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JAN TARSKI

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Model of linearized deviations in the quantum theory of gravity

by

Jan TARSKI

II. Institut für Theoretische Physik der Universität Hamburg.

ABSTRACT. — Linearized deviations from two kinds of hypersurfaces of a de Sitter universe are considered. Ground state functionals of these deviations are described. The approach is the one used by Kuchař in his study of the linearized gravitational field.

RÉSUMÉ. — On considère deux sortes d'hypersurfaces de l'univers de de Sitter et leurs déviations linéarisées. Les fonctionnelles fondamentales de ces déviations sont décrites. On aborde ce problème par la méthode employée par Kuchař dans son étude du champ gravitationnel linéarisé.

1. INTRODUCTION

Since the attempt of DeWitt [*1*] [*2*] to give a systematic presentation of the quantum theory of gravity, various simplified models have been investigated. These are of two kinds. First, one can suppress some, or even nearly all, of the degrees of freedom. Examples can be found in [*1*], [*3*]-[*7*].

The second possibility is to study the linearized deviations (of the metric tensor) from a chosen background space. For the case of the Minkowski background one finds an elaboration of the usual theory of the massless, spin 2 free field. Still, the problem turned out to be rather subtle, and was fully treated only in 1970 by Kuchař [*8*], where various obscurities of an earlier study [9] we clarified.

One can increase the sophistication here by using a curved space rather than a flat space as background. An early investigation of this kind is that of Lichnerowicz [*10*], who confined himself to the determination of propagators and commutators on a curved space.

The present article describes an analysis of the linearized deviations from hypersurfaces of a de Sitter universe as background. In particular, two families of space-like hypersurfaces, indexed by a time parameter, are considered. Members of one family are curved, closed, and the metric is time-independent, while those of the other family are flat, open, and the metric is time-dependent.

The present study, like [*8*], depends in an essential way on the decomposition of tensor fields. Such decompositions can be exhibited in a rather simple way for Einstein spaces, and our hypersurfaces are of this kind. One then finds a natural separation of variables in the functional differential equations, and Gaussian solutions for the ground-state vectors are easily found. (The problem of decomposing tensor fields on a general space continues to be investigated by various authors.)

Our analysis provides an additional perspective for various interesting observations which were made in [*8*]. It brings, moreover, into focus the following problem, which we do not pursue. Extend the present study by admitting hypersurfaces which would interpolate between the two families, and which would consequently give an example of a change in topology.

In sec. 2 we review various useful formulas bearing on the de Sitter universe and on the Einstein equations. Section 3 contains the decomposition of tensor fields, and we give also geometric interpretations of some of the terms. In sec. 4 we give the functional differential equations for state vectors, and construct the ground state functionals, for the two kinds of hypersurfaces. Section 5 contains a brief discussion of inner products for state functionals. Finally, in appendices we point out some simple connections between the present work and the recent geometric approach to the Einstein equations, and also some group-theoretic properties of the de Sitter universe.

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2. THE DE SITTER UNIVERSE AND THE COSMOLOGICAL CONSTANT

We first recall a few basic formulas for the geometry of the de Sitter universe S; a fuller discussion with references can be found in the treatises of Synge [11] and of Tolman [12]. We will suppose that S is represented by the following hyperboloid (with C real and nonzero),

$$S = \{ (u^0, \ldots, u^4) : (u^0)^2 - (u^1)^2 - \ldots - (u^4)^2 + C^{-2} = 0 \}, \quad (2.1)$$

imbedded in the space M^5, with the metric

$$ds^2 = (du^0)^2 - (du^1)^2 - \ldots - (du^4)^2. \quad (2.2)$$

Consequently, the metrics of space-times will have the signature $+ - - -$. For the Ricci tensor, we use $R_{\mu\nu} = \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\mu \Gamma^\alpha_{\nu\alpha} + \Gamma^\alpha_{\mu\nu}\Gamma^\beta_{\alpha\beta} - \Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\nu\alpha}$.

We recall that an Einstein space is one where ([13], p. 136)

$$R_{\mu\nu} = (\text{const.})\, g_{\mu\nu}. \quad (2.3)$$

The number of dimensions and the signature of the metric can be arbitrary. Then S is an Einstein space of constant scalar curvature ${}^4R$, where explicitly,

$${}^4R_{\mu\nu} = (-3C^2)\,{}^4g_{\mu\nu}, \qquad {}^4R = -12C^2. \quad (2.4a, b)$$

In general, we will indicate the four-dimensional tensors by Greek indices (when they are used), and by the prefix ^{4}, while the three-dimensional tensors will be indicated by Latin indices and without prefix.

Let us turn to the parametrization and to the hypersurfaces of S. We introduce the coordinates ρ, θ, α, and τ by

$$u^1 = \rho \sin\theta \cos\alpha, \qquad u^2 = \rho \sin\theta \sin\alpha, \qquad u^3 = \rho\cos\theta,$$
$$u^4 = \pm C^{-1}(1-\rho^2C^2)^{\frac{1}{2}} \cosh(\tau C), \qquad u^0 = C^{-1}(1-\rho^2C^2)^{\frac{1}{2}} \sinh(\tau C). \quad (2.5)$$

The metric of S then becomes

$$ds^2 = (1 - \rho^2C^2)d\tau^2 - (1 - \rho^2C^2)^{-1}d\rho^2 - \rho^2 d\Omega^2, \quad (2.6a)$$

where
$$d\Omega^2 = d\theta^2 + \sin^2\theta\, d\alpha^2. \quad (2.6b)$$

This metric is stationary. For a hypersurface at τ = const. we find

$$R_{jk} = (-2C^2)g_{jk}, \qquad R = -6C^2, \quad (2.7)$$

i. e. again an Einstein space (cf. the formulas in [11], p. 271). Such a hypersurface is a three-sphere, and ρ is restricted to $0 \leqslant \rho \leqslant C^{-1}$. See also Appendix B.

Let us make a further change of variables,

$$r = e^{-\tau C}\rho(1 - \rho^2C^2)^{-\frac{1}{2}}, \qquad t = \tau + (2C)^{-1}\log(1 - \rho^2C^2), \quad (2.8)$$

and then go to the Cartesian coordinates. The metric becomes

$$ds^2 = dt^2 - e^{2t\mathrm{C}}(dx^2 + dy^2 + dz^2). \tag{2.9}$$

This metric is not stationary, and each hypersurface $t = $ const. is an infinite flat space.

We will refer to the hypersurfaces of the two kinds as *C-surfaces* and *F-surfaces* (C for *curved*, F for *flat*), respectively. I. e., let A be a constant, and we set

$$\mathrm{C_A}\text{-surface} = \{ (\tau, \rho, \theta, \alpha) \in \mathrm{S} : \tau = \mathrm{A} \}, \tag{2.10a}$$

$$\mathrm{F_A}\text{-surface} = \{ (t, x, y, z) \in \mathrm{S} : t = \mathrm{A} \}. \tag{2.10b}$$

The subscript A will usually be suppressed.

The universe S satisfies the Einstein free field equations with a cosmological constant Υ (i. e. $\mathrm{G}_{\mu\nu} + g_{\mu\nu}\Upsilon = 0$), given by

$$\Upsilon = 3\mathrm{C}^2. \tag{2.11}$$

We will therefore suppose the same constant for the quantized theory. Indeed, this constant brings about only a minor change in the dynamical structure of the field equations. We introduce, following [*1*] and [*8*],

$$\mathrm{N}_i = -\,{}^4g_{0i}, \qquad \mathrm{N} = ({}^4g^{00})^{-\frac{1}{2}}, \qquad \gamma = \det(-\,g_{ik}), \tag{2.12a-c}$$

and the extrinsic curvature tensor is

$$\mathrm{K}_{ik} = (2\mathrm{N})^{-1}(g_{ik,0} + \nabla_k\mathrm{N}_i + \nabla_i\mathrm{N}_k) \tag{2.13a}$$

with the trace

$$\mathrm{tr}\,\mathrm{K} = g^{ik}\mathrm{K}_{ik}. \tag{2.13b}$$

A comma indicates a partial derivative, and ∇, the covariant derivative (here, on a three-space).

The Lagrangian with $\Upsilon \neq 0$ takes the form,

$$\mathrm{L} = \int d^3\boldsymbol{x}\,\gamma^{\frac{1}{2}}\mathrm{N}[-\,\mathrm{K}_{ik}\mathrm{K}^{ik} + (\mathrm{tr}\,\mathrm{K})^2 + \mathrm{R} + 2\Upsilon]. \tag{2.14}$$

It follows, cf. [*1*], that the canonical momenta retain their usual form,

$$\pi^{ik} = -\,\gamma^{\frac{1}{2}}(\mathrm{K}^{ik} - g^{ik}\,\mathrm{tr}\,\mathrm{K}). \tag{2.15}$$

The supermomentum densities χ^i likewise are not changed,

$$\chi^i = -\,2\nabla_k\pi^{ik} = -\,2\pi^{ik}{}_{,k} - g^{il}(2g_{lm,k} - g_{km,l})\pi^{km}, \tag{2.16}$$

while the superhamiltonian density ν now becomes

$$\nu = \gamma^{-\frac{1}{2}}\left[\pi_i^k\pi_k^i - \frac{1}{2}(\mathrm{tr}\,\pi)^2\right] + \gamma^{\frac{1}{2}}(\mathrm{R} + 2\Upsilon). \tag{2.17a}$$

We see that for a general cosmological constant υ,

$$\nu_{\upsilon=\Upsilon} = \nu_{\upsilon=0} + 2\gamma^{\frac{1}{2}}\Upsilon. \tag{2.17b}$$

The new constraint equations are

$$\chi^i = 0, \qquad \nu_{\upsilon = \Upsilon} = 0. \tag{2.18}$$

We note that the arguments relating the constraints to the variational principles [14] remain valid also if $\Upsilon \neq 0$. We now take, as in [1], Eqs. (2.18) and their quantum analogues as the basic equations of the theory.

3. TENSOR FIELDS AND SMALL DEFORMATIONS

Linearized deviations from given metrics are the object of our investigation. A field of deviations can be considered as a tensor field, the tensors being of second rank, covariant, and symmetric.

For a flat space, one has the familiar decomposition of such a tensor field [8] [15]:

$$k_{ij} = k_{ij}^{\mathrm{T}} + k_{ij}^{\mathrm{T}} + k_{ij}^{\mathrm{L}}. \tag{3.1}$$

The parts k_{ij}^{TT} and k_{ij}^{T} satisfy

$$k_{ij},_j^{\mathrm{TT}} = k_{ii}^{\mathrm{TT}} = 0, \qquad k_{ij},_j^{\mathrm{T}} = 0, \tag{3.2}$$

and the decomposition is given explicitly by [with $\nabla^{-2} = (\Sigma \partial_j^2)^{-1}$]

$$k_{ij}^{\mathrm{T}} = \frac{1}{2}(\delta_{ij} - \nabla^{-2}\partial_i\partial_j)k_{\mathrm{T}}, \qquad k_{\mathrm{T}} = k_{ii} - \nabla^{-2}k_{ij},_{ij}, \tag{3.3a}$$

$$k_{ij}^{\mathrm{L}} = k_i,_j + k_j,_i, \qquad k_i = \nabla^{-2}\left(k_{ij},_j - \frac{1}{2}\nabla^{-2}k_{jl},_{jli}\right), \tag{3.3b}$$

$$k_{ij}^{\mathrm{TT}} = k_{ij} - k_{ij}^{\mathrm{L}} - k_{ij}^{\mathrm{T}}.$$

We now turn to the problem of decomposing tensor fields on more general spaces. Various results bearing on this problem can be found in [16]-[19].

Following [10] [13] [16], we introduce the Laplacian Δ, whose action on a p-fold covariant tensor field α is a follows:

$$(\Delta\alpha)_{j_1 \ldots j_p} = -\nabla^m\nabla_m\alpha_{j_1 \ldots j_p} - \Sigma_k(\nabla_{j_k}\nabla_l - \nabla_l\nabla_{j_k})\alpha_{j_1 \ldots}{}^{l}{}_{\ldots j_p}. \tag{3.4}$$

We set, furthermore,

$$(\delta\alpha)_{j_2 \ldots j_p} = -\nabla^m\alpha_{mj_2 \ldots j_p}, \qquad \mathrm{tr}_{k,l}\alpha = (g^{j_k j_l}\alpha_{j_1 \ldots j_p}). \tag{3.5a, b}$$

Then

$$\Delta\,\mathrm{tr}_{k,l}\,\alpha = \mathrm{tr}_{k,l}\,\Delta\alpha, \tag{3.6}$$

and, in case of an Einstein space and a vector β_1 and a two-tensor β_2,

$$\nabla\Delta\beta_1 = \Delta\nabla\beta_1, \qquad \Delta\delta\beta_2 = \delta\Delta\beta_2. \tag{3.7a, b}$$

The following result is now central [16]. Consider a (smooth) field of symmetric covariant two-tensors σ, on a compact Einstein space of dimension n, with the metric tensor g and the scalar curvature R. One can decompose σ as follows:

$$\sigma = \sigma^{\mathrm{TT}} + \sigma^{\mathrm{T}} + \sigma^{\mathrm{L}} + \sigma^{\mathrm{H}}, \tag{3.8}$$

where σ^{T} is determined by a scalar function ζ, σ^{L} is the Lie derivative of g with respect to a vector field ξ (cf. [13], p. 130), and σ^{H} is harmonic. One has, explicitly,

$$\delta\sigma^{\mathrm{TT}} = 0, \qquad \operatorname{tr}\sigma^{\mathrm{TT}} = 0, \qquad \sigma^{\mathrm{TT}} \perp \operatorname{nul}(\Delta), \tag{3.9}$$

$$\sigma^{\mathrm{T}} = [g(\Delta - (\mathrm{R}/n)) + \nabla\nabla]\zeta, \qquad \delta\sigma^{\mathrm{T}} = 0, \qquad \zeta \perp \operatorname{nul}(\Delta), \tag{3.10}$$

$$\sigma^{\mathrm{L}} = \mathrm{L}_{\xi}g = (\nabla_i\xi_k + \nabla_k\xi_i), \tag{3.11}$$

$$\Delta\sigma^{\mathrm{H}} = 0, \qquad \delta\sigma^{\mathrm{H}} = 0. \tag{3.12}$$

Orthogonality refers to the L_2 inner product defined by the volume element

$$d\mu_g(x) = |\det g|^{\frac{1}{2}} dx^1 \wedge \ldots \wedge dx^n, \tag{3.13}$$

and the four terms in (3.8) are mutually orthogonal.

Let us now suppose that the operator Δ acts on the scalar fields, on a compact manifold with a positive-definite metric. Then it has the following properties [20]: (*a*) It is symmetric and non-negative, hence can be assumed self-adjoint, and (*b*) zero is an isolated eigenvalue with multiplicity one. Thus zero corresponds to the constant functions.

Note that for a negative-definite metric, Δ will be non-positive.

In view of the relations $\nabla g = 0$ and (3.6), one can write

$$\sigma^{\mathrm{H}} = (\text{const.})\, g + \sigma^{\mathrm{HTT}},$$

where σ^{HTT} is traceless and satisfies (3.12). However, one can show that the properties (*a*) and (*b*) remain valid for Δ when it acts on tensor fields such as σ [21]. Thus $\sigma^{\mathrm{HTT}} = 0$, and (3.12) can be strengthened to

$$\sigma^{\mathrm{H}} = (\text{const.})\, g, \qquad \nabla\sigma^{\mathrm{H}} = 0. \tag{3.14a}$$

The relation $\zeta \perp \operatorname{nul}(\Delta)$ can also be strengthened. If $[\Delta - \mathrm{R}/(n-1)]\zeta = 0$, then $\operatorname{tr}\sigma^{\mathrm{T}} = 0$, σ^{T} would be its own TT-part, and by orthogonality, $\sigma^{\mathrm{T}} = 0$. (This conclusion is also implicit in [16].) One can therefore suppose that

$$\zeta \perp \operatorname{nul}(\Delta), \qquad \zeta \perp \operatorname{nul}[\Delta - \mathrm{R}/(n-1)]. \tag{3.14b}$$

Further details about the spectrum of Δ may be found in [20].

The foregoing remarks imply in particular the orthogonal decomposition of scalar functions f,

$$f = (\text{const.}) + f_1, \qquad f_1 \perp \operatorname{nul}(\Delta). \tag{3.15}$$

We will now use this relation to decompose the tensor field fg on a compact Einstein manifold. We observe that if Δ is restricted to scalar functions, then $(\Delta - \mathrm{R}/n)^{-1}$ exists. Indeed, otherwise we would have a function f_2 with $(\Delta - \mathrm{R}/n)f_2 = 0$. Consider $\sigma = {}_{df}\nabla\nabla f_2$. We would have $\sigma = \sigma^{\mathrm{T}} = \sigma^{\mathrm{L}}$, in contradiction with the asserted orthogonality $\sigma^{\mathrm{T}} \perp \sigma^{\mathrm{L}}$. We conclude that

$$fg = (fg)^{\mathrm{T}} + (fg)^{\mathrm{L}} + (fg)^{\mathrm{H}}, \tag{3.16a}$$

where $$(fg)^{\mathrm{T}} = [g(\Delta - \mathrm{R}/n) + \nabla\nabla]\,[(\Delta - \mathrm{R}/n)^{-1}f_1], \tag{3.16b}$$

$$(fg)^{\mathrm{L}} = \nabla\nabla[-(\Delta - \mathrm{R}/n)^{-1}f_1]. \tag{3.16c}$$

The three parts of a tensor, σ^{T}, σ^{L}, and σ^{H}, will now be interpreted geometrically.

Let us consider deformations of coordinate systems, and of C- and of F-surfaces. Let $\boldsymbol{\xi}$ be a vector field on a C-surface. Such a vector field induces an infinitesimal change of coordinate vectors, $\boldsymbol{x} \to \boldsymbol{x} + \varepsilon \boldsymbol{x}_{\xi}$ and the (infinitesimal) change induced on a tensor field σ is given by $\varepsilon \mathrm{L}_{\xi}\sigma$. Thus, if we take for σ the deformation h of the metric tensor, we see that h^{L} corresponds to a change of coordinates. Furthermore, the term h^{H} corresponds to a change of the de Sitter constant C.

In order to interpret σ^{T}, we first change to Gaussian coordinates in (2.5-6), as follows. We select one particular C-surface, and we require that this surface correspond to a constant time τ_{A}, that g and $\mathrm{N}^{-1}\dot{g}$ on this surface remain unchanged, and that $\mathrm{N} = 1$, in these coordinates. (See [*22*], Sec. 10; [*23*], Part III, § 3; and Appendix A.) We denote the new time variable by τ_0, and we make a small deformation of the τ_0-variable and of the chosen C-surface,

$$\tau_0 \to \tau_1 = \tau_0 + \bar{\tau}(\boldsymbol{\rho}), \qquad (\tau_0 = \tau_{\mathrm{A}}) \to [\tau_0 + \bar{\tau}(\boldsymbol{\rho}) = \tau_{\mathrm{A}}], \tag{3.17a}$$

where
$$\boldsymbol{\rho} = (\rho^i) = (\rho, \theta, \alpha). \tag{3.17b}$$

We observe that, to the first order in $\bar{\tau}$,

$$\partial g_{ik}/\partial\tau_1 = 0, \qquad \mathrm{N}_i = \partial\bar{\tau}/\partial\rho^i = \nabla_i\bar{\tau}, \tag{3.18a, b}$$

see (2.12*a*). Consequently

$$(\mathrm{K}_{ik}) = \nabla\nabla\bar{\tau}, \qquad \pi = -\gamma^{\frac{1}{2}}((g^{ik}\Delta + \nabla^i\nabla^k)\bar{\tau}). \tag{3.19a, b}$$

The expression $(g...)\bar{\tau}$ is like σ^{T}, but we must provide the term $(\mathrm{R}/n)g\bar{\tau}$, where now $n = 3$. However, we know that it can be reduced to the parts T, L, and H, see (3.16). The factor $\gamma^{\frac{1}{2}}$ in π characterizes π as a density, so that $\gamma^{-\frac{1}{2}}\pi$ is a tensor π_{tens}, which we can decompose. We conclude that π of (3.19*b*) satisfies

$$\pi = \gamma^{\frac{1}{2}}(\pi^{\mathrm{T}}_{\text{tens}} + \pi^{\mathrm{L}}_{\text{tens}} + \pi^{\mathrm{H}}_{\text{tens}}) \tag{3.20a}$$

$$= {}_{df}\pi^{\mathrm{T}} + \pi^{\mathrm{L}} + \pi^{\mathrm{H}}, \tag{3.20b}$$

and $\pi^{\mathrm{T}}_{\text{tens}}$ can be given in terms of $\bar{\tau}$, cf. (3.15-16),

$$\pi^{\mathrm{T}}_{\text{tens}} = -((g^{ik}\Delta - g^{ik}\mathrm{R}/n + \nabla^i\nabla^k)[1 + (\mathrm{R}/n)(\Delta - \mathrm{R}/n)^{-1}](\bar{\tau} - \text{const.})). \tag{3.21}$$

For an F-surface, the situation is analogous to the conclusions in [*8*]. The change of coordinates $\boldsymbol{x} \to \boldsymbol{x} + \varepsilon\boldsymbol{x}_{\xi}$ induces a contribution h^{L} to the deformed metric. Furthermore, a deformation of the F-surface by the function $\bar{t}(\boldsymbol{x})$ yields $(\pi - \pi^{(0)})^{\mathrm{T}}$, given by

$$((\pi - \pi^{(0)})^{\mathrm{T}})^{ij} = -e^{-t\mathrm{C}}(-\delta_{ij}\nabla^2 + \partial_i\partial_j)\bar{t}. \tag{3.22}$$

Note that for an F-surface, $\pi = \pi^{(0)} \neq 0$ before deformation.

We close this section with some useful formulas for the deformations of geometric objects resulting from a slight change in the metric tensor,

$$g \to g + h, \qquad h = (h_{ik}) \tag{3.23}$$

(see [10], p. 39 ff.). For the contravariant metric tensor, $(g^{ik}) \to (g^{ik}) + (g^{ik})'$, one finds that to the lowest order in h,

$$(g^{ik})' = -(h^{ik}). \tag{3.24}$$

Similarly, for the affine connection,

$$2(\Gamma')^i_{jk} = \nabla_j h^i_k + \nabla_k h^i_j - \nabla^i h_{jk}. \tag{3.25}$$

The result for the Ricci tensor is,

$$2(\mathrm{R}_{ij})' = \Delta h - \mathrm{L}_{\xi(h)} g, \tag{3.26a}$$

where

$$\xi(h) = \delta h + \frac{1}{2}\nabla \operatorname{tr} h, \tag{3.26b}$$

and for the curvature scalar,

$$\mathrm{R}' = \Delta \operatorname{tr} h + \delta\delta h. \tag{3.27}$$

If the background metric is $g = (-\delta_{ij})$, then (3.27) becomes, cf. [8] and (3.3a),

$$\mathrm{R}' = \Delta h_{\mathrm{T}}. \tag{3.28}$$

In the case of a compact Einstein space, h^{TT} and h^{H} do not contribute to R', but the situation with h^{L} is less clear. We write therefore

$$\mathrm{R}' = \Delta \operatorname{tr}(h^{\mathrm{T}} + h^{\mathrm{L}}) + \delta\delta h^{\mathrm{L}}. \tag{3.29}$$

4. FUNCTIONALS OF LINEARIZED DEVIATIONS

In the gravitational theory the g_{jk} and the π^{jk} are canonically conjugate. Thus, in the metric representation of the quantized theory, one takes the g_{jk} as multiplicative operators, and then the π^{jk} should be differentiations,

$$\pi^{jk} = i^{-1}\delta/\delta g_{jk}. \tag{4.1}$$

This relation and (2.16-18) now determine the quantum operators χ^j and v, and the basic constraints on state vectors become

$$\chi^j\Psi = -2i^{-1}(\nabla_k \delta/\delta g_{jk})\Psi = 0, \tag{4.2a}$$

$$v\Psi = \gamma^{\frac{1}{2}}[-\gamma^{-1}(\delta^2/\delta g_{jk}\delta g^{jk}) + (2\gamma)^{-1}(\operatorname{tr}\delta/\delta g_{jk})^2 + (\mathrm{R} + 2\Upsilon)]\Psi = 0. \tag{4.2b}$$

We now follow [8] and expand the variable functions g_{jk} and π^{jk} in the small parameter ε:

$$g_{jk} = g^{(0)}_{jk} + \varepsilon g^{(1)}_{jk} + \varepsilon^2 g^{(2)}_{jk} + \dots, \qquad \pi^{jk} = \pi^{(0)jk} + \varepsilon\pi^{(1)jk} + \dots. \tag{4.3}$$

Here $g^{(0)}_{jk}$ and $\pi^{(0)jk}$ have fixed values, i. e. those of the C- or the F-surfaces. In view of (4.1) we will set $\varepsilon\pi^{(1)} = i^{-1}\delta/\delta(\varepsilon g^{(1)})$, and the derivative is expected to yield a quantity of order ε. We also write

$$\varepsilon g^{(1)}_{jk} = h_{jk}, \qquad \varepsilon\pi^{(1)jk} = i^{-1}\delta/\delta h_{jk} = p^{jk}. \tag{4.4}$$

It is shown in [8] how for each order in ε some of the quantities $g^{(n)}_{jk}$, $\pi^{(m)jk}$ can be chosen arbitrarily, and the others are then determined by the (classical) equations $\chi^j = \nu = 0$. An analogous procedure for the quantized theory is also indicated there. In principle, one could adapt those procedures to deviations from the C- and the F-surfaces, but the details would be more involved. In order to shorten our presentation, we will restrict ourselves primarily to the true gravitational degrees of freedom, in the lowest order. We will ignore, in so far as convenient, the geometrical deformations within the de Sitter universe.

One also knows from [8] the suitability of the extrinsic time representation, in which the relations (4.1) or (4.4) are inverted for the h^{T}-part of the tensor field (h_{ik}), cf. (3.1-3). In particular, p^{T} is taken as the multiplicative operator, and then (component-wise)

$$h^{\mathrm{T}} = -\, i^{-1}\delta/\delta p^{\mathrm{T}}. \tag{4.5}$$

Such a representation will also prove convenient here.

We now consider a C-surface, where

$$\pi^{(0)jk} = 0, \qquad -\, g^{(0)}_{jk} = f_j\delta_{jk}, \tag{4.6a}$$

with

$$f_\rho = (1 - \rho^2 \mathrm{C}^2)^{-1}, \qquad f_\theta = \rho^2, \qquad f_\alpha = \rho^2 \sin^2\theta, \tag{4.6b}$$

and moreover

$$\gamma^{(0)} = f_\rho f_\theta f_\alpha, \qquad \mathrm{R}^{(0)} = -\,6\mathrm{C}^2 = -\,2\Upsilon. \tag{4.7a, b}$$

To the deviation fields h, p we apply the decomposition (3.8). The constraints (2.18) are necessarily satisfied in the zeroth order in ε, and for the first order one finds (note that $\delta p = \delta p^{\mathrm{L}}$),

$$\varepsilon(\chi^j)^{(1)}\Psi = 2\delta p^{\mathrm{L}}\Psi = 0, \tag{4.8a}$$

$$\varepsilon\nu^{(1)}\Psi = \gamma^{(0)\frac{1}{2}}\mathrm{R}^{(1)}\Psi = 0. \tag{4.8b}$$

Equation (4.8a) implies that $p^{\mathrm{L}}\Psi = 0$, and that Ψ is independent of h^{L} (Cf. [1] [8] [17].) We recall also that the tensor field h^{L} constitutes an independent variable, related to the change of coordinates. We will therefore suppose that $h^{\mathrm{L}} = 0$.

Next we consider $\nu^{(1)}$, to which only h^{T} now contributes. We use (3.29), we set

$$h^{\mathrm{T}} = [g(\Delta - \mathrm{R}^{(0)}/3) + \nabla\nabla]\zeta_1, \tag{4.9a}$$

and we find

$$\varepsilon\nu^{(1)}\Psi = \gamma^{(0)\frac{1}{2}}(\mathrm{A}\zeta_1)\Psi = 0, \tag{4.9b}$$

where

$$A = (2\Delta - R^{(0)})\Delta. \qquad (4.9c)$$

In view of (3.14*b*), $\zeta_1 \Psi = 0$, and hence $h^T \Psi = 0$. In the representation (4.5), this means that Ψ is independent of p^T. This latter is an independent variable, which is related to the deformation of a C-surface, cf. eq. (3.21). We will suppose $p^T = 0$.

We recall, in this connection, that the eigenfunctions of Δ with the eigenvalue $R^{(0)}/2$ do not contribute to h^T, and consequently to $v^{(1)}$ likewise not. This appears to be related to the invariance of the dynamical components of the metric under conformal transformations, but we forego a detailed analysis. See [*13*] [*16*] [*19*], and also eqs. (3.16) and Appendix B.

The equations (4.8) say nothing about the harmonic parts. We will suppose that

$$h^H = p^H = 0. \qquad (4.10)$$

This of course corresponds to keeping C fixed.

We have now reduced the problem to the case

$$h = h^{TT}, \qquad p = p^{TT}. \qquad (4.11)$$

These components, i. e. the dynamical ones, are restricted by the Hamiltonian constraint in the second order. It is easy to see what one will get. Second order quantities will give a term like the one in (4.9*b*), i. e. proportional to $\varepsilon^2 g^{(2)T}$, or to ζ_2, and there will be quadratic terms in h^{TT} and p^{TT}, since we suppose the reduction to (4.11). One obtains the form

$$\varepsilon^2 v^{(2)} = \gamma^{(0)\frac{1}{2}} A\zeta_2 + (\gamma^{(0)})^{-\frac{1}{2}} p^{TTjk} p^{TT}_{jk} + Q(h^{TT}), \qquad (4.12)$$

where the (quadratic) form Q contains derivations. One can get Q explicitly by following eqs. (3.24-27) and by modifying the covariant derivatives and the contravariant metric tensor.

The term in (4.12) proportional to ζ_2 is essentially a time derivative, i. e. eqs. (3.21), (4.5), and (4.9*a*) determine a linear operator B such that

$$\gamma^{(0)\frac{1}{2}} A\zeta_2 = B(\delta/i\delta\bar{\tau})\zeta_2 \,. \qquad (4.13)$$

For the ground state we set $\delta/\delta\bar{\tau} = 0$, and the resulting equation (modified by renormalization terms, cf. [*8*]) has a solution of the following form,

$$\Psi^{(0)}\{ h^{TT} \} = (\text{const.}) \exp\left[- \int d^3\rho_1 d^3\rho_2 K^{ijkl}(\rho_1, \rho_2) h^{TT}_{ij}(\rho_1) h^{TT}_{kl}(\rho_2) \right]. \qquad (4.14)$$

We do not determine the kernel K.

In the case of deviations from an F-surface, we can be slightly more explicit. Various formulas of [*8*] carry over directly. However, there is an essential difference, in that one no longer has $\pi^{(0)} = 0$, but rather,

$$\pi^{(0)jk} = 2Ce^{\iota C}\delta^{jk}. \qquad (4.15)$$

The constraints $\chi^j\Psi = 0$ are not affected, so that $p^L\Psi = 0$, and we may again suppose that $h^L = 0$. The Hamiltonian constraint in the first order then becomes a functional differential equation for h_T, or equivalently for p_T [we presuppose here one lower and one upper index in (3.3*a*); $\Delta = e^{-2tC}\nabla^2$]:

$$\varepsilon v^{(1)}\Psi = [2Cp_T + e^{3tC}(\Delta - 6C^2)h_T]\Psi = 0. \tag{4.16}$$

One can use either the metric or the extrinsic time representation. In each case the solution is a Gaussian. If we e. g. take (3.22) into account in the following way,

$$p_T = \mathrm{tr}\,(\pi - \pi^{(0)})^T = e^{tC}2\nabla^2\bar{t}, \tag{4.17}$$

then

$$\Psi\{\bar{t}\,;t\} = (\text{const.})\exp 4iCe^{-tC}\int d^3x[(e^{-2tC}\nabla^2 - 6C^2)^{-\frac{1}{2}}\nabla^2\bar{t}]^2. \tag{4.18}$$

The following problem now arises. One can describe the time evolution of Ψ in terms of the variable function p_T, or $\bar{t}$, and also in terms of t. One would expect that, if t' is a small constant, then

$$\Psi\{\bar{t}\,;t\} = \Psi\{\bar{t} - t'\,;t + t'\}. \tag{4.19}$$

This however seems to be inconsistent with (4.18), if $t' \neq 0$. The difficulty is clearly related to the question of domain of ∇^2. A more careful treatment of the boundary contributions, if $\bar{t}$ does not approach zero at infinity, should clarify the situation.

Let us still consider the second order Hamiltonian. We will set, as before,

$$p_T = h_T = 0, \qquad p^L = h^L = 0. \tag{4.20a, b}$$

Note that (4.20*a*) is consistent with (4.16). One then finds

$$\begin{aligned}\varepsilon^2 v^{(2)}\Psi = \Sigma_{i,j,k}[e^{tC}p^{TTik}p^{TTik} + 6Ch_{ik}^{TT}p^{TTik} + 4C^2e^{-tC}h_{ik}^{TT}h_{ik}^{TT} \\ + (e^{-3tC}/4)h_{ij,k}^{TT}h_{ij,k}^{TT}]\Psi + [2C\pi_T^{(2)} + e^{3tC}(\Delta - 6C^2)g_T^{(2)}]\Psi = 0.\end{aligned} \tag{4.21}$$

One solution of this equation is the following,

$$\Psi\{h^{TT}, \pi_T^{(2)}\,;t\} = \Psi_a\{h^{TT}\,;t\}\Psi_b\{\pi_T^{(2)}\,;t\}, \tag{4.22a}$$

where

$$\Sigma[e^{tC}\ldots]\Psi_a = 0, \qquad [2C\pi_T^{(2)}\ldots]\Psi_b = 0. \tag{4.22b}$$

Here Ψ_b can be obtained by reinterpreting (4.17-18), i. e. $\bar{t}$ should be a second-order quantity, determined by $\pi_T^{(2)}$. As to the equation for Ψ_a, it can be solved by a Gaussian functional of the form

$$\Psi_a\{h^{TT}\,;t\} = (\text{const.})\exp\left[-\int d^3x_1d^3x_2L^{ijkl}(x_1, x_2, t)h_{ij}^{TT}(x_1)h_{kl}^{TT}(x_2)\right]. \tag{4.23}$$

This functional has a kind of « minimal » t-dependence, coming only from the explicit t-dependence of the metric. One may think of Ψ_a as the ground-state functional on an F-surface.

We close this section by suggesting two problems. First, we pointed out that the linearized deviations from the Minkowski space yield a usual free-field theory. We expect therefore that the deviations here described should be in agreement with a spin-two field in a de Sitter universe. Such fields are described in [*24*] [*25*], and a comparison of these works with ours might be of interest.

The second problem is the determination of stability of C-surfaces, if the cosmological constant is assumed in the theory. The stability for the case $\Upsilon = 0$ has been the object of several recent investigations [*26*].

5. REMARKS ON THE INNER PRODUCT

Let us consider more general solutions to the two Hamiltonian constraints. We should like to try to introduce inner products for the respective state vectors, as one does in case of the (Schrödinger) wave mechanics. Such inner products can be given in terms of Gaussian functional integrals, defined with the help of the ground state functionals.

In a set-up like the present one, it is natural to require invariance of the inner product under slight deformations of the surface in question. This type of invariance was discussed (apparently, for the first time) in [*7*]. The arguments of this reference are adapted from the Schrödinger theory, and they carry over directly at least to the case of C-surfaces. We will include a few details, in view of the novelty of the subject.

We now consider a C-surface as background, we suppose $h^{\mathrm{L}} = 0$, etc. as before, and we keep the second-order term $\varepsilon^2 g^{(2)\mathrm{T}}$ in the Hamiltonian constraint. Now, in the extrinsic time representation, $\nu\Psi = 0$ becomes

$$\left[\mathrm{B}\frac{\delta}{i\delta\bar{\tau}} - \frac{1}{\gamma^{(0)\frac{1}{2}}}\frac{\delta}{\delta h^{\mathrm{TT}jk}}\frac{\delta}{\delta h^{\mathrm{TT}}_{jk}} + \mathrm{Q}(h^{\mathrm{TT}}) \right]\Psi = 0. \tag{5.1}$$

We will label the two components of h^{TT} at each point by h^{TT}_j, $j = 1, 2$, and to keep things simple, we will ignore the questions of continuity of each h^{TT}_j. We then express the inner product as follows,

$$\langle \Psi_1, \Psi_2 \rangle = \int \mathscr{D}(h^{\mathrm{TT}}_1)\mathscr{D}(h^{\mathrm{TT}}_2)\Psi_1^* \{ h^{\mathrm{TT}} ; \bar{\tau} \} \Psi_2 \{ h^{\mathrm{TT}} ; \bar{\tau} \} . \tag{5.2}$$

We can define the integral by first choosing const. $= 1$ in eq. (4.14) for the ground state functional $\Psi^{(0)}$, and then by taking $\Psi^{(0)2}$ as the weight factor for the integral, with $\Pi_j \mathscr{D}(h^{\mathrm{TT}}_j)$ as a generalized measure, invariant under translations [*27*]. (This will yield the vacuum sector of the theory.)

In order to examine the deformations $\bar{\tau}(\rho)$, we start with an equation of continuity implied by (5.1),

$$\mathrm{B}\frac{\delta}{i\delta\bar{\tau}(\rho)}(\Psi^*\Psi) = -\sum_j \frac{\delta}{\delta h_j^{\mathrm{TT}}(\rho)}\left(\Psi\frac{\delta}{\delta h_j^{\mathrm{TT}}}\Psi^* - \Psi^*\frac{\delta}{\delta h_j^{\mathrm{TT}}}\Psi\right). \quad (5.3)$$

Note, first, that we have here a kind of point-wise conservation property, like the point-wise conservation of the densities of energy and of momentum. Second, eq. (5.3) remains valid if we make the replacement

$$\Psi^* \to \Psi_1^*, \qquad \Psi \to \Psi_2. \quad (5.4)$$

It follows that the derivative with respect to $\bar{\tau}$ in (5.2),

$$[\delta/\delta\bar{\tau}(\rho)]\int \mathscr{D}(h_1^{\mathrm{TT}}) \ldots \quad (5.5)$$

is equivalent to a sum of derivatives with respect to the h_j^{TT}, and the latter vanish in view of the invariance of the $\mathscr{D}(h_j^{\mathrm{TT}})$ under translation.

(It appears that these heuristic manipulations could easily be made rigorous, but we omit further details.)

The same ideas can also be applied to the F-surfaces. One can take Ψ_a^2 as the weight factor for the functional integral, since Ψ_b is a phase factor, irrelevant here. Furthermore, the conservation law is now determined by (4.21), and becomes

$$\begin{aligned}&\mathrm{B}'\frac{\delta}{i\delta\bar{t}(x)}(\Psi^*\Psi)\\ &\quad = -\sum_j \frac{\delta}{\delta h_j^{\mathrm{TT}}(x)}\left[\Psi\left(\frac{\delta}{\delta h_j^{\mathrm{TT}}} + i6Ce^{-\iota C}h_j^{\mathrm{TT}}\right)\Psi^* - \text{compl. conj.}\right], \quad (5.6)\end{aligned}$$

where B′ is a linear operator. (Cf. the expressions in [7].) We admitted here a formal manipulation of the infinite quantities coming from $(\delta/\delta h)h$. Equation (5.6) now leads by an analogous argument to

$$\frac{\delta}{\delta\bar{t}(x)}\int \mathscr{D}(h_1^{\mathrm{TT}})\mathscr{D}(h_2^{\mathrm{TT}})\Psi_1^*\Psi_2 = 0. \quad (5.7)$$

APPENDIX A

SOME CONNECTIONS TO THE GEOMETRIC APPROACH TO EINSTEIN'S EQUATIONS

There appeared recently some thorough studies of the (classical) Einstein equations, from the point of view of the space of metrics and diffeomorphism groups [22] [23]. We would like to indicate a few simple connections of the present investigation to this geometric or global-analysis approach, in view of the current interest in the latter.

Let us first consider changes of topology. The transformations (2.8) leading from the C-surfaces to the F-surfaces necessarily have singularities. For contrast, in the cited works the functions are usually assumed to be $\mathscr{C}^\infty$, and the problem of changes in topology does not arise. A corresponding extension of these works should be of interest.

We next recall that the cosmological constant Υ in the Einstein equations required a modification of the Hamiltonian. Thus, following [22], we select a smooth function of position N, and write the new total Hamiltonian as

$$\mathrm{H} = \mathrm{T} + \mathrm{V}, \qquad \mathrm{V}(g) = \int_{\mathrm{M}} d^3x\gamma^{\frac{1}{2}}\mathrm{N}[\mathrm{R}(g) + 2\Upsilon]. \tag{A.1}$$

Here M is a compact manifold, to which we associate the space $\mathscr{M}$ of metrics g. The quantity $\mathrm{R}(g)$, and Ric (g) below, are the scalar curvature and the Ricci tensor associated with g, respectively. Finally, T is the kinetic energy associated with the DeWitt metric on $\mathscr{M}$, and its form is implied by the expression (2.17*a*) for v (whose integral gives H).

We find that with reference to the DeWitt metric,

$$-\operatorname{grad} \mathrm{V} = -\mathrm{N}\operatorname{Ric}(g) + \frac{1}{4}\mathrm{N}[\mathrm{R}(g) + 2\Upsilon]g + \operatorname{Hess}\mathrm{N}, \tag{A.2}$$

where Hess indicates the double covariant derivative, $\nabla\nabla$. The calculations are just as in [22], p. 552. Most, perhaps all, of the discussion of this reference can now be adapted to the new situation. In particular, Corollary 10.1, which we used with $\Upsilon \neq 0$ in Sec. 3, remains valid.

Finally, we should like to give an interpretation of linearized quantum deviations. The remarks that follow apply to linearized deviations from a classical path quite generally, and do not depend on the details about $\mathscr{M}$.

In general, a curve $(g(t), \dot{g}(t))$ can be identified with a curve in the tangent bundle $\mathrm{T}\mathscr{M}$. Suppose that this curve is determined by Einstein's equation (with $\Upsilon = 0$ or not, but without lapse nor shift, i. e. $\mathrm{N} = 1$, $\mathrm{N}_i = 0$). Then along this curve, $v(g, \dot{g}) = 0$.

Now, the linearized Hamiltonian operator, to first order in the deviations $(g', \dot{g}')$ from $(g(t), \dot{g}(t))$, can be identified with an element of the bundle $\mathrm{T}^*\mathrm{T}\mathscr{M}$:

$$v' = (\partial_1 v)g' + (\partial_2 v)\dot{g}'. \tag{A.3}$$

We write $g' = h$, and for $\dot{g}'$ we take the functional derivative $i^{-1}\delta/\delta h$. Then the basic equation of linearized quantum theory is

$$v'\Psi\{h\} = 0, \tag{A.4}$$

where a fixed reference point $(g(t), \dot{g}(t))$ is assumed. The functional Ψ is defined on $\mathrm{T}^*\mathscr{M}$.

If t varies, one finds a flow on a subset of $\mathscr{M}$, which induces transformations on subsets of $\mathrm{T}^*\mathscr{M}$ and of $\mathrm{T}^*\mathrm{T}\mathscr{M}$. One can see directly that the relation (A.4) is preserved under the flow. Indeed, for some t_0, the solution to (A.4) will be a Gaussian, cf. (4.17), with the coefficients in the exponent given by the components of $g(t_0)$ and of $\dot{g}(t_0)$. As t varies, both v' and Ψ will retain the same functional form, with $(g(t), \dot{g}(t))$ replacing the values at t_0. Thus (A.4) will remain valid.

(We have ignored here special cases, like those treated in the text, where the deviations had to be examined to the second order.)

APPENDIX B

GROUP-THEORETIC ASPECTS OF THE DE SITTER UNIVERSE

We recall the following expression for the curvature tensor of the universe S [*11*],

$$R_{\alpha\beta\gamma\delta} = C^2(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}). \tag{B.1}$$

We see that this tensor is a covariant constant. Such a situation is characteristic of symmetric spaces.

The available extensive theory of symmetric spaces presupposes a positive-definite metric [*28*]. However, as is emphasized in [*29*], the equations of Riemannian geometry do not depend on the signature of the metric. The global questions like that of completeness, on the other hand, can be answered directly for a specific space like S. We expect therefore that various conclusions of [*28*] can be adapted to the present circumstance.

We now take $C = 1$ in order to avoid irrelevant distractions, and following *loc. cit.*, Chapter IX, we express S as a quotient space,

$$S = SO(1,4)/SO(1,3). \tag{B.2}$$

Then the tangent space at a point of S can be identified with that subspace of the Lie algebra of S(1,4), which is spanned by the generators

$$I_{04}, \qquad I_{14}, \qquad I_{24}, \qquad I_{34}. \tag{B.3}$$

The first generator yields time-like geodesics, while the last three, space-like. These three generators also constitute a Lie triple system (*loc. cit.*, p. 189), and thus generate totally geodesic submanifolds of S. These submanifolds are three-spheres, and the C-surfaces can be obtained in this way. The F-surfaces, on the other hand, are not totally geodesic submanifolds.

Let us now imbed a C-surface in $\mathbf{R}^4$, with center at the origin. We use the spherical coordinates (q, ρ) on $\mathbf{R}^4$, where q is the distance from the origin, and ρ, see (3.17*b*), determines an angle. The Laplacian Δ on the C-surface, when acting on the scalar functions, is related to the operator ∇^2 on $\mathbf{R}^4$ by

$$\nabla^2 = (d^2/dq^2) + 3q^{-1}(d/dq) + q^{-2}\Delta. \tag{B.4}$$

This relation can be used to determine the eigenvalues of Δ.

The spherical functions on S^3 are the elements of the representation matrices of SO(4) [*30*]. We denote these functions by

$$\psi_{nlm}(\rho), \qquad |m| \leqslant l \leqslant n \tag{B.5}$$

(m, l, n integers), and the ψ satisfy

$$\nabla^2(q^n\psi_{nlm}) = 0. \tag{B.6}$$

Equations (B.4) and (B.6) now yield

$$\Delta\psi_{nlm} = -n(n+2)\psi_{nlm}. \tag{B.7}$$

In particular, since $R^{(0)} = -6$ and there are $d = 3$ dimensions, we see that

$$[\Delta - R^{(0)}/(d-1)]\psi_{1lm} = 0. \tag{B.8}$$

This last relation is invariant under a change of scale, and extends also to other dimensions. [Cf. eq. (4.9*c*) and the subsequent discussion.]

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