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E. SCHNIRMAN
C. G. OLIVEIRA

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Section A :
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Conformal invariance of the equations of motion in curved spaces

by

E. SCHNIRMAN and C. G. OLIVEIRA

Centro Brasileiro de Pesquisas Fisicas

ABSTRACT. — The conformal invariance of the classical and quantum equations of motion in curved spaces is studied. It is shown that this invariance may be obtained without considering any mass variation. This is possible due to a suitable modification of the metric of the Riemannian space-time. The arc element of the new geometry depends on the values assumed by a vector field along a trajectory leading to the point under consideration. This new formulation may be called a semi-metrical formulation, in the sense that not only the metric $g_{\mu\nu}(x)$ is a fundamental object, but also the vector field.

INTRODUCTION

In the study of the conformal invariance of the fundamental equations of physics [1] it is assumed, in general, that the rest mass of the particles transform in a specified way in order to assure the conformal invariance of those equations. As we will see, any realization of the rest mass transforming in that way must be dependent on the coordinates. In the particular case of the fifteen parameter conformal group of the flat space, the rest mass has to depend on the scale and acceleration parameters, as well as on the coordinates. Since these quantities may assume any continous range of values, we get a representation of the mass depending on a continuous set of variables. From the point of view of the theory of elementary particles this fact has no physical interpretation. In this connection, it is usually said that conformal symmetry holds only at high energies, the rest energy of the particles being neglectable compared to their kinetic energy [2]. In the classical theory we may, in principle, allow a mass variation of this sort, if for instance we think at the conformal correction to m as a scalar interaction acting on the system, that is in terms of the action integral :

$$S = \int m \, ds + \int \alpha\varphi \, ds = \int (m + \alpha\varphi) \, ds,$$

where $m_{\text{eff}} = (m + \alpha\varphi)$ is the effective mass of the system, which could, eventually, assume a continuous range of values. However this interpretation has no analog in the quantum theory.

Presently we propose a redefinition of the space-time metric in a such a way that the rest mass becomes conformal invariant. This is done for the case of curved manifolds. With the new metric, that means, with the new geometry, the arc element, calculated at any regular point for $g_{\mu\nu}(x)$ (the metric of a Riemannian space), will depend on the line integral of a vector field along a trajectory which leads to this point without crossing any world line of singularity for $g_{\mu\nu}(x)$.

It is important to note that the introduction of this vector field does not absorb the dilatation transformations in flat space-time, so that a conformal invariant scalar mass is only achieved in curved spaces satisfying a particular condition [(1.10), in the text].

The notation used is the following : greek letters denote values going from 0 to 3, latin letters indicate values from 1 to 3. Usual partial derivatives are denoted by a comma, or by $\frac{\partial}{\partial x^\mu}$ if this operates on any quantity independent on the path of integration referred above. The symbol ∂_μ will be used for denoting the derivatives of the path dependent objects. A semi-colon denotes the covariant differentiation for the metric $g_{\mu\nu}(x)$ or any path-independent object and the symbol D_μ indicates the covariant differentiation for the new metric. The metric tensor $g_{\mu\nu}(x)$ is taken with a local signature equal to -2.

1. THE CONFORMAL GROUP : CONFORMAL TRANSFORMATIONS ON THE METRIC

We will think at the conformal group as the group of general transformations which consists of the manifold mapping group (M. M. G.), the group of the general coordinate transformations such that

$$x'^\mu = f^\mu(x), \tag{1.1}$$

and the transformation group of the metric tensor, such that, under the action of this group denoted in the following by C_g [3] :

$$g'_{\mu\nu}(x) = e^{2\sigma(x)} g_{\mu\nu}(x), \tag{1.2}$$

$\sigma(x)$ an arbitrary function at least of the class C^2.

Due to this variation of the metric, the geodesic of a Riemannian space is not conformal invariant, since the Christoffel symbols transform, under C_g, as

$$\left\{ {\mu \atop \alpha\beta} \right\}' = \left\{ {\mu \atop \alpha\beta} \right\} + \delta^{\mu}_{(\alpha} \sigma_{,\beta)} - g_{\alpha\beta}\, g^{\mu\rho}\, \sigma_{,\rho}. \tag{1.3}$$

Weyl has solved this difficulty, in his theory of gravitation and electromagnetism by introducing a conformal invariant semi-metrical symmetric affinity by means of the condition :

$$g_{\mu\nu;\alpha} = A_\alpha\, g_{\mu\nu}(x). \tag{1.4}$$

This condition does not determine uniquely $g_{\mu\nu}(x)$ and $A_\alpha(x)$, due to the fact that, given a pair $(g_{\mu\nu}, A_\alpha)$, which satisfies (1.4), there is an infinity of other possible pairs $[g'_{\mu\nu}(x), A'_\alpha(x)]$ satisfying the same condition [4], provided that :

$$\left\{ \begin{aligned} g'_{\mu\nu}(x) &= e^{2\sigma(x)}\, g_{\mu\nu}(x), \\ A'_\alpha(x) &= A_\alpha(x) + \sigma_{,\alpha}(x) \end{aligned} \right. \tag{1.5}$$

[for $\sigma(x)$ a scalar function]. However, as it may be easily proved, the semi-metrical symmetric affinity of Weyl's formulation is form invariant under the passage from one pair of solutions of the fundamental condition to another. The first half of the above transformation represents just the conformal transformation of the metric and thus we obtain a conformal invariant affinity by considering as fundamental objects the metric $g_{\mu\nu}$ and a vector field A_α. In Weyl's theory A_α is interpreted as the electromagnetic potentials. Other variations of this theory have been proposed in such a form that $A_\alpha(x)$ is interpreted as an object proportional to the metric [5] $g_{\mu\nu}$. However, the determination of conformal invariant equations of motion, in these « purely metrical formulations », seems to us very complicated and, moreover, the conformal invariance is not obtained without considering a conformally covariant variation on the rest mass of the particles.

Since apparently there is no physical interpretation for such variation in the mass, it would be interesting to look for other possible form of obtaining equations of motion which are invariant under the conformal group such that the rest mass is also conformal invariant.

For obtaining this type of formalism, we modify the form of the metric, instead of modifying the form of the affinity. For this modification in $g_{\mu\nu}$ we follow an idea of Mandelstam [6] who showed how to define gauge-invariant quantities with a matter field interacting with the electromagnetic field. The disadvantages of the current formulations of quantum electrodynamics, which use the potentials as the fundamental variables and, therefore, depend on particular gauges (as, for instance, the Coulomb or the Lorentz gauge), are not present in the formulation in terms of operators which keep unchanged under gauge transformations. In this way, the introduction of non-physical states normalized with respect to a non definite metric (Lorentz gauge) or the use of lagrangeans which are not manifestly Lorentz-invariant (Coulomb gauge) is not necessary.

For the electromagnetic field interacting with a scalar field, the gauge-invariant variables of the matter field are defined as

$$(1.6)\qquad \begin{cases} \Phi\ (x,\ \mathrm{P}) = \varphi\ (x) \exp \left\{ - ie \int_{-\infty}^{x} \mathrm{A}_\mu (\xi)\, d\xi^\mu \right\}, \\ \Phi^* (x,\ \mathrm{P}) = \varphi^* (x) \exp \left\{ \quad ie \int_{-\infty}^{x} \mathrm{A}_\mu (\xi)\, d\xi^\mu \right\}. \end{cases}$$

The equations of motion are constructed with the total gauge-invariant Lorentz covariant Lagrangian :

$$(1.7)\qquad \mathcal{L} = - \{ \partial_\mu \Phi^* (x) \} \{ \partial_\mu \Phi (x) \} - m^2 \Phi^* (x) \Phi (x) - \frac{1}{4} \{ \mathrm{F}_{\mu\nu} (x) \}^2,$$

where

$$\mathrm{F}_{\mu\nu} = \frac{\partial \mathrm{A}_\nu}{\partial x^\mu} - \frac{\partial \mathrm{A}_\mu}{\partial x^\nu},$$

$$\partial_\mu = \frac{\partial}{\partial x^\mu} - ie\, \mathrm{A}_\mu.$$

The commutation rules among the gauge invariant operators are determined in the usual way, leading to the quantization of the electromagnetic field which depends only on gauge-independent quantities.

In this paper, we consider the conformal changing on the metric as a gauge transformation on $g_{\mu\nu} (x)$, and then look for a redefinition of $g_{\mu\nu}$ in such a form that it becomes gauge-invariant.

With this aim, we define (in a cartesian system of coordinates, for convenience) :

$$(1.8)\qquad \mathrm{G}_{\mu\nu} (x,\ \mathrm{P}) = g_{\mu\nu} (x) \exp \left\{ - 2 \int_{x_{in}}^{x} k_\alpha (\xi)\, d\xi^\alpha \right\},$$

where the integration is taken over a trajectory P starting at x_{in}, without crossing any world-line of singularity for $g_{\mu\nu}$, and in such a form that $x^0_{in} < \xi^0 < x^0$ (since this is reasonable from the physical point of view-propagation effects of the vector field in the manifold obey the causality law); the k_α in (1.8) is defined as a four vector under the general coordinate transformations, and changes under C_g as

$$(1.9)\qquad k'_\alpha (x) = k_\alpha (x) + \sigma_{,\alpha} (x).$$

It is interesting to note that these two simultaneous requirements do not allow a direct representation for k_α in terms of the metric $g_{\mu\nu}$

and its derivatives $g_{\mu\nu,\sigma}$. Indeed, we may form the quantities :

$$k_\mu = \frac{1}{6} g^{\alpha\beta} (g_{\alpha\beta,\mu} - g_{\beta\mu,\alpha})$$

which obey the transformation law (1.9) but they do not transform as a fourvector under curvilinear coordinate transformations. This implies that we must keep k_α as a set of independent variables, of the same fundamental order as $g_{\mu\nu}$. A conformal invariant formulation, as we intend presently, will need both types of quantities. $G_{\mu\nu}(x, P)$, defined by (1.8) will be conformal-invariant if we impose on the transformation function $\sigma(x)$ the condition :

$$\sigma(x_{in}) = 0. \tag{1.10}$$

It is important to note that our formalism does not absorb the scale transformations (or dilatations) in flat-space time (remember that the special conformal group C_0 contains the dilatations) for, in this case, we have

$$\mathfrak{N}_{\mu\nu}(x, P) = \eta_{\mu\nu}\, e^{-2\int_{x_{in}}^{x} k_\rho(\xi)\, d\xi^\rho} \tag{1.10 a}$$

under C_g,

$$\mathfrak{N}_{\mu\nu}(x, P) = \eta_{\mu\nu}\, e^{2\sigma(x_{in})}.$$

But in flat space-time, under an infinitesimal transformation

$$x'^\mu = x^\mu + \xi^\mu(x),$$

where ξ^μ are the generators of the infinitesimal conformal special group,

$$\xi^\mu(x) = \alpha^\mu + \varepsilon^\mu_\nu x^\nu + \beta x^\mu + a_\alpha (\eta^{\alpha\mu} x^2 - 2 x^\alpha x^\mu),$$

with

α_μ : translation parameters;

ε^μ_ν : parameters of the Lorentz group;

β : dilatation parameter;

a_α : special conformal parameter;

we obtain,

$$\bar{\delta}\eta_{\mu\nu} = -\xi_{(\mu|\nu)} = 2\sigma\eta_{\mu\nu},$$
$$\sigma(x) = -\beta + 2 a_\rho x^\rho.$$

At any fixed point, such as x_{in}, which may, eventually, be taken as $x_{in} = 0$,

$$\sigma(x_{in}) = \beta.$$

Then (1.10 *a*) is conformal invariant only for special transformations not containing dilatations. We conclude that the introduction of a vector field $k_\mu(x)$ does not absorb scale transformations. This is in agreement with the general assumption that conformal C_0-invariance is only verified at high energies (the operator of Hilbert space associated to the generator of scale transformations has a continuous spectrum of eigenfunctions), implying continuous rest-mass values (which are neglectable at high energies) [2]. As we will see in the following, a conformal-invariant scalar mass is only realizable in curved spaces satisfying condition (1.10).

Although having used a cartesian system of coordinates for the definition of $G_{\mu\nu}(x, P)$ and of the limiting conditions, we note that this may be done in any coordinate system, since we have imposed that k_α is a world fourvector. Denoting by $\varphi(x, P)$ the value of the integral present in (1.8), we have

$$G_{\mu\nu}(x, P) = g_{\mu\nu}(x) f(x, P),$$
$$f(x, P) = \exp\{-2\varphi(x, P)\}.$$

Under a coordinate transformation to a curvilinear system of coordinates, we get :

$$(1.11)\quad G'_{\mu\nu}(x', P') = g'_{\mu\nu}(x') f'(x', P') = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x) f(x, P) = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} G_{\alpha\beta}(x, P).$$

The integration in (1.8) will depend on the choice of the trajectory, in each fixed coordinate system. Consider two paths P_1 and P_2 which coincide at all points, except at some point $\bar{x}$ where they differ by a small loop. Calling the area of this infinitesimal closed loop by $\sigma^{\alpha\beta}$, we get from Stoke's theorem :

$$(1.12)\quad G_{\mu\nu}(x, P_2) = G_{\mu\nu}(x, P_1) - 2\, G_{\mu\nu}(x, P_1)\, k_{[\alpha,\beta]}(\bar{x})\, \sigma^{\alpha\beta},$$

where

$$(1.13)\quad k_{[\alpha,\beta]} = k_{\alpha,\beta} - k_{\beta,\alpha}.$$

Thus, there will be independence on the choice of the trajectory, if k_α is a gradient. However, we will not impose any particular form for k_α.

2. THE CONFORMAL INVARIANT DERIVATIVES

The derivatives of $G_{\mu\nu}$ are given by

$$\partial_\mu G_{\alpha\beta}(x, P) = \lim_{\Delta x_\mu \to 0} \frac{G_{\alpha\beta}(x^\mu + \Delta x^\mu, \bar{x}, P + dP) - G_{\alpha\beta}(x, P)}{\Delta x^\mu},$$

where by $\bar{x}$ we indicate the remaining set of variables which do not vary, the variation in the trajectory P being generated by the variation Δx^μ, in the coordinate x^μ. It may be shown that this variation in the trajectory tends to zero in the limit $\Delta x^\mu \to 0$, so that we can define the derivatives of $G_{\alpha\beta}$ by the usual formula

$$\partial_\mu G_{\alpha\beta}(x, P) = \lim_{\Delta x^\mu \to 0} \frac{G_{\alpha\beta}(x^\mu + \Delta x^\mu, \bar{x}, P) - G_{\alpha\beta}(x, P)}{\Delta x^\mu}$$

which gives

$$\partial_\mu G_{\alpha\beta}(x, P) = e^{-2\int_{x_{in}}^{x} k_\alpha(\xi)\, d\xi^\alpha} \left(\frac{\partial}{\partial x^\mu} - 2\, k_\mu(x)\right) g_{\alpha\beta}(x) \tag{2.1}$$

Under a conformal transformation :

$$(\partial_\mu G_{\alpha\beta}(x, D))' = \partial_\mu G_{\alpha\beta}(x, P). \tag{2.2}$$

This shows explicitly that the first kind Christoffel symbols constructed with the $G_{\alpha\beta}(x, P)$ are conformal invariant :

$$\{\rho, \sigma\nu\}(x, P) = \frac{1}{2}\{\partial_\sigma G_{\rho\nu}(x, P) + \partial_\nu G_{\rho\sigma}(x, P) - \partial_\rho G_{\sigma\nu}(x, P)\}. \tag{2.3}$$

The quantities $\{\rho, \sigma\nu\}$ depend on the choice of the trajectory P. However the second kind Christoffel symbols :

$$\Gamma^\mu_{\nu\sigma} = G^{\mu\rho}(x, P)\{\rho, \nu\sigma\}(x, P) \tag{2.4}$$

are also conformal-invariant and independ on the choice of the trajectory P, since they can be written entirely in terms of path independent quantities

$$\Gamma^\mu_{\nu\sigma}(x) = \frac{1}{2} g^{\mu\rho}(x)\left[\left(\frac{\partial}{\partial x^\sigma} - 2\, k_\sigma\right) g_{\rho\nu}(x) + \left(\frac{\partial}{\partial x^\nu} - 2\, k_\nu\right) g_{\rho\sigma}(x) + \left(\frac{\partial}{\partial x^\rho} - 2\, k_\sigma\right) g_{\sigma\nu}(x)\right]. \tag{2.5}$$

For the definition of $\Gamma^\mu_{\nu\sigma}$ we have used the convention that the tensor index of any path dependent quantity, e. g., $\varphi(x, P)$ is raised (or lowered) by $G^{\lambda\rho}$ ($G_{\lambda\rho}$) associated to the same trajectory :

$$\varphi^\lambda(x, P) = G^{\lambda\rho}(x, P)\, \varphi_\rho(x, P).$$

In particular,

$$G^{\lambda\rho}(x, P)\, G_{\rho\sigma}(x, P) = g^{\lambda\rho}(x)\, g_{\rho\sigma}(x) = \delta^\lambda{}_\sigma$$

but

$$G^{\lambda\rho}(x, P')\, G_{\rho\sigma}(x, P) = \delta^\lambda{}_\sigma (1 + 2\, k_{[\tau, \alpha]}(\bar{x})\, \sigma^{\tau\alpha})$$

if

$$P' = P + \delta P.$$

In general, all path independent conformal invariant quantities are equivalent to the corresponding objects in Weyl's theory. For instance, the affinity $\Gamma^{\mu}_{\nu\sigma}$ is the same as the Weyl affinity. The only difference is that now $\Gamma^{\mu}_{\nu\sigma}$ is the second kind Christoffel symbol for $G_{\mu\nu}$ and its reciprocal. This interpretation is not obtained in Weyl's geometry. By the other hand, objects like $G_{\mu\nu}(x, P)$ and $\{\alpha, \beta\gamma\}(x, P)$ which are conformal invariant, but depend on the trajectory P, do not exist in Weyl's formulation. As it will be shown in the next section, is just the existence of such objects that will allow a formulation of the equations of motion in a conformal invariant form, without requiring any transformation on the rest mass of the particles.

The order in which conformal invariant derivatives appear cannot be interchanged. A straightforward calculation shows that :

$$(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) G_{\alpha\beta}(x, P) = 2(k_{\mu,\nu} - k_{\nu,\mu}) G_{\alpha\beta}(x, P). \tag{2.6}$$

The tensor standing on the right hand side of (2.6) is a fourth rank conformal invariant tensor.

We saw that $\partial_\rho G_{\alpha\beta}(x, P)$ is conformal invariant, nevertheless it is not a third rank tensor with respect to curvilinear coordinate transformations. For obtaining a conformal invariant third rank tensor out of $\partial_\rho G_{\alpha\beta}$ we introduce the conformal invariant covariant derivative $D_\rho G_{\alpha\beta}$ by

$$\begin{aligned} D_\rho G_{\alpha\beta}(x, P) &= \partial_\rho G_{\alpha\beta}(x, P) \\ &\quad - \Delta^{\lambda}_{\alpha\rho}(x, P) G_{\lambda\beta}(x, P) - \Delta^{\lambda}_{\beta\rho} G_{\alpha\lambda}(x, P), \end{aligned} \tag{2.7}$$

where $\Delta^{\lambda}_{\alpha\rho}$ is the corresponding affinity, which, by definition, has to be conformal invariant. We impose the condition

$$D_\rho G_{\alpha\beta}(x, P) = 0. \tag{2.8}$$

For another trajectory $P' = P + \delta P$:

$$\begin{aligned} & D_\rho G_{\alpha\beta}(x, P + \delta P) \\ &\quad = \partial_\rho G_{\alpha\beta}(x, P + \delta P) - \Delta^{\lambda}_{\alpha\rho}(x, P + \delta P) G_{\lambda\beta}(x, P + \delta P) \\ &\qquad - \Delta^{\lambda}_{\beta\rho}(x, P + \delta P) G_{\alpha\lambda}(x, P + \delta P), \end{aligned}$$

$$\begin{aligned} & D_\rho G_{\alpha\beta}(x, P + \delta P) \\ &\quad = \partial_\rho G_{\alpha\beta}(x, P) - 2 k_{[\tau,\sigma]}(\bar{x}) \sigma^{\tau\sigma} \partial_\rho G_{\alpha\beta}(x, P) \\ &\qquad - \Delta^{\lambda}_{\alpha\rho}(x, P + \delta P)[G_{\lambda\beta}(x, P) - 2 G_{\lambda\beta}(x, P) k_{[\tau,\sigma]}(\bar{x}) \sigma^{\tau\sigma}] \\ &\qquad - \Delta^{\lambda}{}_{\rho}(x, P + \delta P)[G_{\alpha\lambda}(x, P) - 2 G_{\alpha\lambda}(x, P) k_{[\tau,\sigma]}(\bar{x}) \sigma^{\tau\sigma}]. \end{aligned}$$

Since for any choice of the affinity, we have

$$\Delta^{\lambda}_{\rho\beta}(x,\ P+\delta P) = \Delta^{\lambda}_{\rho\beta}(x,\ P) + \delta\Delta^{\lambda}_{\rho\beta}(x,\ \bar{x}),$$

we get, to first order in the variations :

$$\begin{aligned} D_\rho G_{\alpha\beta}(x,\ P+\delta P) = {} & \partial_\rho G_{\alpha\beta}(x,\ P) - 2\,k_{[\tau,\sigma]}(\bar{x})\,\sigma^{\tau\sigma}\,\partial_\rho G_{\alpha\beta}(x,\ P) \\ & - \Delta^{\lambda}{}_{\rho}(x,\ P)\,G_{\lambda\beta}(x,\ P) - 2\,\Delta^{\lambda}_{\alpha\rho}(x,\ P)\,G_{\lambda\beta}(x,\ P)\,k_{[\tau,\sigma]}(\bar{x})\,\sigma^{\tau\sigma} \\ & - \delta\Delta^{\lambda}{}_{\rho}(x,\ \bar{x})\,G_{\lambda\beta}(x,\ P) - \Delta^{\lambda}_{\beta\rho}(x,\ P)\,G_{\alpha\lambda}(x,\ P) \\ & - 2\,\Delta^{\lambda}_{\beta\rho}(x,\ P)\,G_{\alpha\lambda}(x,\ P)\,k_{[\tau,\sigma]}(\bar{x})\,\sigma^{\tau\sigma} \\ & - \delta\Delta^{\lambda}_{\beta\rho}(x,\ \bar{x})\,G_{\alpha\lambda}(x,\ P), \end{aligned}$$

$$\begin{aligned} D_\rho G_{\alpha\beta}(x,\ P+\delta P) = {} & D_\rho G_{\alpha\beta}(x,\ P) - 2\,k_{[\tau,\sigma]}(\bar{x})\,\sigma^{\tau\sigma}\,D_\rho G_{\alpha\beta}(x,\ P) \\ & - \delta\,\Delta^{\lambda}_{\alpha\rho}(x,\ \bar{x})\,G_{\lambda\beta}(x,\ P) - \delta\Delta^{\lambda}_{\beta\rho}(x,\ \bar{x})\,G_{\alpha\lambda}(x,\ P). \end{aligned}$$

Then, the condition (2.8) will hold independently of the trajectory only if

$$\delta\Delta^{\lambda}_{\rho\beta}(x,\ \bar{x}) = 0 \tag{2.9}$$

that is,

$$\Delta^{\lambda}_{\rho\beta} = \Delta^{\lambda}_{\rho\beta}(x). \tag{2.10}$$

The afinity which is obtained from the conditions (2.8) and (2.10) is the second kind Christoffel symbols calculated for $G_{\mu\nu}(x,\ P)$ and its reciprocal :

$$\Delta^{\lambda}_{\rho\beta} = \Gamma^{\lambda}_{\rho\beta}. \tag{2.11}$$

3. THE CONFORMAL INVARIANT CURVATURE TENSOR

The conformal invariant curvature tensor associated to $G_{\mu\nu}(x,\ P)$ is

$$\mathbf{R}^{\mu}{}_{\nu\rho\sigma} = \partial_\sigma \Gamma^{\mu}_{\nu\rho} - \partial_\rho \Gamma^{\mu}_{\nu\sigma} + \Gamma^{\lambda}_{\nu\rho}\Gamma^{\mu}_{\lambda\sigma} - \Gamma^{\lambda}_{\nu\sigma}\Gamma^{\mu}_{\lambda\rho}. \tag{3.1}$$

$\mathbf{R}^{\mu}{}_{\nu\rho\sigma}$ is independent on the choice of the trajectory and as can be easily verified, antisymmetric only on the last pair of indices, contrarily to the Riemannian curvature tensor, which is antisymmetric on both pair of indices.

The Ricci tensor

$$\mathbf{R}_{\mu\nu} \equiv \mathbf{R}^{\rho}{}_{\mu\nu\rho}, \tag{3.2}$$

contains a symmetric and an antisymmetric part, the last being,

$$\mathbf{R}_{\mu\nu} = k_{\mu,\nu} - k_{\nu,\mu} = \varphi_{\mu\nu}. \tag{3.3}$$

Starting from the identity :

$$(D_\mu D_\nu - D_\nu D_\mu) G_{\alpha\beta}(x, P) \equiv 0$$

we arrive at the following relation

$$\mathbf{R}_{\beta\alpha\rho\sigma} + \mathbf{R}_{\alpha\beta\rho\sigma} = -2\,\varphi_{\rho\sigma}\, G_{\alpha\beta}(x, P), \tag{3.4}$$

where

$$\mathbf{R}_{\beta\alpha\rho\sigma} = G_{\lambda\beta}\, \mathbf{R}^{\lambda}{}_{\alpha\rho\sigma}.$$

The affinities being symmetric, the Bianchi identities are verified :

$$D_\sigma \mathbf{R}^{\tau}{}_{\mu\nu\rho} + D_\rho \mathbf{R}^{\tau}{}_{\mu\sigma\nu} + D_\nu \mathbf{R}^{\tau}{}_{\mu\rho\sigma} \equiv 0. \tag{3.5}$$

Then,

$$D_\sigma \mathbf{R}^{\rho\sigma}{}_{\nu\rho} + D_\rho \mathbf{R}^{\rho\sigma}{}_{\sigma\nu} + D_\nu \mathbf{R}^{\rho\sigma}{}_{\rho\sigma} = 0.$$

Recalling that

$$-2\,\varphi_{\mu\nu}\, G^{\alpha\beta} = \mathbf{R}^{\alpha\tau}{}_{\mu\nu} + \mathbf{R}^{\tau\alpha}{}_{\mu\nu}$$

a straightforward calculation leads to

$$D_\sigma\left(\underline{\mathbf{R}^{\sigma}{}_{\nu}} - \frac{1}{2}\delta^{\sigma}{}_{\nu}\,\mathbf{R} + \varphi^{\sigma}{}_{\nu}\right) \equiv 0, \tag{3.6}$$

where

$$\underline{\mathbf{R}^{\sigma}{}_{\nu}} = G^{\rho\sigma}\,\underline{\mathbf{R}_{\rho\nu}}.$$

Writing

$$\mathbf{G}^{\sigma}{}_{\nu} = \mathbf{R}^{\sigma}{}_{\nu} - \frac{1}{2}\delta^{\sigma}{}_{\nu}\,\mathbf{R},$$

we have for equation (3.6) :

$$D^{\sigma}(\mathbf{G}^{\sigma}{}_{\nu} + \varphi^{\sigma}{}_{\nu}) \equiv 0 \tag{3.7}$$

which are the contracted Bianchi identities [7] for this case.

4. THE CLASSICAL AND QUANTUM EQUATIONS OF MOTION

The classical equation of motion for electromagnetic interactions is

$$u^{\sigma} u_{\nu;\sigma} = \frac{e}{m} g^{\sigma\rho}\, F_{\rho\nu}\, u_{\sigma} \tag{4.1}$$

where e and m are respectively the charge and rest mass of the particle. The velocity four-vector is $u^{\sigma} = \frac{dx^{\sigma}}{ds}$, ds being the line element of the Riemannian space-time geometry.

This equation is not form-invariant under C_g, mainly due to the behaviour of the metrical affinity, the Christoffel symbols for $g_{\mu\nu}(x)$. To make (4.1) conform invariant, we may try, as a first approximation, to replace the metrical affinity $\left\{ {\alpha \atop \beta\gamma} \right\}$ by the semi-metrical affinity $\Gamma^{\alpha}_{\beta\gamma}$ given by (2.5). But the conformal covariance of the equation of motion will only be attained if the rest mass varies under C_g. Indeed, denoting by

$$U_\nu = u^\sigma u_{\nu;\sigma},$$

$$W_\nu = \frac{e}{m} g^{\sigma\rho} F_{\rho\nu} u_\sigma,$$

we see that, under C_g :

$$U'_\nu = e^{-2\sigma} u_\nu,$$

$$W'_\nu = e^{-3\sigma} W_\nu$$

(use that the electromagnetic field strength do not vary). Thus, in order to obtain conformal covariance, the rest mass has to transform as

$$m' = e^{-\sigma(x)} m. \tag{4.2}$$

A variation of this type in the rest mass was primarily proposed by Schouten and Haantjes [1]. Since our initial aim was to avoid such a transformation in the mass, we will look for a conformal invariant equation of motion for electromagnetic interactions, starting from conformal invariant field equations, plus the Bianchi identities (3.7). (In analogy with the theory of General Relativity, where, as it is well known, we can get the correct equations of motion for the particle, represented as point singularities, from the field equations.)

In our case, the field equations will be formally derived from a variational principle, where $G_{\mu\nu}$ is the variant quantity which vanishes at the boundary of the domain of integration.

Consider the following conformal invariant Lagrangian density for empty space-time :

$$\mathcal{L} = \sqrt{-G}\,\mathbf{R} + \alpha \sqrt{-G}\,(u_1 + u_2), \tag{4.3}$$

where α is a constant, and u_1, u_2 are the two field invariants constructed out of the $\varphi_{\mu\nu}$,

$$u_1 = G^{\mu\alpha} G^{\nu\beta} \varphi_{\mu\nu} \varphi_{\alpha\beta},$$

$$u_2 = \frac{\varepsilon^{\alpha\beta\rho\sigma}}{\sqrt{-G}} \varphi_{\alpha\beta} \varphi_{\rho\sigma}$$

upon variations on $G_{\mu\nu}$ we get the field equations

$$\mathbf{R}_{\mu\nu} - \frac{1}{2} G_{\mu\nu} \mathbf{R} = \beta\, T^{(\varphi)}_{\mu\nu}. \tag{4.4}$$

In these equations, $\mathbf{R}_{\mu\nu}$ represents the symmetric part of the generalized Ricci tensor, and $T^{(\varphi)}_{\mu\nu}$ is the analog of the Maxwell stress tensor for the field variables $\varphi_{\mu\nu}$:

$$T^{(\varphi)}_{\mu\nu} = \frac{1}{4}\varphi_{\alpha\beta}\,\varphi^{\alpha\beta}\,G_{\mu\nu} - \varphi_{\mu\rho}\,\varphi^{\rho}_{\nu},$$

$$\beta = 2\,\alpha.$$

Since

$$T^{(\varphi)} = G_{\mu\nu}\,T^{\mu\nu(\varphi)} = 0.$$

we obtain, from (4.4), the following condition

$$\mathbf{R} = 0. \tag{4.5}$$

Or, in terms of $g_{\mu\nu}(x)$ and $k_{\mu}(x)$, we have

$$R = 6\,(k_{\alpha}\,k^{\alpha} - k^{\alpha}{}_{;\alpha}), \tag{4.6}$$

where R is the Ricci scalar of the Riemannian geometry.

As (4.4) represent ten equations for the fourteen quantities $g_{\mu\nu}$ and k_{μ}, we impose four subsidiary conditions :

$$D_{\mu}\,\varphi^{\mu\nu} = 0. \tag{4.7}$$

There is no way of obtaining these equations from the Lagrangean (4.3), since the variations are taken on the $G_{\mu\nu}$.

For a space with charged matter, we have the Lagrangean density :

$$\mathcal{L} = \mathcal{L}_{f} + \chi\sqrt{-G}\,G^{\mu\alpha}\,G^{\nu\beta}\,F_{\mu\nu}\,F_{\alpha\beta} + \lambda\sqrt{-G}\,\Lambda, \tag{4.8}$$

where $\mathcal{L}_{f}$ denotes the free Lagrangean (4.3),

$F_{\mu\nu}$: electromagnetic field strength;
Λ : density characterizing the matter;
χ, λ : constants.

The field equations are

$$\mathbf{G}_{\mu\nu} = \beta\,T^{(\varphi)}_{\mu\nu} + \lambda\,T^{(\mu)}_{\mu\nu} + \chi\,T^{(e)}_{\mu\nu}. \tag{4.9}$$

Since :

$$D^{\nu}\,T^{(\mu)}_{\mu\nu} = \mu_{(0)}\,u^{\nu}\,D_{\nu}\,u_{\mu},$$

where $u^{\nu} = \dfrac{dx^{\nu}}{d\sigma}$ is the velocity four-vector for our geometry with line element $d\sigma = (G_{\alpha\beta}\,dx^{\alpha}\,dx^{\beta})^{1/2}$, and using that

$$D^{\nu}\,T^{(e)}_{\mu\nu} = \rho_{(0)}\,F^{\mu}_{\nu}\,u_{\nu},$$

$$D^{\nu}\,T^{(\varphi)}_{\mu\nu} = 0 \qquad \text{[due to the conditions (4.7)]}$$

for $\rho_{(0)}$ and $\mu_{(0)}$ the charge and matter densities in the rest frame. We obtain from the identities (3.7) :

$$u^\sigma \mathrm{D}_\sigma u_\mu = \frac{e}{m} \mathrm{F}_\mu^\sigma u_\sigma, \tag{4.10}$$

where we have put

$$\frac{\rho_{(0)}}{\mu_{(0)}} = \frac{e}{m},$$

(4.10) are the conformal invariant equations of motion for particles of mass m and electric charge e.

We may also present (4.10) in the form

$$\frac{dx^\mu}{d\sigma^2} + \Gamma^\mu_{\rho\sigma} \frac{dx^\rho}{\sigma} \frac{dx^\sigma}{d\sigma} + k_\lambda u^\lambda u^\mu = \frac{e}{m} \mathrm{F}^{\mu\sigma} u_\sigma.$$

We see, from this equation, that even in the case where $\mathrm{F}_{\mu\sigma} = 0$ and $\Gamma^\mu_{\rho\sigma} = 0$, the particle does not move along a straight line, in the local tangent plane, due to the presence of the vector field $k_\mu(x)$ [1].

Observe that in (4.10) m is a conformal invariant scalar, since all quantities that enter into this equation are invariant inder C_g. But we should note that this invariance for m holds only in a non Riemannian space, resembling close the so called Weyl space which appears in the unitary field theory proposed by Weyl.

Now we treat the case for the quantum equation of motion, the Dirac equation. From the fundamental relation connecting the generalized Dirac matrices $\gamma_\mu(x)$ to the metric $g_{\mu\nu}(x)$,

$$\gamma_\mu(x)\gamma_\nu(x) + \gamma_\nu(x)\gamma_\mu(x) = 2\, g_{\mu\nu}(x) \tag{4.11}$$

it follows that $\gamma_\mu(x)$ transforms under C_g according to

$$\gamma'_\mu(x) = \gamma_\mu(x) \exp(\sigma(x)). \tag{4.12}$$

Thus, to $\gamma_\mu(x)$ may be corresponded the new object

$$\Gamma_\mu(x, \mathrm{P}) = \gamma_\mu(x)\, e^{-\int_{x_{in}}^{x} k_\rho d\xi^\rho}. \tag{4.13}$$

Such that from (1.8) we obtain the new equation replacing (4.11),

$$\Gamma_{(\mu}(x, \mathrm{P})\, \Gamma_{\nu)}(x, \mathrm{P}) = 2\, \mathrm{G}_{\mu\nu}(x, \mathrm{P}). \tag{4.14}$$

[1] This was to be expected, since the local vanishing of the conformal invariant affinity does not imply that the Christoffel symbols vanish, but rather that they are proportional to the local value assumed by the vector k_μ.

The covariant derivative of the Dirac spinor $\psi(x)$, in the Riemannian space, is given by

$$(4.15)\qquad \psi_{;\mu} = \psi_{,\mu} + S_\mu \psi.$$

Similarly the covariant derivatives of $\gamma_\mu(x)$ will be given by

$$(4.16)\qquad \gamma_{\mu;\nu} = \gamma_{\mu,\nu} - \left\{ {\rho \atop \mu\nu} \right\} \gamma_\rho + S_\nu \gamma_\mu - \gamma_\mu S_\nu.$$

The explicit expression for the spin-affine connection $S_\nu(x)$ is given by the conditions

$$\gamma_{\mu;\nu} = 0$$

its explicit form being (*see* Appendix) :

$$(4.17)\qquad S_\nu = \frac{1}{8}\left[\gamma^\mu \gamma_{\mu;\nu} - \gamma_{\mu,\nu}\gamma^\mu - \left\{ {\rho \atop \mu\nu} \right\}(\gamma^\mu \gamma_\rho - \gamma_\rho \gamma^\mu)\right].$$

From (4.13) we obtain the conformal invariant derivative of $\Gamma_\mu(x, P)$ in a form similar to (2.1) :

$$(4.18)\qquad \partial_\mu \Gamma_\alpha(x, P) = e^{-\int_{x_{in}}^{x} k_\varrho d\xi^\varrho} \left(\frac{\partial}{\partial x^\mu} - k_\mu\right)\gamma_\alpha(x).$$

As before, we note that this derivative is not a second rank tensor under the curvilinear coordinate transformations. Here the $\partial_\mu \Gamma_\alpha$ also do not transform as a Dirac spin-tensor under the internal transformations. These two difficulties may be removed by introducing the conformal-invariant covariant derivative $D_\mu \Gamma_\alpha(x, P)$ by

$$(4.19)\qquad \begin{aligned} D_\rho \Gamma_\alpha(x, P) = {} & \partial_\rho \Gamma_\alpha(x, P) - \Gamma^\lambda_{\alpha\rho}(x)\Gamma_\lambda(x, P) \\ & + \Sigma_\rho(x)\Gamma_\alpha(x, P) - \Gamma_\alpha(x, P)\Sigma_\rho(x), \end{aligned}$$

where we have taken directly the affinity $\Gamma^\lambda_{\alpha\rho}(x)$ as given by the conformal invariant expression (2.5), and have considered that the internal affinity $\Sigma_\rho(x)$ is independent of trajectory [by the same reason as in the case for $G_{\mu\nu}(x, P)$]. We now set, accordingly to (2.6) and (4.14),

$$(4.20)\qquad D_\rho \Gamma_\alpha(x, P) = 0.$$

It is important to note that the expression (4.13), as well as the expression (4.18), for the contravariant $\gamma^\alpha(x)$ take the form

$$(4.21)\qquad \Gamma^\mu(x, P) = \gamma^\mu(x)\, e^{\int_{x_{in}}^{x} k_\varrho d\xi^\varrho},$$

$$(4.22)\qquad \partial_\mu \Gamma^\alpha(x, P) = e^{\int_{x_{in}}^{x} k_\varrho d\xi^\varrho} \left(\frac{\partial}{\partial x^\mu} + k_\mu\right)\gamma^\alpha(x).$$

In compact notation, introducing

$$\pi_\mu = \frac{\partial}{\partial x^\mu} - k_\mu, \qquad \tilde{\pi}_\mu = \frac{\partial}{\partial x^\mu} + k_\mu$$

we rewrite (4.18) and (4.22) as

$$\partial_\mu \Gamma_\alpha (x, P) = e^{-\int_{x_{in}}^{x} k_\rho d\xi^\rho} \pi_\mu \gamma_\alpha (x), \tag{4.23}$$

$$\partial_\mu \Gamma^\alpha (x, P) = e^{\int_{x_{in}}^{x} k_\rho d\xi^\rho} \tilde{\pi}_\mu \gamma^\alpha (x). \tag{4.24}$$

The equation that is obtained from (4.19) and (4.20), taken into consideration the notation introduced in (4.23) and (4.24), will be

$$\Sigma_\nu (x) = \frac{1}{8} \left[\gamma^\mu . \pi_\nu \gamma_\mu - (\pi_\nu \gamma_\mu) . \gamma^\mu - \Gamma^\rho_{\mu\nu} (\gamma^\mu \gamma_\rho - \gamma_\rho \gamma^\mu)\right]. \tag{4.25}$$

Which is similar to (4.17), except that we replace the usual derivative $\frac{\partial}{\partial x^\mu}$ by π_μ, since it operates on the γ_μ, and replace $\left\{ \begin{matrix} \alpha \\ \beta\mu \end{matrix} \right\}$ by the $\Gamma^\alpha_{\beta\mu}$. This new internal affinity $\Sigma_\mu (x)$, given by (4.25), is conformal invariant. That is, under C_g we obtain

$$\Sigma'_\mu (x) = \Sigma_\mu (x).$$

Thus, the equation (4.15) now becomes

$$D_\mu \psi (x) = \psi_{,\mu} (x) + \Sigma_\mu (x) \psi (x) \tag{4.26}$$

for the conformal invariant coderivative of the Dirac spinor. From this equation it follows directly the form for Dirac's equation. In the usual versions this equation is presented as

$$\gamma^\mu (x) \psi_{;\mu} (x) - m \psi (x) = 0.$$

Now, we write it under the form

$$\Gamma^\mu (x) D_\mu \psi (x) - m \psi (x) = 0$$

which is conformal invariant, without mass variation, that means, m is conformal invariant. Again we note that this equation depends on variables which contain explicitly the trajectory P, namely the Γ^μ. If we use instead of the Γ^μ the usual matrices $\gamma^\mu (x)$, the mass m has to transform in the form given before in order to maintain the form invariance of the equation.

Note 1. — Contrarily to the general assumption that a spin $\frac{1}{2}$ field changes under C_g as [2] :

$$\psi'(x) = e^{-3/2\,\sigma(x)}\,\psi(x)$$

(this is obtained by requiring that the current density $J^\mu = \sqrt{-g}\,\bar{\psi}\,\gamma^\mu\,\psi$ is conformal invariant), we consider that in our case $\psi(x)$ does not change. That means, the generalized current density vector :

$$J^\mu(x, P) = \sqrt{-G}\,\bar{\psi}\,\Gamma^\mu(x, P)\,\psi$$

is conformal invariant, without requiring any transformation on $\psi(x)$.

Note 2. — We emphasize that the conformal invariant mass m is not an observable, in the usual sense of a direct observation, since its conformal invariance is only attained in a strong gravitational field [due to the condition (1.10)]. The mass obtained by direct observations is measured in weak gravitational fields and thus, conformal variant.

APPENDIX

From (4.16), along with $\gamma_{\mu;\nu} = 0$ we have

$$\gamma_{\mu,\nu} - \left\{ {\rho \atop \mu\nu} \right\} \gamma_\rho + S_\nu\,\gamma_\mu - \gamma_\mu\,S_\nu = 0. \tag{A.1}$$

Using the vierbeine formalism, where the coordinate-dependent matrices $\gamma_\mu(x)$ are given as linear combination of the usual Dirac's matrices $\gamma_{(\mu)}$,

$$\gamma_\rho(x) = h_\rho^{(\mu)}(x)\,\gamma_{(\mu)}.$$

We may expand the matrix $S(x)$ as a linear combination of the sixteen Dirac matrices

$$S_\nu(x) = a_\nu(x).1 + a_{\nu(\lambda)}(x)\,\gamma^{(\lambda)} + \frac{i}{4}\,b_{\nu[(\lambda),(\sigma)]}(x)\,(\gamma^{(\lambda)}\gamma^{(\sigma)} - \gamma^{(\sigma)}\gamma^{(\lambda)}) + c_\nu(x)\,\gamma_{(5)} + d_{\nu(\rho)}(x)\,\gamma_{(5)}\,\gamma^{(\rho)}. \tag{A.2}$$

Therefore, the commutator standing in the left hand side of (A.1) takes the form

$$[S_\nu, \gamma_\mu] = a_{\nu(\lambda)}\,[\gamma^{(\lambda)}, \gamma_\mu] + \frac{1}{2}\,b_{\nu[(\lambda),(\sigma)]}\,[\sigma^{(\lambda)(\sigma)}, \gamma_\mu] + c_\nu\,[\gamma_{(5)}, \gamma_\mu] + d_{\nu(\rho)}\,[\gamma_{(5)}\,\gamma^{(\rho)}, \gamma_\mu],$$

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[2] *Cf.* J. Ellis, CERN, Genebra, Ref. Th.-1245. CERN, 22 October 1970.

where

$$\sigma^{(\lambda)(\sigma)} = \frac{i}{2}(\gamma^{(\lambda)}\gamma^{(\sigma)} - \gamma^{(\sigma)}\gamma^{(\lambda)})$$

Using

$$[\gamma_{(5)}, \gamma_\mu] = 2\, h_{\mu(\rho)}\, \gamma_{(5)}\, \gamma^{(\rho)},$$

$$[\gamma_{(5)}\gamma^{(\rho)}, \gamma_\mu] = 2\, h_\mu^{(\mu)}\, \gamma_{(5)},$$

$$[\sigma^{(\lambda)(\sigma)}, \gamma_\mu] = -2\, i\, (\eta^{\lambda\tau}\gamma^{(\sigma)} - \eta^{\sigma\tau}\gamma^{(\lambda)})\, h_{\mu(\tau)},$$

$$\eta^{\lambda\tau} = \text{diag.}\ (+,\ -,\ -,\ -).$$

We get

$$[S_\nu, \gamma_\mu] = \frac{2}{i}\, a_{\nu(\lambda)}\, h_{\mu(\rho)}\, \sigma^{(\lambda)(\rho)} - i h_{\mu(\tau)}\, b_{\nu[(\lambda),(\sigma)]}\, (\eta^{\lambda\tau}\gamma^{(\sigma)} - \eta^{\sigma\tau}\gamma^{(\lambda)})$$
$$+ 2\, c_\nu\, h_{\mu(\rho)}\, \gamma_{(5)}\, \gamma^{(\rho)} + 2\, d_{\nu(\rho)}\, h^{(\rho)}\, \gamma_{(5)}.$$

Substituting this into (A.1), we find after some easy steps

$$\left\{ h^\alpha_{\mu,\nu} - \left\{ {\rho \atop \mu\nu} \right\} h^{(\alpha)}_\mu - i h_{\mu(\tau)}\, b_{\nu[(\lambda)^{(\alpha)}]}\, \eta^{\lambda\tau} + i b_\nu{}^{[(\alpha)}{}_{(\sigma)]}\, h_{\mu(\tau)}\, \eta^{\sigma\tau} \right\} \gamma\,(\alpha)$$
$$+ \frac{2}{i}\, a_{\nu(\lambda)}\, h_{\mu(\rho)}\, \sigma^{(\lambda)(\rho)} + 2\, c_\nu\, h_{\mu(\rho)}\, \gamma_{(5)}\, \gamma^{(\rho)} + 2\, d_{\nu(\rho)}\, h^{(\rho)}_\mu\, \gamma_{(5)} = 0.$$

From the linear independence of the Dirac operators, it follows that

$$a_{\nu(\lambda)} = 0, \qquad c_\nu = 0 \qquad \text{and} \qquad d_{\nu(\rho)} = 0,$$

and that

$$i h_{\mu(\tau)}\, b_\nu^{[(\tau),(\alpha)]} = \frac{1}{2}\left[h^{(\alpha)}_{\mu\nu} - \left\{ {\rho \atop \mu\nu} \right\} h^{(\alpha)}_\rho \right]$$

(where the fact that $b_{\nu[(\tau),(\alpha)]}$ is skew-symmetric over the indices τ, α was used). This last equation is solved for the $b_\nu^{[(\tau),(\alpha)]}$,

$$i b_\nu^{(\tau)(\alpha)} = \frac{1}{2}\, h^{\mu(\tau)} \left[h^{(\alpha)}_{\mu\nu} - \left\{ {\rho \atop \mu\nu} \right\} h^{(\alpha)}_\rho \right].$$

Substituting the coefficients a, b, c and d back into (A.2) we get the explicit form the matrix $S_\nu(x)$ satisfying (A.1) :

$$\text{(A.3)}\quad S_\nu = a_\nu.1 + \frac{1}{8}\left[\gamma^\mu \gamma_{\mu,\nu} - \gamma_{\mu,\nu}\gamma^\mu - \left\{ {\rho \atop \mu\nu} \right\} (\gamma^\mu \gamma_\rho - \gamma_\rho \gamma^\mu) \right]$$

which except for a multiple of the identity matrix is the formula (4.17) of the text. The term proportional to the identity matrix is of no use for us presently ([3]), so that we set $a_\nu = 0$.

([3]) Usually this term is connected to the electromagnetic interaction by means of the requirement of the minimal coupling.

The formula (A.3) was first derived by Fock and Ivanenko [8], and by this reason the S_ν are sometimes called as the Fock-Ivanenko coefficients.

We also note that the derivation of the formula (4.25) is similar to the derivation which we have done here. However, we will include here some details of this derivation which are of interest due to the comments done at the end. From (4.19) and (4.20) we have

$$\partial_\rho \Gamma_\alpha - \Gamma^\lambda_{\alpha\rho} \Gamma_\lambda + \Sigma_\rho \Gamma_\alpha - \Gamma_\alpha \Sigma_\rho = 0.$$

Now,

$$\Gamma_\lambda = h^{(\alpha)}_\lambda \Gamma_{(\alpha)}, \qquad \Gamma_{(\alpha)} = \gamma_{(\alpha)} e^{-\int_{x_{in}}^{x} k_\rho d\xi^\rho}$$

and a formula similar to (A.2) holds for the Σ_ρ. Thus,

$$[\Sigma_\rho, \Gamma_\alpha] = \frac{2}{i} a_{\rho(\lambda)} h_{\alpha(\tau)} e^{-\int_{x_{in}}^{x} k_\beta d\xi^\beta} \sigma^{(\lambda)(\tau)} + 2 c_\rho h_{\alpha(\beta)} e^{-\int_{x_{in}}^{x} k_\nu d\xi^\nu} \gamma_{(5)} \gamma^{(\beta)}$$
$$- i h_{\alpha(\tau)} e^{-\int_{x_{in}}^{x} k_\nu d\xi^\nu} b_{[(\lambda),(\sigma)]} (\eta^{\lambda\tau} \gamma^{(\sigma)} - \eta^{\sigma\tau} \gamma^{(\lambda)})$$
$$+ 2 d_{\rho(\beta)} h^{(\beta)}_\alpha e^{-\int_{x_{in}}^{x} k_\nu d\xi^\nu} \gamma_{(5)}$$

and

$$\partial_\rho \Gamma_\alpha = e^{-\int_{x_{in}}^{x} k_\nu d\xi^\nu} H^{(\beta)}_{\alpha\rho} \gamma_{(\beta)}$$

where

$$H^\beta_{\alpha\rho} = h^{(\beta)}_{\alpha,\rho} - k_\rho h^{(\beta)}_\alpha$$

a calculus similar to before gives as result

$$c_\rho = 0, \qquad d_{\rho(\beta)} = 0, \qquad a_{\rho(\lambda)} = 0;$$
$$i b_\rho^{[(\nu),(\beta)]} = \frac{1}{2} [h^{\alpha(\nu)} H^{(\beta)}_{\alpha\rho} - h^{\alpha(\nu)} \Gamma^\lambda_{\alpha\rho} h^{(\beta)}_\lambda].$$

Then,

$$\Sigma_\rho = a_\rho . 1 + \frac{1}{8} [\gamma^\alpha \pi_\rho \gamma_\alpha - (\pi_\rho \gamma_\alpha) \gamma^\alpha - \Gamma^\lambda_{\alpha\rho} (\gamma^\alpha \gamma_\lambda - \gamma_\lambda \gamma^\alpha)].$$

It should be observed that the first two terms inside the bracket may also be written simply as

$$\gamma^\alpha \pi_\rho \gamma_\alpha - (\pi_\rho \gamma_\alpha) \gamma^\alpha = \gamma^\alpha \gamma_{\alpha,\rho} - \gamma_{\alpha,\rho} \gamma^\alpha.$$

In this form, the only difference between Σ_ρ and S_ρ is due to the change of $\left\{ {\rho \atop \mu\nu} \right\}$ by $\Gamma^\rho_{\mu\nu}$. As it may be easily checked, the Σ_ρ will be conformal invariant independently if we use $\gamma^\alpha \pi_\rho \gamma_\alpha - (\pi_\rho \gamma_\alpha) \gamma^\alpha$ or $\gamma^\alpha \gamma_{\alpha,\rho} - \gamma_{\alpha,\rho} \gamma^\alpha$ as the first term inside of the bracket in (4.25). In other works, the combination $\gamma^\alpha \gamma_{\alpha,\rho} - \gamma_{\alpha,\rho} \gamma^\alpha$ is conformal invariant.

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