We further define

$$\overset{p}{\nabla}\mathrm{H}_{p+k+1} = \left(\overset{\circ}{\mathrm{H}}_{ab;a_1\ldots a_p,a_{p+1}\ldots a_{p+k}}x^{ba_1\ldots a_pa_{p+1}\ldots a_{p+k}}\right), \tag{4.12}$$

where $p = 0, 1, 2, \ldots$. The object will be to find the relations for $p \leq 4$ between the $\overset{p}{\nabla}\mathrm{H}_{p+1}$ and the $\mathrm{H}_{p+1} = \overset{\circ}{\nabla}\mathrm{H}_{p+1}$. This is achieved by expanding the $\overset{p}{\nabla}\mathrm{H}$ to fifth order for $p = 0, \ldots 4$:

$$\mathrm{H} = \mathrm{H}_1 + \mathrm{H}_2 + \frac{1}{2}\mathrm{H}_3 + \frac{1}{6}\mathrm{H}_4 + \frac{1}{24}\mathrm{H}_5, \tag{4.13}$$

$$\overset{1}{\nabla}\mathrm{H} = \overset{1}{\nabla}\mathrm{H}_2 + \overset{1}{\nabla}\mathrm{H}_3 + \frac{1}{2}\overset{1}{\nabla}\mathrm{H}_4 + \frac{1}{6}\overset{1}{\nabla}\mathrm{H}_5, \tag{4.14}$$

$$\overset{2}{\nabla}\mathrm{H} = \overset{2}{\nabla}\mathrm{H}_3 + \overset{2}{\nabla}\mathrm{H}_4 + \frac{1}{2}\overset{2}{\nabla}\mathrm{H}_5, \tag{4.15}$$

$$\overset{3}{\nabla}\mathrm{H} = \overset{3}{\nabla}\mathrm{H}_4 + \overset{3}{\nabla}\mathrm{H}_5, \tag{4.16}$$

$$\overset{4}{\nabla}\mathrm{H} = \overset{4}{\nabla}\mathrm{H}_5, \tag{4.17}$$

and using the recurrence relation (4.11). For each $p = 1, \ldots, 4$ the left and right hand sides of (4.11) are expanded to fifth order by substituting from the appropriate formulae (4.13) to (4.17). The required form is obtained on noting that

$$\mathrm{X}(\overset{p}{\nabla}\mathrm{H}_{p+k+1}) = (p + k + 1)\overset{p}{\nabla}\mathrm{H}_{p+k+1} \tag{4.18}$$

and that

$$\mathrm{X}(\mathrm{G})\mathrm{G}^{-1} = \frac{2}{3}\mathrm{R}_2\overset{\circ}{\mathrm{G}}{}^{-1} + \frac{1}{3}\overset{1}{\nabla}\mathrm{R}_3\overset{\circ}{\mathrm{G}}{}^{-1} + \frac{1}{5}\overset{2}{\nabla}\mathrm{R}_4\overset{\circ}{\mathrm{G}}{}^{-1} - \frac{2}{45}(\mathrm{R}_2\overset{\circ}{\mathrm{G}}{}^{-1})^2, \tag{4.19}$$

where

$$\overset{p}{\nabla}\mathrm{R}_{p+2} = \left(\overset{\circ}{\mathrm{R}}_{acdb;a_1\ldots a_p}x^{cda_1\ldots a_p}\right). \tag{4.20}$$

On equating terms of the same order on opposite sides in the equations just described the following four sets of equations are obtained:

for $p = 1$:

$$\begin{aligned}
&\overset{1}{\nabla}\mathrm{H}_2 = \mathrm{H}_2 \quad , \quad \overset{1}{\nabla}\mathrm{H}_3 = \mathrm{H}_3 - \frac{1}{3}\mathrm{R}_2\overset{\circ}{\mathrm{G}}{}^{-1}\mathrm{H}_1, \\
&\overset{1}{\nabla}\mathrm{H}_4 = \mathrm{H}_4 - \frac{2}{3}\mathrm{R}_2\overset{\circ}{\mathrm{G}}{}^{-1}\mathrm{H}_2 - \frac{1}{2}\overset{1}{\nabla}\mathrm{R}_3\overset{\circ}{\mathrm{G}}{}^{-1}\mathrm{H}_1, \\
&\overset{1}{\nabla}\mathrm{H}_5 = \mathrm{H}_5 - \mathrm{R}_2\overset{\circ}{\mathrm{G}}{}^{-1}\mathrm{H}_3 - \frac{3}{2}\overset{1}{\nabla}\mathrm{R}_3\overset{\circ}{\mathrm{G}}{}^{-1}\mathrm{H}_2 - \frac{3}{5}\overset{2}{\nabla}\mathrm{R}_4\overset{\circ}{\mathrm{G}}{}^{-1}\mathrm{H}_1 + \frac{2}{15}(\mathrm{R}_2\overset{\circ}{\mathrm{G}}{}^{-1})^2\mathrm{H}_1;
\end{aligned} \tag{4.21}$$

for $p = 2$:

$$\overset{2}{\nabla}H_3 = \overset{1}{\nabla}H_3 \quad , \quad \overset{2}{\nabla}H_4 = \overset{1}{\nabla}H_4 - \frac{1}{3}R_2\overset{\circ}{G}{}^{-1}\overset{1}{H}_2 \, ,$$
$$\overset{2}{\nabla}H_5 = \overset{1}{\nabla}H_5 - \frac{2}{3}R_2\overset{\circ}{G}{}^{-1}\overset{1}{\nabla}H_3 - \frac{1}{2}\overset{1}{\nabla}R_3\overset{\circ}{G}{}^{-1}\overset{1}{\nabla}H_2 \, ; \tag{4.22}$$

for $p = 3$:

$$\overset{3}{\nabla}H_4 = \overset{2}{\nabla}H_4 \quad , \quad \overset{3}{\nabla}H_5 = \overset{2}{\nabla}H_5 - \frac{1}{3}R_2\overset{\circ}{G}{}^{-1}\overset{2}{\nabla}H_3 \, ; \tag{4.23}$$

for $p = 4$:

$$\overset{4}{\nabla}H_5 = \overset{3}{\nabla}H_5 \, . \tag{4.24}$$

These four sets of equations may now be solved for the H_{p+1} in terms of the $\overset{p}{\nabla}H_{p+1}$. The solutions are

$$\begin{aligned}
H_2 &= \overset{1}{\nabla}H_2 \, , \\
H_3 &= \overset{2}{\nabla}H_3 + \frac{1}{3}R_2\overset{\circ}{G}{}^{-1}H_1 \, , \\
H_4 &= \overset{3}{\nabla}H_4 + R_2\overset{\circ}{G}{}^{-1}\overset{1}{\nabla}H_2 + \frac{1}{2}\overset{1}{\nabla}R_3\overset{\circ}{G}{}^{-1}H_1 \, , \\
H_5 &= \overset{4}{\nabla}H_5 + 2R_2\overset{\circ}{G}{}^{-1}\overset{2}{H}_3 + \frac{1}{5}(R_2\overset{\circ}{G}{}^{-1})^2H_1 + 2\overset{1}{\nabla}R_3\overset{\circ}{G}{}^{-1}\overset{1}{\nabla}H_2 \\
&\qquad + \frac{3}{5}\overset{2}{\nabla}R_4\overset{\circ}{G}{}^{-1}H_1 \, .
\end{aligned} \tag{4.25}$$

From (4.9), (4.20) and the above we may construct the following covariant expansion for A_a :

$$\begin{aligned}
A_a \overset{*}{=} \overset{\circ}{H}_{ab}x^b &+ \frac{2}{3}\overset{\circ}{H}_{ab;c}x^{bc} + \frac{1}{4}\left(\overset{\circ}{H}_{ab;cd} + \frac{1}{3}\overset{\circ}{R}_{abc}{}^k\overset{\circ}{H}_{kd}\right)x^{bcd} \\
&+ \frac{1}{15}\left(\overset{\circ}{H}_{ab;cde} + \overset{\circ}{R}_{abc}{}^k\overset{\circ}{H}_{kd;e} + \frac{1}{2}\overset{\circ}{R}_{abc}{}^k{}_{;d}\overset{\circ}{H}_{ke}\right)x^{bcde} \\
&+ \frac{1}{72}\left(\overset{\circ}{H}_{ab;cdef} + 2\overset{\circ}{R}_{abc}{}^k\overset{\circ}{H}_{kd;ef} + \frac{1}{5}\overset{\circ}{R}_{abc}{}^l\overset{\circ}{R}_{lde}{}^k\overset{\circ}{H}_{kf}\right. \\
&\left. + 2\overset{\circ}{R}_{abc}{}^k{}_{;d}\overset{\circ}{H}_{ke;f} + \frac{3}{5}\overset{\circ}{R}_{abc}{}^k{}_{;de}\overset{\circ}{H}_{kf}\right)x^{bcdef} \, .
\end{aligned} \tag{4.26}$$

According to (3.34) we actually require $A^a{}_{,a}$; we shall also need A_aA^a.

Using (4.2) and (4.26) we obtain to fourth order the following expansions, assuming the conformal transformation (3.24) has been made:

$$
\begin{aligned}
A^{a}{}_{,a} \overset{*}{=} \frac{2}{3}\overset{\circ}{H}{}^{c}{}_{b;c}x^{b} + \frac{3}{4}(\overset{\circ}{H}{}^{b}{}_{(b;cd)} - \overset{\circ}{C}{}^{b}{}_{(bc}{}^{k}\overset{\circ}{H}{}_{|k|d)})x^{cd} \\
+ \frac{4}{15}\left(\overset{\circ}{H}{}^{b}{}_{(b;cde)} - \frac{7}{3}\overset{\circ}{C}{}^{b}{}_{(bc}{}^{k}\overset{\circ}{H}{}_{|k|d;e)} - 2\overset{\circ}{R}{}^{b}{}_{(bc}{}^{k}{}_{;d}\overset{\circ}{H}{}_{|k|e)}\right)x^{cde} \\
+ \frac{5}{72}(\overset{\circ}{H}{}^{b}{}_{(b;cdef)} - 4\overset{\circ}{C}{}^{b}{}_{(bc}{}^{k}\overset{\circ}{H}{}_{|k|d;ef)} + 3\overset{\circ}{C}{}^{b}{}_{(bc}{}^{l}\overset{\circ}{C}{}_{|l|de}{}^{k}\overset{\circ}{H}{}_{|k|f)}) \\
- 6\overset{\circ}{R}{}^{b}{}_{(bc}{}^{k}{}_{;d}\overset{\circ}{H}{}_{|k|e;f)} - 3\overset{\circ}{R}{}^{b}{}_{(bc}{}^{k}{}_{;de}\overset{\circ}{H}{}_{|k|f)}x^{cdef}\,,
\end{aligned}
\tag{4.27}
$$

$$
\begin{aligned}
A_{a}A^{a} \overset{*}{=} \overset{\circ}{H}{}_{ka}\overset{\circ}{H}{}^{k}{}_{b}x^{ab} + \frac{4}{3}\overset{\circ}{H}{}_{ka}\overset{\circ}{H}{}^{k}{}_{b;c}x^{abc} \\
+ \frac{1}{18}(9\overset{\circ}{H}{}_{ka}\overset{\circ}{H}{}^{k}{}_{b;cd} - 3\overset{\circ}{H}{}_{ka}\overset{\circ}{C}{}^{k}{}_{bc}{}^{l}\overset{\circ}{H}{}_{ld} + 8\overset{\circ}{H}{}_{ka;b}\overset{\circ}{H}{}^{k}{}_{c;d})x^{abcd}\,.
\end{aligned}
\tag{4.28}
$$

On account of (3.24) we have to second order

$$
A^{a}g^{bc}g_{bc,a} \overset{*}{=} \frac{2}{3}\overset{\circ}{H}{}^{a}{}_{b}\overset{\circ}{R}{}_{ac}x^{b}x^{c} = 0\,. \tag{4.29}
$$

Finally the expansion of the scalar C is to second order

$$
C \overset{*}{=} \overset{\circ}{C} + \overset{\circ}{C}{}_{;a}x^{a} + \frac{1}{2}\overset{\circ}{C}{}_{;ab}x^{ab}\,. \tag{4.30}
$$

The required covariant Taylor expansion for σ may now be obtained from (3.35) upon making the appropriate substitutions from (4.27), (4.29), (4.3) and (4.30).

5. DERIVATION OF NECESSARY CONDITIONS

Five sets of necessary conditions for the validity of Huygens' principle expressed in covariant form will now be obtained from the condition (3.36) and the covariant Taylor expansion for σ constructed in § 4. The following derivation is similar to that employed by the author in [*23*], the basic method being the same as that used by Mathisson [*22*], Hadamard [*15*] and Günther [*11*]. We shall proceed as follows: Huygens' principle is assumed for (1.1) and an arbitrary point x_0 is chosen. Consequently $[\sigma(x_0, x)] = 0$ or $\sigma(x_0, x) = 0 \ \forall x \in C(x_0)$, which implies the following conditions hold at x_0(see [*23*], p. 144):

$$
\overset{\circ}{\sigma} = 0\,, \tag{5.1a}
$$

$$
\overset{\circ}{\sigma}{}_{,a} = 0\,, \tag{5.1b}
$$

$$\mathrm{TS}(\overset{\circ}{\sigma}_{;ab}) = 0\,, \tag{5.1c}$$

$$\mathrm{TS}(\overset{\circ}{\sigma}_{;abc}) = 0\,, \tag{5.1d}$$

$$\mathrm{TS}(\overset{\circ}{\sigma}_{;abcd}) = 0\,, \tag{5.1e}$$

where TS( ), one recalls, denotes the operation which forms the trace free symmetric tensor from the enclosed tensor ([9]). From the Taylor series expansion for σ we are able to evaluate $\overset{\circ}{\sigma}$, $\overset{\circ}{\sigma}_{,a}$ ... in terms of $\overset{\circ}{\mathrm{R}}_{abcd}$, $\overset{\circ}{\mathrm{A}}_a$, $\overset{\circ}{\mathrm{H}}_{ab}$; $\overset{\circ}{\mathrm{C}}$ and their covariant derivatives. Beginning with (5.1*a*) we obtain a condition, invariant under the trivial transformations, expressed in terms of the above quantities. The condition must hold for all $x = x_0 \in \mathrm{V}_4$ since x_0 may be chosen arbitrarily. The first condition and the subsequent ones obtained in a similar way from (5.1*b*), (5.1*c*), ... are used to simplify the ones which follow. In every case we give the conditions in a form invariant under the trivial transformation.

It is hoped that a relatively small number of these necessary conditions will be sufficient to characterize the coefficients modulo the trivial transformations, of all the Huygens' differential equations. This was the case for the equations considered by Mathisson [*22*] and by the author [*23*].

We shall now derive the conditions which arise from the successive powers of x^a in σ.

I. **Order of magnitude:** [0]

From (3.35), (4.3), (4.27), (4.29) and (4.30) we find that

$$\overset{\circ}{\sigma} = 4\overset{\circ}{\mathrm{C}}\,.$$

Thus, in view of (5.1*a*), the first condition is

$$\overset{\circ}{\mathrm{C}} = 0 \tag{5.2}$$

with our special choice of the trivial transformations (see § 3). We desire an expression of this condition in a form invariant under the trivial transformations. Hence we are looking for an invariant which reduces to $\overset{\circ}{\mathrm{C}}$

([9]) For an arbitrary tensor $\mathrm{M}_{a_1 \dots a_p}$ one has explicitly

$$\mathrm{TS}(\mathrm{M}_{a_1\dots a_p}) = \mathrm{M}_{(a_1\dots a_p)} - \sum_{r=1}^{[\frac{p}{2}]} g_{(a_1 a_2} \cdots g_{a_{2r-1}a_{2r}} \overset{p}{\mathrm{M}}_{a_{2r+1}\dots a_p)}\,.$$

The $\overset{p}{\mathrm{M}}_{a_{2r+1}\dots a_p}$ are obtained by solving the $\left[\frac{p}{2}\right]$ equations which result from contracting both sides of the above successively with $g^{a_1a_2}, g^{a_3a_4}, \dots, g^{a_{2[\frac{p}{2}]-1}a_{2[\frac{p}{2}]}}$.

when the special trivial transformations have been made. Such a quantity is the Cotton invariant $\mathscr{C}$ defined in (3.3c) which obeys the transformation law $\bar{\mathscr{C}} = e^{-2\phi}\mathscr{C}$. Under the special trivial transformations one has in view of (4.27), (4.28), (4.29) and (3.25)

$$\overset{\circ}{\mathscr{C}} = \overset{\circ}{\mathrm{C}} .$$

Thus, in general, the condition (5.2) is at x_0

$$\overset{\circ}{\mathscr{C}} = \overset{\circ}{\mathrm{C}} - \frac{1}{2}\overset{\circ}{\mathrm{A}}{}^a{}_{;a} - \frac{1}{4}\overset{\circ}{\mathrm{A}}_a\overset{\circ}{\mathrm{A}}{}^a - \frac{1}{6}\overset{\circ}{\mathrm{R}} = 0 .$$

Since x_0 has been chosen arbitrarily, we obtain *the first necessary condition*

$$\mathscr{C} = \mathrm{C} - \frac{1}{2}\mathrm{A}^a{}_{;a} - \frac{1}{4}\mathrm{A}_a\mathrm{A}^a - \frac{1}{6}\mathrm{R} = 0 \tag{5.3}$$

which is, in effect, the condition (1.4) obtained by Günther.

II. **Order of magnitude:** [1]

The condition (5.3) permits a simplification of the quantity σ. Noting this in (3.35) one obtains the new value for

$$\sigma \overset{*}{\equiv} \gamma - \frac{2}{3}\mathrm{R} + 2\mathrm{A}^a{}_{,a} - \mathrm{A}_a\mathrm{A}^a . \tag{5.4}$$

From (4.3), (4.27), (4.28) and (5.4) we obtain

$$\overset{\circ}{\sigma}_{,a} = \frac{4}{3}\overset{\circ}{\mathrm{H}}{}^k{}_{a;k} .$$

Thus from (5.1*b*) one has at x_0

$$\overset{\circ}{\mathrm{H}}{}^k{}_{a;k} = 0 \tag{5.5}$$

for the special choice of trivial transformations. Now $\mathrm{H}^k{}_{a;k}$ is vector which obeys the transformation law

$$\bar{\mathrm{H}}{}^k{}_{a\tilde{;}k} = e^{-2\phi}\mathrm{H}^k{}_{a;k} \tag{5.6}$$

under a general trivial transformation, where $\tilde{;}k$ denotes the covariant derivative with respect to $\tilde{g}_{ab}$. Thus (5.5) is the general form of the condition at x_0. Since x_0 may be chosen arbitrarily we recover the second necessary condition of Günther (1.5) which must be valid $\forall x \in \mathrm{V}_4$.

III. **Order of magnitude:** [2]

From (4.3) we have to second order

$$\gamma - \frac{2}{3}\mathrm{R} \overset{*}{=} \frac{3}{5}\left(\mathring{\mathrm{R}}_{(cd;ef)} - \frac{2}{9}\mathring{\mathrm{C}}_{k(cd|l}\mathring{\mathrm{C}}^{k}{}_{ef)}{}^{l}\right)\mathring{g}^{cd}x^{ef} - \frac{1}{3}\mathring{\mathrm{R}}_{;cd}x^{cd}\,. \tag{5.7}$$

As a result of the special conformal transformation (3.24c)

$$\mathring{\mathrm{R}}_{(cd;ef)} = \frac{1}{6}\mathring{g}_{(cd}\mathring{\mathrm{R}}_{;ef)}\,. \tag{5.8}$$

Taking account of this and (3.25) in (5.7) and expanding the symmetrization brackets yields

$$\gamma - \frac{2}{3}\mathrm{R} \overset{*}{=} \frac{1}{10}\Big[-2\mathring{\mathrm{R}}_{;cd} - \frac{2}{9}(\mathring{\mathrm{C}}_{kmcl}\mathring{\mathrm{C}}^{km}{}_{d}{}^{l} + \mathring{\mathrm{C}}_{kmcl}\mathring{\mathrm{C}}^{k}{}_{d}{}^{ml} + \mathring{\mathrm{C}}_{kcml}\mathring{\mathrm{C}}^{km}{}_{d}{}^{l} + \mathring{\mathrm{C}}_{kcml}\mathring{\mathrm{C}}^{k}{}_{d}{}^{ml})\Big]x^{cd}\,.$$

The last four terms on the right may be simplified using the symmetries of the Weyl tensor allowing us to write

$$\gamma - \frac{2}{3}\mathrm{R} \overset{*}{=} \frac{1}{10}\left(-2\mathring{\mathrm{R}}_{;cd} - \frac{2}{3}\mathring{\mathrm{C}}_{klmc}\mathring{\mathrm{C}}^{klm}{}_{d}\right)x^{cd}\,.$$

On account of the identity

$$\mathrm{C}_{klmc}\mathrm{C}^{klm}{}_{d} = \frac{1}{4}g_{cd}\mathrm{C}_{klmn}\mathrm{C}^{klmn} \tag{5.9}$$

valid when $n = 4$, we obtain finally

$$\gamma - \frac{2}{3}\mathrm{R} \overset{*}{=} -\frac{1}{10}\left(2\mathring{\mathrm{R}}_{;cd} + \frac{1}{6}\mathring{g}_{cd}\mathring{\mathrm{C}}_{klmn}\mathring{\mathrm{C}}^{klmn}\right)x^{cd}\,. \tag{5.10}$$

Next we turn our attention to $\mathrm{A}^{a}{}_{,a}$. By expanding the symmetrization brackets in the second term on the right of (4.27) one obtains, if (5.7) and the symmetries of H_{ab} and C_{abcd} are taken into account,

$$\mathrm{A}^{a}{}_{,a} \overset{*}{=} \frac{1}{2}\mathring{\mathrm{H}}^{b}{}_{c;db}x^{cd}\,. \tag{5.11}$$

We recall that Ricci's identities are

$$\mathrm{X}_{ab;[cd]} = \frac{1}{2}[\mathrm{R}^{k}{}_{acd}\mathrm{X}_{kb} + \mathrm{R}^{k}{}_{bcd}\mathrm{X}_{ak}] \tag{5.12}$$

for an arbitrary second rank tensor X_{ab}. Applying these identities to H_{ab} on the right-hand side of (5.11) and noting (1.5) one has to second order

$$A^a{}_{,a} \overset{*}{=} 0 . \tag{5.13}$$

Combining (5.10), (5.13) and the first term on the right of (4.28) with (5.4) we have to second order

$$\sigma \overset{*}{=} -\frac{1}{10}\left(2\overset{\circ}{R}_{;cd} + 10\overset{\circ}{H}_{kc}\overset{\circ}{H}^k{}_d + \frac{1}{6}g_{cd}\overset{\circ}{C}_{klmn}\overset{\circ}{C}^{klmn}\right)x^{cd} .$$

Consequently (5.1*c*) becomes

$$TS(\overset{\circ}{\sigma}_{;ab}) = -\frac{2}{5}TS(\overset{\circ}{R}_{;cd} + 5\overset{\circ}{H}_{kc}\overset{\circ}{H}^k{}_d) = 0 ,$$

which on account of (3.25) may be rewritten as

$$\overset{\circ}{S}_{cdk;}{}^k = -5\left(\overset{\circ}{H}_{kc}\overset{\circ}{H}^k{}_d - \frac{1}{4}\overset{\circ}{g}_{cd}\overset{\circ}{H}_{kl}\overset{\circ}{H}^{kl}\right) . \tag{5.14}$$

We note that (5.14) is valid at x_0 for the special choice of trivial transformations. It remains to find the form of the condition invariant under these transformations. We remark that under a trivial transformation

$$\bar{H}_{kc}\bar{H}^k{}_d - \frac{1}{4}\tilde{g}_{cd}\bar{H}_{kl}\bar{H}^{kl} = e^{-2\phi}\left(H_{kc}H^k{}_d - \frac{1}{4}g_{cd}H_{kl}H^{kl}\right) , \tag{5.15}$$

while

$$\tilde{S}_{cdk\tilde{;}}{}^k = e^{-2\phi}(S_{cdk;}{}^k - \phi_{k;l}C^k{}_{cd}{}^l + \phi_k C^k{}_{cd}{}^l\phi_l) .$$

However, in view of (3.23) the tensor

$$B_{cd} = S_{cdk;}{}^k - \frac{1}{2}C^k{}_{cd}{}^l L_{kl} , \tag{5.16}$$

called the Bach tensor (see [*27*], p. 313), transforms as follows:

$$\tilde{B}_{cd} = e^{-2\phi}B_{cd} . \tag{5.17}$$

Furthermore at x_0 for the special conformal transformation

$$\overset{\circ}{B}_{cd} = \overset{\circ}{S}_{cdk;}{}^k .$$

Thus the general form of the condition (5.14) at x_0 is

$$\overset{\circ}{S}_{cdk;}{}^k - \frac{1}{2}\overset{\circ}{C}{}^k{}_{cd}{}^l\overset{\circ}{L}_{kl} = -5\left(\overset{\circ}{H}_{kc}\overset{\circ}{H}^k{}_d - \frac{1}{4}\overset{\circ}{g}_{cd}\overset{\circ}{H}_{kl}\overset{\circ}{H}^{kl}\right) .$$

Again since x_0 may be chosen arbitrarily we recover the condition (1.6) which is the third necessary condition found by Günther and is valid $\forall x \in V_4$.

IV. Order of magnitude: [3]

We obtain the third order contribution to $\gamma - \frac{2}{3}\mathrm{R}$ from the second term on the right of (4.3). Expanding the symmetrization brackets yields

$$\left(\gamma - \frac{2}{3}\mathrm{R}\right)[3] \stackrel{*}{=} \frac{1}{45}\big(-4\mathring{\mathrm{R}}_{;cde} + 2\mathring{\mathrm{R}}_{kc;}{}^{k}{}_{de} + 2\mathring{\mathrm{R}}_{kc;d}{}^{k}{}_{e} + 2\mathring{\mathrm{R}}_{kc;de}{}^{k} + \mathring{\mathrm{R}}_{cd;k}{}^{k}{}_{e}$$
$$+ \mathring{\mathrm{R}}_{cd;ke}{}^{k} + \mathring{\mathrm{R}}_{cd;ek}{}^{k} - 4\mathring{\mathrm{R}}_{klmc}\mathring{\mathrm{R}}^{klm}{}_{d;e} - \mathring{\mathrm{R}}_{kcdl}\mathring{\mathrm{R}}^{kl}{}_{;e} - 2\mathring{\mathrm{R}}_{kcdl}\mathring{\mathrm{R}}^{k}{}_{me\,;}{}^{l\,m}\big)x^{cde}\,. \quad (5.18)$$

To obtain (5.18) the following identities are required:

$$\mathrm{R}_{klmc;c_1\ldots c_p}\mathrm{R}^{mlk}{}_{d;d_1\ldots d_q} = \frac{1}{2}\mathrm{R}_{klmc;c_1\ldots c_p}\mathrm{R}^{klm}{}_{d;d_1\ldots d_q}\,, \quad (5.19)$$

$$\mathrm{R}_{kcml;c_1\ldots c_p}\mathrm{R}^{k}{}_{de}{}^{l\,m}{}_{;\,e_1\ldots e_p} = \frac{1}{2}\mathrm{R}_{klmc;c_1\ldots c_p}\mathrm{R}^{klm}{}_{d;ee_1\ldots e_p}\,. \quad (5.20)$$

The first of these identities follows from the cyclical identity

$$\mathrm{R}_{[abc]d} = 0 \quad (5.21)$$

satisfied by the Riemann tensor while the second is a consequence of Bianchi's identitites

$$\mathrm{R}_{ab[cd;e]} = 0\,. \quad (5.22)$$

The first seven terms on the right-hand side of (5.18) may be considerably simplified by applying the Ricci identities. Furthermore, on taking account of the contracted Bianchi identities,

$$\mathrm{R}^{k}{}_{abc;k} = 2\mathrm{R}_{a[b;c]}\,, \quad (5.23)$$

$$\mathrm{R}_{ak;}{}^{k} = \frac{1}{2}\mathrm{R}_{,a}\,, \quad (5.24)$$

one finds that (5.18) reduces to

$$\left(\gamma - \frac{2}{3}\mathrm{R}\right)[3] \stackrel{*}{=} \frac{1}{45}\big(3\mathring{\mathrm{R}}_{cd;k}{}^{k}{}_{e} - \mathring{\mathrm{R}}_{;cde} - 2\mathring{\mathrm{R}}^{k}{}_{cd}{}^{l}\mathring{\mathrm{R}}_{kl;e} - 4\mathring{\mathrm{R}}_{klmc}\mathring{\mathrm{R}}^{klm}{}_{d;e}\big)x^{cde}\,. \quad (5.25)$$

Further simplification occurs when condition III, (1.6), is taken into account. It may be put into the following form in terms of the Riemann tensor and its contractions:

$$3\mathrm{R}_{cd;k}{}^{k} - \mathrm{R}_{;cd} = 6\mathrm{R}_{kcdl}\mathrm{R}^{kl} - 2\mathrm{R}\mathrm{R}_{cd} + 30\mathrm{H}_{kc}\mathrm{H}^{k}{}_{d} + \mathrm{Q}g_{cd}\,, \quad (5.26)$$

where

$$Q = \frac{1}{2}(\Box R + R^2 - 3R_{kl}R^{kl} - 15H_{kl}H^{kl}) .$$

Differentiation of (5.26) and symmetrization yields at x_0 on account of the special trivial transformations

$$3\overset{\circ}{R}_{(cd;|k|}{}^{k}{}_{e)} - \overset{\circ}{R}_{;(cde)} = 6\overset{\circ}{R}_{k(cd|l|}\overset{\circ}{R}^{kl}{}_{;e)} + 60\overset{\circ}{H}_{k(c;d}\overset{\circ}{H}^{k}{}_{e)} + \overset{\circ}{g}_{(cd}\overset{\circ}{Q}_{,e)} . \quad (5.27)$$

Finally we shall need to take into account the identities (5.9). Writing them in terms of the Riemann tensor and its contractions one obtains

$$R_{klmc}R^{klm}{}_{d} = 2R_{kcdl}R^{kl} + 2R_{kc}R^{k}{}_{d} - RR_{cd} + Mg_{cd} , \quad (5.28)$$

where

$$M = \frac{1}{4}(R_{klmn}R^{klmn} - 4R_{kl}R^{kl} + R^2) .$$

Differentiation of (5.28) and symmetrization yields at x_0 on account of the special trivial transformations

$$\overset{\circ}{R}_{klm(c}\overset{\circ}{R}^{klm}{}_{d;e)} = \overset{\circ}{R}_{k(cd|l|}\overset{\circ}{R}^{kl}{}_{;e)} + \overset{\circ}{g}_{(cd}\overset{\circ}{M}_{,e)} . \quad (5.29)$$

Noting (5.27) and (5.29) the equation (5.25) simplifies to

$$\left(\gamma - \frac{2}{3}R\right)[3] \overset{*}{=} \frac{4}{3}(\overset{\circ}{H}_{kc;d}\overset{\circ}{H}^{k}{}_{e} + \overset{\circ}{g}_{cd}\overset{\circ}{M}'_{,e})x^{cde} , \quad (5.30)$$

where the exact form of $\overset{\circ}{M}'_{,e}$ is irrelevant.

We next consider the contribution to σ from $A^{a}{}_{,a}$. Expanding the symmetrization brackets in the third term on the left-hand side of (4.27) yields on taking account of (1.5) and (5.23)

$$A^{a}{}_{,a}[3] \overset{*}{=} \frac{2}{15}\left(\overset{\circ}{H}^{b}{}_{c;dbe} + \overset{\circ}{H}^{b}{}_{c;deb} + \frac{7}{3}\overset{\circ}{C}^{b}{}_{cd}{}^{g}\overset{\circ}{H}_{eg;b} + 4\overset{\circ}{R}_{cg;d}\overset{\circ}{H}^{g}{}_{e} - 2\overset{\circ}{R}_{cd;g}\overset{\circ}{H}^{g}{}_{e}\right)x^{cde} .$$

The right-hand side of the above may be put in the desired form by applying Ricci's identities to the first two terms and taking into account the conditions (3.24*b*) and (3.25) noting in particular that $\overset{\circ}{R}_{gc;d} = -\frac{4}{3}\overset{\circ}{S}_{(gc)d}$. One obtains

$$A^{a}{}_{,a}[3] \overset{*}{=} \frac{8}{45}(3\overset{\circ}{S}_{cdk}\overset{\circ}{H}^{k}{}_{e} + \overset{\circ}{C}^{k}{}_{cd}{}^{l}\overset{\circ}{H}_{ek;l})x^{cde} . \quad (5.31)$$

Finally by combining (4.28), (5.30) and (5.31) with (5.4) we find

$$\sigma[3] \overset{*}{=} \frac{16}{45}\left(3\overset{\circ}{S}_{cdk}\overset{\circ}{H}^{k}{}_{e} + \overset{\circ}{C}^{k}{}_{cd}{}^{l}\overset{\circ}{H}_{ek;l} + \frac{15}{4}\overset{\circ}{g}_{cd}\overset{\circ}{M}'_{,e}\right)x^{cde} .$$

Therefore, in view of (5.1d), we have the condition

$$\mathrm{TS}(3\mathring{\mathrm{S}}_{cdk}\mathring{\mathrm{H}}^{k}{}_{e} + \mathring{\mathrm{C}}^{k}{}_{cd}{}^{l}\mathring{\mathrm{H}}_{ek;l}) = 0 \tag{5.32}$$

which is valid at x_0 for the special choice of trivial transformations. Now it may be shown that under a general trivial transformation

$$\mathrm{TS}(3\tilde{\mathrm{S}}_{cdk}\tilde{\mathrm{H}}^{k}{}_{e} + \tilde{\mathrm{C}}^{k}{}_{cd}{}^{l}\tilde{\mathrm{H}}_{ek\tilde{;}l}) = e^{-2\phi}\mathrm{TS}(3\mathrm{S}_{cdk}\mathrm{H}^{k}{}_{e} + \mathrm{C}^{k}{}_{cd}{}^{l}\mathrm{H}_{ek;l}) . \tag{5.33}$$

Thus the tensor on the right-hand side of (5.33) is an invariant under the trivial transformations and consequently (5.32) is the general form of the condition at x_0. Noting again that x_0 may be chosen arbitrarily we recover (1.7) which is Günthers fourth necessary condition.

V. Order of magnitude: [4]

Lengthy calculations are required to simplify the fourth order contributions to σ. For this reason we shall break the work down into a number of parts which will be treated separately. The procedures used here are an extension of those used to obtain the first four conditions. Consequently we shall only indicate the steps required and give the final result in the different steps of the calculation.

We shall first consider the contribution to σ of fourth order from $\gamma - \frac{2}{3}\mathrm{R}$. This involves the third and fourth sets of terms on the right-hand side of (4.3). We shall work out the contribution from each set of terms separately. We start by considering the two terms

$$\frac{5}{84}\mathring{\mathrm{R}}_{(cd;efgh)}\mathring{g}^{gh}x^{cdef} - \frac{1}{36}\mathring{\mathrm{R}}_{;cdef}x^{cdef} . \tag{5.34}$$

If the symmetrization bracket is expanded and the Ricci identities are applied systematically one obtains

$$\frac{1}{126}(3\mathring{\mathrm{R}}_{cd;k}{}^{k}{}_{ef} - \mathring{\mathrm{R}}_{;cdef} + 6\mathring{\mathrm{C}}^{k}{}_{cd}{}^{l}\mathring{\mathrm{R}}_{kl;ef} + 16\mathring{\mathrm{C}}^{k}{}_{cd}{}^{l}\mathring{\mathrm{R}}_{ek;lf}$$
$$+ 2\mathring{\mathrm{C}}^{k}{}_{cd}{}^{l}\mathring{\mathrm{R}}_{ef;kl} - 12\mathring{\mathrm{R}}_{kc;d}\mathring{\mathrm{R}}^{k}{}_{e;f} + 3\mathring{\mathrm{R}}^{k}{}_{cd}{}^{l}{}_{;e}\mathring{\mathrm{R}}_{kl;f})x^{cdef} . \tag{5.35}$$

The fourth order derivatives can be transformed to second order derivatives by means of condition III. If one differentiates (5.26) and symmetrizes, one obtains at x_0, on account of the special trivial transformations, the relation

$$3\mathring{\mathrm{R}}_{(cd;|k|}{}^{k}{}_{ef)} - \mathring{\mathrm{R}}_{;(cdef)} = 12\mathring{\mathrm{R}}^{k}{}_{(cd}{}^{l}{}_{;e}\mathring{\mathrm{R}}_{|kl|;f)} + 6\mathring{\mathrm{C}}^{k}{}_{(cd}{}^{l}\mathring{\mathrm{R}}_{|kl|;ef)}$$
$$+ 60\mathring{\mathrm{H}}_{k(c;de}\mathring{\mathrm{H}}^{k}{}_{f)} + 60\mathring{\mathrm{H}}_{k(c;d}\mathring{\mathrm{H}}^{k}{}_{e;f)} + g_{(cd}\mathring{\mathrm{Q}}_{;ef)} .$$

On account of this the expression (5.35) may be written as follows:

$$\frac{1}{126}(12\mathring{C}^{k}{}_{cd}{}^{l}\mathring{R}_{kl;ef} + 16\mathring{C}^{k}{}_{cd}{}^{l}\mathring{R}_{ek;lf} + 2\mathring{C}^{k}{}_{cd}{}^{l}\mathring{R}_{ef;kl} + 15\mathring{R}^{k}{}_{cd}{}^{l}{}_{;e}\mathring{R}_{kl;f} - 12\mathring{R}_{kc;d}\mathring{R}^{k}{}_{e;f} + 60\mathring{H}_{kc;de}\mathring{H}^{k}{}_{f} + 60\mathring{H}_{kc;d}\mathring{H}^{k}{}_{e;f} + \mathring{g}_{cd}\mathring{Q}_{;ef})x^{cdef} \,. \tag{5.36}$$

We now consider the terms

$$-\frac{2}{7}\mathring{R}_{k(cd|l|}\mathring{R}^{k}{}_{ef}{}^{l}{}_{;gh)}\mathring{g}^{gh}x^{cdef} \,. \tag{5.37}$$

Writing out the symmetrization bracket one gets on account of (5.19)

$$-\frac{2}{105}(3\mathring{R}_{klmc}\mathring{R}^{klm}{}_{d;ef} + 2\mathring{R}_{kmcl}\mathring{R}^{k}{}_{de}{}^{l}{}_{;}{}^{m}{}_{f} + 2\mathring{R}_{kmcl}\mathring{R}^{k}{}_{de}{}^{l}{}_{;f}{}^{m} + \mathring{C}^{k}{}_{cd}{}^{l}\mathring{R}_{kl;ef} + 2\mathring{C}^{k}{}_{cd}{}^{l}\mathring{R}_{kmel;}{}^{m}{}_{f} + 2\mathring{C}^{k}{}_{cd}{}^{l}\mathring{R}_{kmel;f}{}^{m} + \mathring{C}^{k}{}_{cd}{}^{l}\mathring{R}_{kefl;m}{}^{m})x^{cdef} \,. \tag{5.38}$$

The following identities derived by applying the Ricci identitites to the Riemann tensor will aid us in simplifying (5.38):

$$\mathring{R}_{km(c|l|}\mathring{R}^{k}{}_{de}{}^{l}{}_{;f)}{}^{m} = \mathring{R}_{km(c|l|}\mathring{R}^{k}{}_{de}{}^{l}{}_{;}{}^{m}{}_{f)} - 3\mathring{C}_{(cd|k|}\mathring{C}^{k}{}_{e|ml|}\mathring{C}^{l}{}_{f)}{}^{m}{}_{p} + 3\mathring{C}^{p}{}_{(cd|k}\mathring{C}^{k}{}_{m|e|l)}\mathring{C}^{l}{}_{f)}{}^{m}{}_{p} \,, \tag{5.39}$$

$$\mathring{C}^{k}{}_{(cd}{}^{l}\mathring{R}_{|km|e|l|;f)}{}^{m} = \mathring{C}^{k}{}_{(cd}{}^{l}\mathring{R}_{|km|e|l|;}{}^{m}{}_{f)} + 2\mathring{C}^{p}{}_{(cd|k|}\mathring{C}^{k}{}_{e|ml|}\mathring{C}^{l}{}_{f)}{}^{m}{}_{p} - \mathring{C}^{p}{}_{(cd|k}\mathring{C}^{k}{}_{m|e|l|}\mathring{C}^{l}{}_{f)}{}^{m}{}_{p} + \mathring{C}^{k}{}_{(cd}{}^{l}\mathring{C}^{m}{}_{ef)}{}^{p}\mathring{C}_{kmpl} \,, \tag{5.40}$$

$$\mathring{C}^{k}{}_{(cd}{}^{l}\mathring{R}_{|k|ef)l;m}{}^{m} = \mathring{C}^{k}{}_{(cd}{}^{l}\mathring{R}_{|kl|;ef)} - 2\mathring{C}^{k}{}_{(cd}{}^{l}\mathring{R}_{e|k;l|f)} + \mathring{C}^{k}{}_{(cd}{}^{l}\mathring{R}_{ef);kl} + 4\mathring{C}^{p}{}_{(cd|k|}\mathring{C}^{k}{}_{e|ml|}\mathring{C}^{l}{}_{f)}{}^{m}{}_{p} - 2\mathring{C}^{p}{}_{(cd|k|}\mathring{C}^{k}{}_{|m|e|l|}\mathring{C}^{l}{}_{f)}{}^{m}{}_{p} + 2\mathring{C}^{k}{}_{(cd}{}^{l}\mathring{C}^{m}{}_{ef)}{}^{p}\mathring{C}_{kmpl} \,. \tag{5.41}$$

We shall also need the following identity which is obtained from (5.28) by differentiating twice and symmetrizing:

$$\mathring{R}_{klm(c}\mathring{R}^{klm}{}_{d;ef)} = -\mathring{R}_{klm(c;d}\mathring{R}^{klm}{}_{e;f)} + \mathring{C}^{k}{}_{(cd}{}^{l}\mathring{R}_{|kl|;ef)} + 2\mathring{R}^{k}{}_{(cd}{}^{l}{}_{;e}\mathring{R}_{|kl|;f)} + 2\mathring{R}_{k(c;d}\mathring{R}^{k}{}_{e;f)} + \mathring{g}_{(cd}\mathring{M}_{;ef)} \,. \tag{5.42}$$

If one makes the appropriate substitutions from (5.39), (5.40), (5.41) and (5.42) and takes into account (5.20) (with $p = 0$ and $q = 2$) and (5.23), the expression (5.38) takes the form

$$-\frac{2}{105}(-5\mathring{R}_{klmc;d}\mathring{R}^{klm}{}_{e;f} + 11\mathring{C}^{k}{}_{cd}{}^{l}\mathring{R}_{kl;ef} - 6\mathring{C}^{k}{}_{cd}{}^{l}\mathring{R}_{ek;lf} + \mathring{C}^{k}{}_{cd}{}^{l}\mathring{R}_{ef;kl} + 10\mathring{R}^{k}{}_{cd}{}^{l}{}_{;e}\mathring{R}_{kl;f} + 10\mathring{R}_{kc;d}\mathring{R}^{k}{}_{e;f} + 2\mathring{C}^{p}{}_{cdk}\mathring{C}^{k}{}_{eml}\mathring{C}^{l}{}_{f}{}^{m}{}_{p} + 2\mathring{C}^{p}{}_{cdk}\mathring{C}^{k}{}_{mel}\mathring{C}^{l}{}_{f}{}^{m}{}_{p} + 4\mathring{C}^{k}{}_{cd}{}^{l}\mathring{C}^{m}{}_{ef}{}^{p}\mathring{C}_{kmpl} + 5\mathring{g}_{cd}\mathring{M}_{;ef})x^{cdef} \,. \tag{5.43}$$

We next examine the undifferentiated terms

$$\frac{4}{2835}\mathring{C}^{p}{}_{(cd|k|}\mathring{C}^{k}{}_{ef|l|}\mathring{C}^{l}{}_{gh)p}g^{gh}x^{cdef} + \frac{2}{45}\mathring{C}_{k(ef|l|}\mathring{C}^{k}{}_{gh)}{}^{l}\mathring{C}^{g}{}_{cd}{}^{h}x^{cdef} . \quad (5.44)$$

This expression takes the following simpler form after one expands the symmetrization bracket and takes into account the symmetries of the Weyl tensor and the cyclical identity (5.21):

$$\frac{1}{945}(2\mathring{C}^{p}{}_{cdk}\mathring{C}^{k}{}_{eml}\mathring{C}^{l}{}_{f}{}^{m}{}_{p} + 32\mathring{C}^{p}{}_{cdk}\mathring{C}^{k}{}_{mel}\mathring{C}^{l}{}_{f}{}^{m}{}_{p} + 14\mathring{C}^{k}{}_{cd}{}^{l}\mathring{C}^{m}{}_{ef}{}^{p}\mathring{C}_{kmpl)}x^{cdef} . \quad (5.45)$$

We now turn our attention to the terms

$$-\frac{5}{56}\mathring{R}_{k(cd|l|;e}\mathring{R}^{k}{}_{fg}{}^{l}{}_{;h)}\mathring{g}^{gh}x^{cdef} . \quad (5.46)$$

When one writes out explicitly the sum implied by the presence of the symmetrization brackets and employs the identities (5.19), (5.20) (with appropriate choices of p and q) (5.23) and the conditions (3.25), the expression (5.46) simplifies to

$$-\frac{1}{168}(8\mathring{R}^{k}{}_{cd}{}^{l}{}_{;e}\mathring{R}_{kl;f} + 5\mathring{R}_{klmc;d}\mathring{R}^{klm}{}_{e;f} + \mathring{R}_{kcdl;m}\mathring{R}^{k}{}_{ef}{}^{l}{}_{;m})x^{cdef} . \quad (5.47)$$

Finally on account of (5.8) the contribution to $\left(\gamma - \frac{2}{3}R\right)[4]$ from the terms

$$-\frac{1}{5}\mathring{R}_{(ef;gh)}\mathring{C}^{g}{}_{cd}{}^{h}x^{cdef} \quad (5.47)$$

is

$$-\frac{1}{180}\mathring{g}_{cd}\mathring{R}_{;kl}\mathring{C}^{k}{}_{ef}{}^{l}x^{cdef} , \quad (5.48)$$

We may now combine the results of (5.36), (5.43), (5.45), (5.47) and (5.48) to obtain

$$\left(\gamma - \frac{2}{3}R\right)[4] \stackrel{*}{=} \frac{1}{22680}(576\mathring{C}^{k}{}_{cd}{}^{l}\mathring{R}_{kl;ef} + 3744\mathring{C}^{k}{}_{cd}{}^{l}\mathring{R}_{ek;lf} + 216\mathring{C}^{k}{}_{cd}{}^{l}\mathring{R}_{ef;kl}$$
$$+ 180\mathring{R}^{k}{}_{cd}{}^{l}{}_{;e}\mathring{R}_{kl;f} - 3600\mathring{R}_{kc;d}\mathring{R}^{k}{}_{e;f} + 45\mathring{R}_{klmc;d}\mathring{R}^{klm}{}_{e;f}$$
$$- 135\mathring{R}_{kcdl;m}\mathring{R}^{k}{}_{ef}{}^{l}{}_{;}{}^{m} - 240\mathring{C}^{p}{}_{cdk}\mathring{C}^{k}{}_{eml}\mathring{C}^{l}{}_{f}{}^{m}{}_{p} + 480\mathring{C}^{p}{}_{cdk}\mathring{C}^{k}{}_{mel}\mathring{C}^{l}{}_{f}{}^{m}{}_{p}$$
$$- 240\mathring{C}^{k}{}_{cd}{}^{l}\mathring{C}^{m}{}_{ef}{}^{n}\mathring{C}_{kmnl} + 10800\mathring{H}_{kc;de}\mathring{H}^{k}{}_{f} + 10800\mathring{H}_{kc;d}\mathring{H}^{k}{}_{e;f}$$
$$+ \mathring{g}_{cd}\mathring{P}_{ef})x^{cdef} , \quad (5.49)$$

where

$$\mathring{P}_{ef} = 180\mathring{Q}_{;ef} - 720\mathring{M}_{;ef} - 126\mathring{R}_{;kl}\mathring{C}^{k}{}_{ef}{}^{l} .$$

The following relations allow us to re-express (5.49) in terms of the tensors C_{abcd} and S_{abc} :

$$\mathring{R}_{klm(c;d}\mathring{R}^{klm}{}_{e;f)} = \mathring{C}_{klm(c;d}\mathring{C}^{klm}{}_{e;f)} - \frac{8}{3}\mathring{C}^{k}{}_{(cd}{}^{l}{}_{;e}\mathring{S}_{|kl|f)} + \frac{1}{2}g_{(cd}\mathring{L}^{kl}{}_{;e}\mathring{L}_{|kl|;f)}, \quad (5.50)$$

$$\mathring{R}_{k(cd|l;m|}\mathring{R}^{k}{}_{ef)}{}^{lm}{}_{;} = \mathring{C}_{k(cd|l;m|}\mathring{C}^{k}{}_{ef)}{}^{lm}{}_{;} + \frac{8}{9}\mathring{S}_{(cd}{}^{k}\mathring{S}_{ef)k} - g_{(cd}\Big(\mathring{C}_{|k|ef)l;m}\mathring{L}^{kl\,m}{}_{;}$$
$$+ \frac{1}{4}\mathring{L}_{e}{}^{k}{}_{;}{}^{l}\mathring{L}_{f)k;l} - \frac{1}{4}g_{ef)}\mathring{L}_{kl;m}\mathring{L}^{kl\,m}{}_{;}\Big), \quad (5.51)$$

$$\mathring{R}^{k}{}_{(cd}{}^{l}\mathring{R}_{|kl|;f)} = -\frac{4}{3}\mathring{C}^{k}{}_{(cd}{}^{l}{}_{;e}\mathring{S}_{|kl|f)} - \frac{4}{9}\mathring{S}_{(cd}{}^{k}\mathring{S}_{ef)k} + \frac{1}{2}g_{(cd}\mathring{L}^{kl}{}_{;e}\mathring{L}_{|kl|;f)}. \quad (5.52)$$

Taking account also of (3.25) one finds that (5.49) takes the form

$$\left(\gamma - \frac{2}{3}R\right)[4] \stackrel{*}{=} \frac{1}{504}(8\mathring{C}^{k}{}_{cd}{}^{l}\mathring{S}_{kle,f} + 24\mathring{C}^{k}{}_{cd}{}^{l}\mathring{S}_{efk;l} - 8\mathring{C}^{k}{}_{cd}{}^{l}{}_{;e}\mathring{S}_{klf}$$
$$- 40\mathring{S}_{cd}{}^{k}\mathring{S}_{efk} - 3\mathring{C}_{kcdl;m}\mathring{C}^{k}{}_{ef}{}^{lm}{}_{;} + \mathring{C}_{klmc;d}\mathring{C}^{klm}{}_{e;f}$$
$$- \frac{14}{3}\mathring{C}^{p}{}_{cdk}\mathring{C}^{k}{}_{eml}\mathring{C}^{l}{}_{f}{}^{m}{}_{p} + \frac{28}{3}\mathring{C}^{p}{}_{cdk}\mathring{C}^{k}{}_{mel}\mathring{C}^{l}{}_{f}{}^{m}{}_{p}$$
$$- \frac{14}{3}\mathring{C}_{kcdl}\mathring{C}^{m}{}_{ef}{}^{n}\mathring{C}^{k}{}_{mn}{}^{l} + 240\mathring{H}_{kc;de}\mathring{H}^{k}{}_{f} + 240\mathring{H}_{kc;d}\mathring{H}^{k}{}_{e;f}$$
$$+ g_{cd}\mathring{N}_{ef})x^{cdef}, \quad (5.53)$$

where the $\mathring{N}_{ef}$ are quantities whose exact form is unimportant.

We next consider the fourth order contribution to σ from $A^{a}{}_{,a}$. Expanding the symmetrization brackets in the last term on the right-hand side of (4.27) yields, when (1.5) and the Ricci identities have been taken into account,

$$A^{a}{}_{,a}[4] \stackrel{*}{=} \frac{1}{45}(3\mathring{R}_{c}{}^{k}{}_{;de}\mathring{H}_{kf} - 3\mathring{R}_{cd;}{}^{k}{}_{e}\mathring{H}_{kf} + 8\mathring{R}^{k}{}_{c;d}\mathring{H}_{ke;f} - 2\mathring{R}^{k}{}_{cdl;e}\mathring{H}^{l}{}_{f;k}$$
$$- 2\mathring{R}^{k}{}_{cdl}\mathring{H}^{l}{}_{e;kb})x^{cdef}. \quad (5.54)$$

If now one substitutes for R_{abcd} and $R_{ab;c}$ from (1.11) and (3.25), the relation (5.54) assumes the following form:

$$A^{a}{}_{,a}[4] \stackrel{*}{=} \frac{2}{45}\Big(3\mathring{S}_{cdk;e}\mathring{H}^{k}{}_{f} + 3\mathring{S}_{cdk}\mathring{H}^{k}{}_{e;f} + \mathring{C}^{k}{}_{cd}{}^{l}{}_{;e}\mathring{H}_{fk;l} + \mathring{C}^{k}{}_{cd}{}^{l}\mathring{H}_{ek;fl}$$
$$- \frac{1}{2}g_{cd}\mathring{L}_{kl;e}\mathring{H}^{k}{}_{f;}{}^{l}\Big)x^{cdef}. \quad (5.55)$$

On comparing (5.55) with the derivative of the fourth necessary condition (1.7), one sees that required contribution from $A^a{}_{,a}$ is

$$A^a{}_{,a}[4] \stackrel{*}{=} \frac{2}{45}\overset{\circ}{g}_{(cd}\left(\overset{\circ}{K}_{e;f)} - \frac{1}{2}\overset{\circ}{L}_{kl;e}\overset{\circ}{H}^k{}_{f;}{}^l\right)x^{cdef} . \qquad (5.56)$$

Finally on combining (5.53), (5.56) and (4.28) with (5.4) we obtain

$$\begin{aligned}\sigma[4] \stackrel{*}{=} \frac{1}{504}(&8\overset{\circ}{C}{}^k{}_{cd}{}^l\overset{\circ}{S}_{kle;f} + 24\overset{\circ}{C}{}^k{}_{cd}{}^l\overset{\circ}{S}_{efk;l} - 8\overset{\circ}{C}{}^k{}_{cd}{}^l{}_{;e}\overset{\circ}{S}_{klf} - 40\overset{\circ}{S}_{cd}{}^k\overset{\circ}{S}_{efk} \\ &- 3\overset{\circ}{C}_{kcdl;m}\overset{\circ}{C}{}^k{}_{ef}{}^l{}_;{}^m + \overset{\circ}{C}_{klmc;d}\overset{\circ}{C}{}^{klm}{}_{e;f} - \frac{14}{3}\overset{\circ}{C}{}^p{}_{cdk}\overset{\circ}{C}{}^k{}_{eml}\overset{\circ}{C}{}^l{}_f{}^m{}_p \\ &+ \frac{28}{3}\overset{\circ}{C}{}^p{}_{cdk}\overset{\circ}{C}{}^k{}_{mel}\overset{\circ}{C}{}^l{}_f{}^m{}_p - \frac{14}{3}\overset{\circ}{C}_{kcdl}\overset{\circ}{C}{}^m{}_{ef}{}^n\overset{\circ}{C}{}^k{}_{mn}{}^l - 12\overset{\circ}{H}_{kc;de}\overset{\circ}{H}{}^k{}_f \\ &+ 16\overset{\circ}{H}_{kc;d}\overset{\circ}{H}{}^k{}_{e;f} + 84\overset{\circ}{H}{}^k{}_c\overset{\circ}{C}_{kdel}\overset{\circ}{H}{}^l{}_f + \overset{\circ}{g}_{cd}\overset{\circ}{N}'_{ef})x^{cdef} , \end{aligned} \qquad (5.57)$$

where the $\overset{\circ}{N}'_{ef}$ are quantities whose exact form is not important.

The following identities proved in [*23*] are required to complete the derivation of condition V:

$$TS(C_{klmc;d}C^{klm}{}_{e;f}) = 0 , \qquad (5.58)$$

$$\begin{aligned}&TS(C^p{}_{cdk}C^k{}_{eml}C^l{}_f{}^m{}_p) \\ &= \frac{1}{2}TS(C^p{}_{cdk}C^k{}_{mel}C^l{}_f{}^m{}_p) = \frac{1}{3}TS(C_{kcdl}C^m{}_{ep}{}^nC^k{}_{mn}{}^l) . \end{aligned} \qquad (5.59)$$

When these identities are taken into account, (5.1*e*) becomes on account of (5.57)

$$\begin{aligned}&TS(3\overset{\circ}{C}_{kcdl;m}\overset{\circ}{C}{}^k{}_{ef}{}^l{}_;{}^m + 8\overset{\circ}{C}{}^k{}_{cd}{}^l{}_{;e}\overset{\circ}{S}_{klf} + 40\overset{\circ}{S}_{cd}{}^k\overset{\circ}{S}_{efk} - 8\overset{\circ}{C}{}^k{}_{cd}{}^l\overset{\circ}{S}_{kle;f} \\ &- 24\overset{\circ}{C}{}^k{}_{cd}{}^l\overset{\circ}{S}_{efk;l} + 12\overset{\circ}{H}_{kc;de}\overset{\circ}{H}{}^k{}_f - 16\overset{\circ}{H}_{kc;d}\overset{\circ}{H}{}^k{}_{e;f} - 84\overset{\circ}{H}{}^k{}_c\overset{\circ}{C}_{kdel}\overset{\circ}{H}{}^l{}_f) = 0 . \end{aligned} \qquad (5.60)$$

This is the fifth necessary condition which must hold at x_0 for the special choice of trivial transformations. To find the general form of this condition we must find a fourth rank tensor which is invariant under the trivial transformations and which reduces to the left-hand side of (5.60) at x_0 when the special trivial transformations are made. As shall be proven in § 6 the following is the only such tensor:

$$\begin{aligned}\mathscr{H}_{cdef} = TS(&3C_{kcdl;m}C^k{}_{ef}{}^l{}_;{}^m + 8C^k{}_{cd}{}^l{}_{;e}S_{klf} + 40S_{cd}{}^kS_{efk} - 8C^k{}_{cd}{}^lS_{kle;f} \\ &- 24C^k{}_{cd}{}^lS_{efk;l} + 4C^k{}_{cd}{}^lC_l{}^m{}_{ek}L_{fm} + 12C^k{}_{cd}{}^lC^m{}_{efl}L_{km} \\ &+ 12H_{kc;de}H^k{}_f - 16H_{kc;d}H^k{}_{e;f} - 84H^k{}_cC_{kdel}H^l{}_f - 18H_{kc}H^k{}_dL_{ef}) . \end{aligned} \qquad (5.61)$$

Under a trivial transformation it transforms as follows:

$$\overline{\mathscr{H}}_{cdef} = e^{-2\phi}\mathscr{H}_{cdef} . \qquad (5.62)$$

Thus, on account of (5.60), (5.61) and (3.25), the general form of condition V at x_0 is

$$\overset{\circ}{\mathscr{H}}_{cdef} = 0 .$$

Finally, since x_0 has been chosen arbitrarily, *we conclude that the fifth necessary condition for the validity of Huygens' principle for the general equation* (1.1) *with* $n = 4$ *is*

$$\mathscr{H}_{cdef} = 0 , \tag{5.63}$$

where $\mathscr{H}_{cdef}$ *is given by* (5.61).

The condition (5.63) reduces to Wünsch's condition (1.12) when $A^a = 0$ is assumed. It is remarkable that Wünsch's result agrees with ours when one considers that he was led to it by studying Huygens' differential equations infinitesimally close to the wave equation while we obtained it as a consequence of Hadamard's necessary and sufficient condition for the equation (1.1).

It also should be noted that (5.63) reduces to the condition (1.12) obtained by the author when $R_{ab} = 0$. This is because $R_{ab} = 0$ implies $H_{ab} = 0$ on account of (1.6).

6. INVARIANCE OF THE NECESSARY CONDITIONS UNDER TRIVIAL TRANSFORMATIONS

In this section we wish to justify our procedure of obtaining the necessary conditions using the special choice of trivial transformations (3.24) and (3.26). In particular we wish to establish the invariance (5.62), under trivial transformations, of the tensor $\mathscr{H}_{abcd}$ which appears in condition V. We shall also exhibit a second fourth rank conformally invariant tensor.

In order to justify our procedure it is sufficient to show that each necessary condition may be expressed by the vanishing of some tensor which is invariant under the trivial transformations. This is a consequence of the transformation law (3.37) for σ. If a fixed point x_0 is chosen and both sides of (3.37) are expanded in normal coordinates ($[\sigma] = 0$ is not assumed), one obtains the following relations valid at x_0 (10):

$$\overset{\circ}{\bar{\sigma}} = e^{-2\phi}\overset{\circ}{\sigma} , \tag{6.1a}$$

$$\overset{\circ}{\bar{\sigma}}_{,a} = e^{-2\phi}(\overset{\circ}{\sigma}_{,a} - 2\overset{\circ}{\sigma}\overset{\circ}{\phi}_{,a}) , \tag{6.1b}$$

$$\mathrm{TS}(\overset{\circ}{\bar{\sigma}}_{;ab}) = e^{-2\phi}\mathrm{TS}(\overset{\circ}{\sigma}_{;ab} - 6\overset{\circ}{\sigma}_{,a}\overset{\circ}{\phi}_{,b} + 2\overset{\circ}{\sigma}(4\overset{\circ}{\phi}_{,a}\overset{\circ}{\phi}_{,b} - \overset{\circ}{\phi}_{;ab})) , \tag{6.1c}$$

$$\vdots$$

(10) The above relations are obtained by noting that normal coordinates ($\tilde{x}^a$) with respect to the metric $\tilde{g}_{ab}$ are related to normal coordinates (x^a) with respect to g_{ab} on $C(x_0)$ by the following formula:

$$\tilde{x}^a = e^{-2\phi}a_1 x^a ,$$

where a_1 is given by (3.15).

The succeeding formulae in the above series have a similar form: the highest derivatives of σ on the right hand side are followed by terms linear in the lower derivatives. From (6.1*a*) we note that $\overset{\circ}{\sigma}$ is invariant, modulo the factor $e^{-2\phi}$, under the trivial transformations. If $\overset{\circ}{\sigma} = 0$, then from (6.1*b*) we have $\overset{\circ}{\bar{\sigma}}_{,a} = e^{-2\phi}\overset{\circ}{\sigma}_{,a}$ so $\overset{\circ}{\sigma}_{,a}$ is an invariant vector under the trivial transformations. Similarly if $\overset{\circ}{\sigma} = \overset{\circ}{\sigma}_{,a} = 0$, (6.1*c*) implies $\mathrm{TS}(\overset{\circ}{\bar{\sigma}}_{\bar{;}ab}) = e^{-2\phi}\mathrm{TS}(\overset{\circ}{\sigma}_{;ab})$ and we conclude that $\mathrm{TS}(\overset{\circ}{\sigma}_{;ab})$ is invariant under the trivial transformations. In general if $\overset{\circ}{\sigma} = \overset{\circ}{\sigma}_{,a} = \mathrm{TS}(\overset{\circ}{\sigma}_{;ab}) = \ldots = \mathrm{TS}(\overset{\circ}{\sigma}_{;a_1 \ldots a_p}) = 0$, we shall have $\mathrm{TS}(\overset{\circ}{\bar{\sigma}}_{\bar{;}a_1 \ldots a_{p+1}}) = e^{-2\phi}\mathrm{TS}(\overset{\circ}{\sigma}_{;a_1 \ldots a_{p+1}})$. It should further be remarked that if the quantities $\mathrm{TS}(\overset{\circ}{\sigma}_{;a_1 \ldots a_{p+1}})$ are simplified by using the previous conditions assumed now to be valid in a *neighbourhood* of x_0, then the simplified expression for $\mathrm{TS}(\overset{\circ}{\sigma}_{;a_1 \ldots a_{p+1}})$ remains invariant under the trivial transformations since the preceeding conditions are themselves invariant under the trivial transformations. *Thus we may conclude that each necessary condition may be expressed by the vanishing of a tensor which is invariant under the trivial transformations.*

The above result justifies the use of a special choice of trivial transformations at x_0 to simplify the calculations. When the special trivial transformations have been made at x_0, the necessary conditions appear in a reduced form which are not always invariant under the trivial transformations (see for example (5.2), (5.14) or (5.60)). It thus remains to extend such a condition to the invariant form. It should be pointed out that such extensions are unique. To make things more precise consider, for example, the derivation of condition V. With the special choice of trivial transformations this condition is given at x_0 by (5.60). The tensor on the right hand side of (5.60) is not invariant under trivial transformations (this will be seen later in the section). Thus one must find a tensor $\mathscr{H}_{cdef}$ which is invariant under the trivial transformations and which reduces to the left hand side of (5.60) at x_0 when the special trivial transformations have been made ([11]). The tensor satisfying these conditions is unique for suppose $\mathscr{H}'_{cdef}$ is a second such tensor. Then their difference $\Delta_{cdef} = \mathscr{H}_{cdef} - \mathscr{H}'_{cdef}$ will be invariant under the trivial transformations: $\bar{\Delta}_{cdef} = e^{-2\phi}\Delta_{cdef}$. However $\overset{\circ}{\Delta}_{cdef} = 0$ since $\mathscr{H}_{cdef}$ and $\mathscr{H}'_{cdef}$ both reduce to the same tensor at x_0 when the special trivial transformation is made. Therefore $\mathscr{H}'_{cdef} = \mathscr{H}_{cdef}$ since x_0 has been chosen arbitrarily.

It now remains to verify the invariance of $\mathscr{H}_{cdef}$ under trivial transformations. It should first be remarked that $\mathscr{H}_{cdef}$ can be broken up into

([11]) This conformally invariant tensor is of fourth order in the g_{ab} and of third order in the A^a. Thus, by an obvious extension of a result of Szekeres [*31*], it must be composed of C_{abcd}, H_{ab} and their first and second covariant derivatives and the tensors L_{ab} and g^{ab}.

three sets of terms: (*i*) terms involving only g_{ab} and its derivatives (*ii*) the terms $TS(H^k{}_c C_{kdel} H^l{}_f)$ (*iii*) the rest of the terms involving the H_{ab} 's. We shall denote these terms respectively by the tensors $\underset{1}{\mathcal{H}}_{cdef}$, $\underset{2}{\mathcal{H}}_{cdef}$ and $\underset{3}{\mathcal{H}}_{cdef}$. Explicitly we have

$$\begin{aligned}\underset{1}{\mathcal{H}}_{cdef} = TS(3C_{kcdl;m}C^k{}_{ef}{}^{l\,m}{}_{;} + 8C^k{}_{cd}{}^l{}_{;e}S_{klf} + 40S_{cd}{}^kS_{efk} \\ - 8C^k{}_{cd}{}^lS_{kle;f} - 24C^k{}_{cd}{}^lS_{efk;l} + 4C^k{}_{cd}{}^lC_l{}^m{}_{ek}L_{fm} \\ + 12C^k{}_{cd}{}^lC^m{}_{efl}L_{km})\,, \qquad (6.2)\end{aligned}$$

$$\underset{2}{\mathcal{H}}_{cdef} = -\,84TS(H^k{}_cC_{kdel}H^l{}_f)\,, \qquad (6.3)$$

$$\underset{3}{\mathcal{H}}_{cdef} = TS(12H_{kc;de}H^k{}_f - 16H_{kc;d}H^k{}_{e;f} - 18H_{kc}H^k{}_dL_{ef})\,. \qquad (6.4)$$

Each one of the above tensors is separately invariant under the trivial transformations. This is easily seen to be the case for $\underset{2}{\mathcal{H}}_{cdef}$ since from (3.2*a*) and (3.4*a*, *b*) one has

$$\overline{\underset{2}{\mathcal{H}}}_{cdef} = e^{-2\phi}\underset{2}{\mathcal{H}}_{cdef}\,. \qquad (6.5)$$

The verification of the invariance of $\underset{3}{\mathcal{H}}_{cdef}$ involves a straightforward but heavy calculation. We shall require the transformation larw for the Christoffel symbols under conformal transformations [27]:

$$\overline{\left\{\begin{matrix} a \\ b\ c\end{matrix}\right\}} = \left\{\begin{matrix} a \\ b\ c\end{matrix}\right\} + 2\delta_{(b}{}^a\phi_{c)} - g_{bc}\phi^a\,. \qquad (6.6)$$

By repeated use of (6.6), (3.4*b*) and (3.23) one obtains the following transformation formulae:

$$\begin{aligned}TS(\bar{H}_{kc\bar{;}de}\bar{H}^k{}_f) = e^{-2\phi}TS(H_{kc;de}H^k{}_f - 3H_{kc}H^k{}_d\phi_{e;f} - 8H_{kc;d}H^k{}_e\phi_f \\ + 15H_{kc}H^k{}_d\phi_e\phi_f)\,, \quad (6.7)\end{aligned}$$

$$TS(\bar{H}_{kc\bar{;}d}\bar{H}^k{}_{e\bar{;}f}) = e^{-2\phi}TS(H_{kc;d}H^k{}_{e;f} - 6H_{kc;d}H^k{}_e\phi_f + 9H_{kc}H^k{}_d\phi_e\phi_f)\,, \quad (6.8)$$

$$TS(\bar{H}_{kc}\bar{H}^k{}_d\bar{L}_{ef}) = e^{-2\phi}TS(H_{kc}H^k{}_dL_{ef} - 2H_{kc}H^k{}_d\phi_{e;f} + 2H_{kc}H^k{}_d\phi_e\phi_f)\,. \quad (6.9)$$

From these formulae and (6.4) it is easy to show that

$$\overline{\underset{3}{\mathcal{H}}}_{cdef} = e^{-2\phi}\underset{3}{\mathcal{H}}_{cdef}\,. \qquad (6.10)$$

We now prove the conformal invariance of the tensor $\underset{1}{\mathcal{H}}_{cdef}$. The basic formula is that giving the conformal transformation of $C_{abcd;e}$. From (3.4*a*) and (6.6) one obtains

$$\begin{aligned}\tilde{C}_{abcd\tilde{;}e} = -\,e^{2\phi}(C_{abcd;e} - 2C_{abcd}\phi_e + 2\phi_{[a}C_{b]ecd} + 2C_{abe[c}\phi_{d]} \\ + 2g_{e[a}C^m{}_{b]cd}\phi_m + 2C_{ab[c}{}^mg_{d]e}\phi_m)\,. \quad (6.11)\end{aligned}$$

It follows from (6.11) that the conformal transformation law for S_{abc} is

$$\tilde{S}_{abc} = S_{abc} - \phi_kC^k{}_{abc}\,, \qquad (6.12)$$

when one notes that for $n = 4$ Bianchi's identities have the form

$$C^{d}{}_{abc;d} = - S_{abc} \,. \tag{6.13}$$

Using (3.2*a*), (3.4*a*), (3.23), (5.9), (6.6), (6.11), (6.12) and (6.13) we find the following transformation formulae for the various terms on the right hand side of (6.2):

$$\begin{aligned} \mathrm{TS}(\tilde{C}_{kcdl\tilde{;}m}\tilde{C}^{k}{}_{ef}{}^{l}{}_{\tilde{;}}{}^{m}) = e^{-2\phi}\mathrm{TS}(&C_{kcdl;m}C^{k}{}_{cd}{}^{l}{}_{;}{}^{m} - 4C_{kcml;d}C^{k}{}_{ef}{}^{l}\phi^{m} \\ &- 8\phi_{k}C^{k}{}_{cdl;m}C^{l}{}_{ef}{}^{m} + 4C_{kcdl;e}C^{k}{}_{mf}{}^{l}\phi^{m} + 8S_{klc}C^{k}{}_{de}{}^{l}\phi_{f} \\ &+ 6C_{kcdl}C^{k}{}_{ef}{}^{l}\phi_{m}\phi^{m} + 4\phi_{k}C^{k}{}_{cdl}C^{l}{}_{efm}\phi^{m} - 12C_{kcdl}C^{k}{}_{e}{}^{ml}\phi_{f}\phi_{m}) \,, \end{aligned} \tag{6.14}$$

$$\begin{aligned} \mathrm{TS}(\tilde{C}_{kcdl\tilde{;}e}\tilde{S}^{kl}{}_{f}) = e^{-2\phi}(&C_{kcdl;e}S^{kl}{}_{f} - C_{kcdl;e}C^{mkl}{}_{f}\phi_{m} \\ &- 4C_{kcdl}S^{kl}{}_{e}\phi_{f} + 4C_{kcdl}C^{mkl}{}_{e}\phi_{f}\phi_{m} - \phi_{m}C^{m}{}_{cdl}S_{ef}{}^{l} \\ &+ \phi_{m}C^{m}{}_{cdl}C^{l}{}_{efk}\phi^{k}) \,, \end{aligned} \tag{6.15}$$

$$\mathrm{TS}(\tilde{S}_{cd}{}^{k}\tilde{S}_{efk}) = e^{-2\phi}\mathrm{TS}(S_{cd}{}^{k}S_{efk} - 2S_{cdk}C^{k}{}_{ef}{}^{l}\phi_{l} + \phi_{k}C^{k}{}_{cdl}C^{l}{}_{efm}\phi^{m}) \,, \tag{6.16}$$

$$\begin{aligned} \mathrm{TS}(\tilde{C}^{k}{}_{cd}{}^{l}\tilde{S}_{kle\tilde{;}f}) = e^{-2\phi}\mathrm{TS}(&C^{k}{}_{cd}{}^{l}S_{kle;f} - C^{k}{}_{cd}{}^{l}C_{ke}{}^{m}{}_{l}\phi_{f;m} \\ &- C_{kcml;d}C^{k}{}_{ef}{}^{l}\phi^{m} - 4S_{klc}C^{k}{}_{de}{}^{l}\phi_{f} + S_{cdk}C^{k}{}_{efl}\phi^{l} + 4C^{k}{}_{cd}{}^{l}C_{k}{}^{m}{}_{el}\phi_{f}\phi_{m} \\ &- \phi_{k}C^{k}{}_{cdl}C^{l}{}_{efm}\phi^{m}) \,, \end{aligned} \tag{6.17}$$

$$\begin{aligned} \mathrm{TS}(\tilde{C}^{k}{}_{cd}{}^{l}\tilde{S}_{efk\tilde{;}l}) = e^{-2\phi}\mathrm{TS}(&C^{k}{}_{cd}{}^{l}S_{efk;l} - C^{k}{}_{cd}{}^{l}C^{m}{}_{efk}\phi_{l;m} - \phi_{m}C^{m}{}_{cdk;l}C^{k}{}_{ef}{}^{l} \\ &- 4S_{cdk}C^{k}{}_{ef}{}^{l}\phi_{l} + S_{klc}C^{k}{}_{de}{}^{l}\phi_{f} + 4\phi_{k}C^{k}{}_{cdl}C^{l}{}_{efm}\phi^{m} - C^{k}{}_{cd}{}^{l}C_{l}{}^{m}{}_{ek}\phi_{f}\phi_{m}) \,, \end{aligned} \tag{6.18}$$

$$\begin{aligned} \mathrm{TS}(\tilde{C}^{k}{}_{cd}{}^{l}\tilde{C}_{k}{}^{m}{}_{el}\tilde{L}_{fm}) = e^{-2\phi}\mathrm{TS}(&C^{k}{}_{cd}{}^{l}C_{k}{}^{m}{}_{el}L_{fm} - 2C^{k}{}_{cd}{}^{l}C_{k}{}^{m}{}_{el}\phi_{f;m} \\ &+ 2C^{k}{}_{cd}{}^{l}C_{k}{}^{m}{}_{el}\phi_{f}\phi_{m} - C^{k}{}_{cd}{}^{l}C_{kefl}\phi_{m}\phi^{m}) \,, \end{aligned} \tag{6.19}$$

$$\begin{aligned} \mathrm{TS}(\tilde{C}^{k}{}_{cd}{}^{l}\tilde{C}^{m}{}_{efk}\tilde{L}_{lm}) = e^{-2\phi}\mathrm{TS}(&C^{k}{}_{cd}{}^{l}C^{m}{}_{efk}L_{lm} - 2C^{k}{}_{cd}{}^{l}C^{m}{}_{efk}\phi_{l;m} \\ &+ 2\phi_{k}C^{k}{}_{cdl}C^{l}{}_{efm}\phi^{m} - C^{k}{}_{cd}{}^{l}C_{lefk}\phi_{m}\phi^{m}) \,. \end{aligned} \tag{6.20}$$

Employing the formulae (6.14) to (6.20) one finds that

$$\begin{aligned} \underset{1}{\tilde{\mathscr{H}}}_{cdef} = e^{-2\phi}\underset{1}{\mathscr{H}}_{cdef} + 2\mathrm{TS}(&2C_{kcdl;e}C^{k}{}_{mf}{}^{l}\phi^{m} - 2C_{kcml;d}C^{k}{}_{ef}{}^{l}\phi^{m} \\ &+ C_{kcdl}C^{k}{}_{ef}{}^{l}\phi_{m}\phi^{m} - 2\phi_{k}C^{k}{}_{cdl}C^{l}{}_{efm}\phi^{m} - 2C_{kcdl}C^{k}{}_{e}{}^{ml}\phi_{f}\phi_{m}) \,. \end{aligned} \tag{6.21}$$

To show that $\underset{1}{\mathscr{H}}_{cdef}$ is a conformally invariant tensor we must prove that the second term on the right hand side of (6.21) vanishes. The fact that it does follows from two further identities valid when $n = 4$ that may be derived by the method of Lovelock [*20*]. This method for producing identities follows from the fact that in an n-dimensional space the generalized Kronecker delta $\delta^{a_1 \ldots a_{n+1}}_{b_1 \ldots b_{n+1}}$ vanishes identically. Thus when $n = 4$ the expression

$$\delta^{klmnc}_{pqrsd}C_{km}{}^{pr}{}_{;e}C^{qs}{}_{fn}\phi_{l} \tag{6.22}$$

vanishes identically. If one expands the Kronecker delta in (6.22) using the definition

$$\delta^{a_1 \ldots a_n}_{b_1 \ldots b_n} = n!\,\delta^{[a_1}_{b_1} \ldots \delta^{a_n]}_{b_n}$$

and takes the trace-free symmetric part, one finds, on account of (5.9), the identity

$$\mathrm{TS}(C_{kcm;d}C^{k}{}_{ef}{}^{l}\phi^{m} - C_{kcdk;e}C^{k}{}_{mf}{}^{l}\phi^{m}) = 0 \,. \tag{6.23}$$

A similar procedure, when applied to the expression

$$\delta^{klmnc}_{pqrsd}C_{k}{}^{r}{}_{em}C_{n}{}^{s}{}_{f}{}^{p}\phi_{l}\phi^{q} \,,$$

yields the further identity

$$\mathrm{TS}(C_{kcdl}C^{k}{}_{ef}{}^{l}\phi_{m}\phi^{m} - 2\phi_{k}C^{k}{}_{cdl}C^{l}{}_{efm}\phi^{m} - 2C_{kcdl}C^{k}{}_{e}{}^{ml}\phi_{f}\phi_{m}) = 0 \,. \tag{6.24}$$

The identities (6.23) and (6.24) guarantee the vanishing of the second term on the right hand side of (6.21) which thus becomes

$$\tilde{\underset{1}{\mathcal{H}}}_{cdef} = e^{-2\phi}\underset{1}{\mathcal{H}}_{cdef} \,. \tag{6.25}$$

This establishes the conformal invariance of the tensor $\underset{1}{\mathcal{H}}_{cdef}$. *It now follows that* $\mathcal{H}_{cdef}$ *is invariant under trivial transformations since*

$$\mathcal{H}_{cdef} = \underset{1}{\mathcal{H}}_{cdef} + \underset{2}{\mathcal{H}}_{cdef} + \underset{3}{\mathcal{H}}_{cdef} \,. \tag{6.26}$$

It is worth pointing out that one can construct a second fourth rank conformally invariant tensor from the transformation formulae (6.14) to (6.20). This tensor is

$$\begin{aligned} H_{cdef} = \mathrm{TS}(C_{kcdl;m}C^{k}{}_{ef}{}^{l}{}_{;}{}^{m} + 4C^{k}{}_{cd}{}^{l}{}_{;e}S_{klf} + 12S_{cd}{}^{k}S_{efk} \\ - 4C^{k}{}_{cd}{}^{l}S_{kle;f} - 8C^{k}{}_{cd}{}^{l}S_{efk;l} + 2C^{k}{}_{cd}{}^{l}C_{k}{}^{m}{}_{el}L_{fm} \\ + 4C^{k}{}_{cd}{}^{l}C^{m}{}_{efk}L_{lm}) \,. \end{aligned} \tag{6.27}$$

Under conformal transformations it transforms as follows:

$$\tilde{H}_{cdef} = e^{-2\phi}H_{cdef} \,. \tag{6.28}$$

The tensors $\underset{1}{\mathcal{H}}_{cdef}$ and H_{cdef} appear to be unrelated and provided explicit examples of the polynomial conformal tensors studied by du Plessis [*9*] and Szekeres [*31*].

7. DETERMINATION OF THE SELF-ADJOINT HUYGENS' DIFFERENTIAL EQUATIONS ON A SYMMETRIC SPACE

In this section we apply the conditions I to V to determine the self-adjoint Huygens' differential equations on a symmetric V_4. A pseudo-Riemannian space V_4 is symmetric in the sense of Cartan if

$$R_{abcd;e} = 0 \,. \tag{7.1}$$

These spaces, in the case of Lorentzian signature for $n = 4$, have been determined by Cahen and McLenaghan [*3*]. They may be classified according to the Petrov type ([24] [25]) of the Weyl tensor C_{abcd}. In [3] it has been shown that symmetric V_4 's are of Petrov type N, D or [−] (conformally flat). We shall consider each of these cases separately.

Type N: In this case there exists a coordinate system $(v, z, \bar{z}, u)$ in which the metric of symmetric V_4 has the form

$$ds^2 = 2dv(du + (\alpha(z^2 + \bar{z}^2) + \beta z\bar{z})dv) - 2dzd\bar{z} , \tag{7.2}$$

where α and β are real constants. If one defines

$$k_a = v_{,a} , \tag{7.3}$$

one has

$$k_a C^a{}_{bcd} = 0 , \tag{7.4}$$

$$R_{ab} = - 2\beta k_a k_b , \tag{7.5}$$

$$k_a k^a = 0 . \tag{7.6}$$

We shall also need the following relations which follow from (7.1) and (1.11):

$$C_{abcd;e} = 0 , \tag{7.7a}$$

$$R_{ab;c} = 0 , \tag{7.7b}$$

$$R_{,a} = 0 . \tag{7.7c}$$

Now it follows from (7.4), (7.5) and (7.7) that condition III becomes in a type N symmetric space

$$H_{ka}H^k{}_b - \frac{1}{4}g_{ab}H_{kl}H^{kl} = 0 .$$

This implies that

$$H_{ab} = 0 , \tag{7.8}$$

from which it follows on account of (1.8) that there exists a function A such that $A_a = A_{,a}$. Thus on account of condition I a Huygens' equation on a type N symmetric space is equivalent to the equation

$$g^{ab}u_{;ab} = 0 . \tag{7.9}$$

Now it follows from (7.4), (7.5), (7.7) and (7.8) that conditions IV and V are *identically satisfied. Furthermore, it is implicit in the work of Günther [13] ([12]) that* (7.9) *is a Huygens' differential equation on a type* N *symmetric space. Since* $C_{abcd} \neq 0$, (7.9) *is not equivalent to the wave equation* (1.2). *This shows that Hadamard's conjecture is not true for the equation* (7.9) *on a symmetric space. Thus the conclusion of Helgason [16], p. 68 is false.*

([12]) This is because (1.3) is the metric of a symmetric space if and only if $a_{12} = 0$, $a_{11} = \sin^2((a + b)^{1/2}x^0)$ and $a_{22} = \sin h^2((a - b)^{1/2}x^0)$ with $|b| < a$ constant and suitable restrictions on x^0. This has been remarked by the author [23] for the case $R_{ab} = 0$. The metrics (1.3) and (7.2) are related by a coordinate transformation with $\alpha = a$ and $\beta = 2b$.

Type D: In this case there exists a coordinate system $(u, z, \bar{z}, v)$ in which the metric takes the form

$$ds^2 = \frac{2dudv}{(1 - (R + \beta)uv/8)^2} - \frac{2dzd\bar{z}}{(1 + (R - \beta)\bar{z}z/8)^2}, \tag{7.10}$$

where R, the curvature scalar, and β are constant.

In this case we restrict ourselves to the self-adjoint equation (1.13). Condition I tells us immediately that $C = \frac{1}{6}R$, while condition III reduces to

$$C^k{}_{ab}{}^l L_{kl} = 0 \tag{7.11}$$

on account of (7.7). We need to determine C_{abcd} and L_{ab} for the metric (7.10). To achive this we introduce the null tetrad $(k_a, m_a, \bar{m}_a, n_a)$ defined as follows:

$$k_a = u_{,a}, \tag{7.12a}$$

$$m_a = (1 + (R - \beta)z\bar{z}/8)^{-1}z_{,a}, \tag{7.12b}$$

$$n_a = (1 - (R + \beta)uv/8)^{-2}v_{,a} \tag{7.12c}$$

It is easy to see that

$$g^{ab}k_a n_n = - g^{ab}m_a\bar{m}_b = 1 \tag{7.13}$$

with all other inner products being zero. One can now show that

$$R_{ab} - \frac{1}{4}g_{ab}R = \frac{\beta}{2}(k_{(a}n_{b)} + m_{(a}\bar{m}_{b)}), \tag{7.14}$$

$$C_{abcd} = \frac{R}{3}(\bar{m}_{[a}n_{b]}k_{[c}m_{d]} + k_{[a}m_{b]}\bar{m}_{[c}l_{d]} + (k_{[a}n_{b]} - m_{[a}\bar{m}_{b]})(k_{[c}n_{d]} - m_{[c}\bar{m}_{d]})) + \text{complex conjugate}. \tag{7.15}$$

Using (7.13), (7.14) and (7.15) one can show that (7.11) implies $\beta R = 0$ from which we have $R = 0$ or $\beta = 0$. If $R = 0$, (7.15) implies that $C_{abcd} = 0$ which is inconsistant with our hypothesis that our space be of Petrov type D. Thus we must have $\beta = 0$. On account of this and (7.14) condition V reduces to

$$TS(RC^k{}_{cd}{}^l C_{kefl}) = 0. \tag{7.16}$$

Employing (7.13) and (7.15) one finds that (7.16) implies $R = 0$ which is impossible. *We conclude from the above that the differential equation*

$$\Box u + \frac{R}{6}u = 0 \tag{7.17}$$

cannot satisfy Huygens' principle on a Type D symmetric space.

Type [–]: In this case the symmetric space is conformally flat. According to Cahen and McLenaghan [3] canonical forms for the metrics are (7.2) with $\alpha = 0$, (7.10) with $R = 0$ and the following:

$$ds^2 = dt^2 - \frac{dx^2 + dy^2 + dz^2}{\left[1 + \frac{R}{24}(x^2 + y^2 + z^2)\right]^2}, \tag{7.18}$$

$$ds^2 = -dx^2 + \frac{dt^2 - dy^2 - dz^2}{\left[1 - \frac{R}{24}(t^2 - y^2 - z^2)\right]^2} \tag{7.19}$$

$$ds^2 = \frac{dt^2 - dx^2 - dy^2 - dz^2}{\left[1 - \frac{R}{48}(t^2 - x^2 - y^2 - z^2)\right]^2}. \tag{7.20}$$

As for the case of a type N symmetric space, condition III implies $H_{ab} = 0$. Thus in view of (1.4), a Huygens' differential equation on a conformally flat symmetric space is equivalent to the self-adjoint equation (7.17) which, on account of (3.3), is equivalent to the wave equation. *This shows that Huygens' principle holds for the self-adjoint equation* (7.17) *on a conformally flat symmetric space but not for the pure equation* $\square u = 0$ *unless the curvature scalar vanishes.*

In a subsequent paper further consequences of the conditions I to V will be discussed.

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