A. O. Barut
R. Raczka

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<http://www.numdam.org/item?id=AIHPA_1972__17_2_111_0>
Properties of Non-unitary Zero Mass Induced Representations of the Poincaré Group on the Space of Tensor-valued Functions (*)

by

A. O. BARUT and R. RACZKA (**) 

Institute for Theoretical Physics, 
University of Colorado, Boulder, Colorado 80302

ABSTRACT. — The theory of induced representations is used to discuss a class of indecomposable representations of the Poincaré group in Hilbert space with an indefinite metric which occur in theories with zero-mass particles. The formalism provides a number of further generalizations to representations with \( m^2 < 0 \), and to infinite-component tensor fields.

The purpose of this note is to present a concise group theoretical origin for and a proof of the quantization procedure in quantum-electrodynamics (and in linearized general relativity) using an indefinite metric. A number of detailed studies have appeared and are appearing [1] which, by direct and lengthy calculations, show how the indefinite metric comes about ('). The method of induced representations provides,

(*) Supported in part by the Air Force Office of Scientific Research under Grant AFOSR-71-1959.

(**) NSF Visiting Scientist, on leave from the Institute of Nuclear Research, Warsaw, Poland.

(') Remark : Let \( H \) be the carrier Hilbert space of a representation of a group \( G \), with the scalar product \( \langle u, v \rangle, u, v \in H \). A certain subclass of non-unitary representations of \( G \) in \( H \) may be defined which leave the bilinear form \( \langle U, v \rangle = \langle u, v \rangle \) invariant where \( \Gamma \) is an indefinite metric tensor. This subclass of representations may be called "\( \Gamma \)-unitary" representations, or "representations of \( G \) in a Hilbert space with the indefinite metric \( \Gamma \)". This is the connection between certain non-unitary representations and the indefinite metric.
we believe, an elegant statement and a simple proof of this problem. Furthermore, the new formulation makes it possible to state a number of generalizations.

We consider the representations of the Poincaré group $P = T^\ast \ltimes SL(2, \mathbb{C})$ [i.e. semi-direct product of $T^\ast$ and $SL(2, \mathbb{C})$] and use the general theory of induced representations for regular semidirect product groups ([2], [3], [6]).

Let $\Pi$ be an orbit in the momentum space which may be a hyperboloid, a cone, or the point $p_\mu = 0$. The stability subgroup $K$ of an arbitrary point $p$ of the orbit is isomorphic to a subgroup $T^\ast \ltimes \tilde{K}$, where $\tilde{K}$ is a subgroup of $SL(2, \mathbb{C})$. The construction of induced representations of the Poincaré group is carried along the following steps:

1° Choose a representation $k \rightarrow L_k$ of $K$ in a carrier space $\Phi$, which conserves in $\Phi$ a bilinear form $(\varphi, \psi)_\Phi$.

2° Form the space $H$ of function over the orbit $\Pi$ with values in $\Phi$ satisfying the condition

$$ (\varphi, \varphi_\mu) = \int_\Pi (\varphi(p), \varphi(p))_\Phi \ d\mu(p) < \infty. $$

3° Consider the map $P \ni g = \{a, \Lambda\} \rightarrow T_g$ in $H$ defined by

$$ T_{\{a, \Lambda\}} \varphi(p) = L_k^{-1}(\Lambda^{-1} p), $$

where $k$ is an element of $K$ corresponding to the Mackey decomposition of the element $\{a, \Lambda\}^{-1} x_g$ and $x_g$ is an element of $P$ corresponding to the momentum $p$ (1)

Then equation (2) provides a representation of $P$ in $H$ which conserves the scalar product (1).

If the representation $k \rightarrow L_k$ of $K$ is irreducible and unitary then the resulting induced representation $\{a, \Lambda\} \rightarrow T_{\{a, \Lambda\}}$ given by (2) is also irreducible and unitary. Moreover this construction provides all irreducible unitary representations of $P$.

However, if we use functions $\varphi \in \Phi$ which transform under the Lorentz group in a covariant manner as vectors, spinors, tensors, etc., we are in effect using non-unitary finite-dimensional representations of $SL(2, \mathbb{C})$, which in some cases imply also non-unitary representations of the stability sub-group $\tilde{K}$ of $\Pi$. For instance, in the case of massless particles

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(1) Let $K$ be a closed subgroup of a locally compact group $G$. Then the Mackey decomposition theorem states that there exists a Borel set $X$ in $G$ such that every element $g$ in $G$ has the unique decomposition $g = x_g k_g, x_g \in X, k_g \in K$. Because every coset $g K$ intersects with $X$ at one point every element $p \simeq g K$ in the quotient space $\Pi = K/G$ may be uniquely represented by the element $x_g \in X$, i.e. $p \simeq x_g k_g K = x_g K$. 

(II = p; p^2 = 0), the stability subgroup K has the form

\[ K = T^i \otimes \tilde{E}(2), \]

where \( \tilde{E}(2) \) is a covering group of the Euclidean group \( E(2) \). The group in equation (3) is connected and solvable. Hence, every finite-dimensional irreducible representation of K is one-dimensional by Lie's theorem. An arbitrary \( n \)-dimensional representation of K in (3) may be, moreover, reduced again by Lie's theorem, to a triangular form

\[ k \rightarrow L_k = \begin{pmatrix} \chi_i(k) & \star \\ & 0 \end{pmatrix}, \]

where \( \chi_i(k), i = 1, \ldots, n, \) are the complex characters of K. Consequently an arbitrary \( n \)-dimensional representation of K is either a direct sum of one-dimensional irreducible representations, or is indecomposable. In addition, if we demand that the representation \( k \rightarrow L_k \) is a faithful finite-dimensional representation, which conserves a bilinear form \( (\varphi, \Psi)_\Phi \) in the carrier space \( \Phi \), then, because K is non-compact, the bilinear form \( (\ldots, \ldots)_\Phi \) must be necessarily indefinite. By virtue of equation (1), the indefinite form \( (\varphi, \Phi)_\Phi \) makes the scalar product \( (\ldots, \ldots)_H \) in \( H \) also indefinite. Consequently, the representation defined by (2) is also non-unitary, but \( \Gamma \)-unitary. Note that this result holds for any choice of the bilinear form \( (\varphi, \Psi)_\Phi \) in the carrier space \( \Phi \) of the representation of K.

This is the idea of our proof. The precise statement, proof and generalizations now follow:

**Theorem.** — *Every representation of the Poincaré group P with \( m = 0 \), on the space of tensor-valued functions is non-unitary. Each representation is realized as a \( \Gamma \)-unitary representation in the Hilbert space \( H \) of tensor-valued functions, with domain on the momentum cone, by the formula*

\[ T(a, \Lambda) \varphi_{\mu_1 \ldots \mu_N} (p) = e^{ip\alpha} D_{\mu_1 \ldots \mu_N}^{-1} (k) \varphi^{\mu'_1 \ldots \mu'_N} (k) \varphi_{\mu_1 \ldots \mu_N} (\Lambda^{-1} p), \]

*where \( k \rightarrow D(k) \) is a finite-dimensional indecomposable representation of \( \tilde{E}(2) \) obtained by the reduction of the representation \( \bigotimes_X (D^{00} \oplus D^{10}) \) of \( SL(2, C) \) to \( \tilde{E}(2) \); \( k \) is the element of \( \tilde{E}(2) \) obtained from the Mackey decomposition \( P = XK, X \subset P, K = T^i \otimes \tilde{E}(2) \), of the product

\[ \{ a, \Lambda \}^{-1} x_g = \tilde{x} k, \quad \tilde{x} \in X, \quad k \in \tilde{E}(2), \]

*where \( x_g \) is the unique element of the Poincaré group \( P \) characterizing the coset*

\[ p \sim g K = x_g k_g K = x_g K. \]
The indefinite invariant scalar product in $H$ is given by the formula

$$\langle \varphi, \psi \rangle_H = \int_{\{p^2 = 0\}} d^3 p \frac{p_0}{p_0} \varphi_{\mu_1 \ldots \mu_N} (p) \psi_{\nu_1 \ldots \nu_N} (p).$$

Proof. — Let $\Phi$ be the linear space of all tensors $\varphi_{\mu_1 \ldots \mu_N}$ of order $N$ which carries a finite-dimensional representation $k \to L_k$ of $K$. Every representation of $P$ in the space of tensor valued functions may be obtained by induction from a corresponding tensor representation of $SL(2, C)$. Consequently we take the representation $k \to L_k$ in the form

$$k = (a, \mathbf{k}) \to L_k = e^{-i p_0 \mathbf{a}} D_{\mathbf{k}}, \quad p = \omega (1, 0, 0, 1),$$

where $\mathbf{k} = D_{\mathbf{k}}$ is a finite-dimensional faithful indecomposable representation of $\tilde{E}(2)$ obtained by the reduction of a finite-dimensional representation $\otimes_{i=1}^N (D^{00} \oplus D^{10})$ of $SL(2, C)$ to the subgroup $\tilde{E}(2)$. The representation (6) of $K$ conserves the following sesquilinear form in $\Phi$:

$$\langle \varphi, \psi \rangle_\Phi \equiv \varphi_{\mu_1 \ldots \mu_N} \psi_{\nu_1 \ldots \nu_N}.\]

Clearly, the form (7) is indefinite.

Now let $H$ be the space of all functions on $P$ with values in $\Phi$ satisfying the conditions

$$\begin{cases} 
1^o & (\varphi (g), \Psi) \Phi \text{ is a measurable function on } P \text{ for all } \Psi \in \Phi, \\
2^o & \varphi (gk) = L_k^{-1} \varphi (g), \quad g \in P, \\
3^o & \int_{\mathbf{p}/K} (\varphi, \varphi_\Phi) d\mu (\gamma) < \infty, \quad \gamma = gK.
\end{cases}$$

Then the action of $g_0 \to T_{g_0}$ of $P$ in $H$ is given by the formula

$$T_{g_0} \varphi (g) = \varphi (g_0^{-1} g).$$

Let

$$g = x_\mathbf{g} k_\mathbf{g}, \quad k_\mathbf{g} \in K, \quad x_\mathbf{g} \in X \subset P$$

be the Mackey decomposition $P = X K$ for the Poincaré group implied by the subgroup $K$. Then the action of the operators $T_{g_0}$ in the space $\Phi$ of tensor functions with domain $C_0 = K/P$ is given by

$$\langle T_{(a, \Lambda)} \varphi (p) = e^{i p a} D_{\mathbf{k}}^{-1} \varphi (\Lambda^{-1} p),$$
where $\mathbf{k}$ is the element of $\mathbf{E}(2)$ corresponding to the Mackey decomposition \( \Lambda \), \( x_\mathbf{g} = \mathbf{x} \mathbf{k}, \ x_\mathbf{g} \sim p \) is the unique element of $G$ corresponding to $p$,

\[ p \sim g K = x_\mathbf{g} k_\mathbf{g} K = x_\mathbf{g} K, \quad x_\mathbf{g} \in X. \]

Because the invariant bilinear form (7) is indefinite, (8.3°) is also indefinite. Moreover, because $K$ is noncompact any conserved form $(\varphi, \Psi)_\Phi$ for a faithful representation (6) of $K$ must be indefinite.

**Remarks and Generalizations.** — 1° In the massive case, $m^2 > 0$, the little group $K$ is $T^4 \otimes SU(2)$. The finite-dimensional tensor representations of $SL(2, \mathbb{C})$ reduced with respect to $SU(2)$ yield representations of SU(2) equivalent to a direct sum of unitary irreducible representations. Hence the induced representations (2) of the Poincaré group $P$ in this case will be also unitary.

2° As a special case, consider the representations of the Poincaré group in the space of vector functions $\varphi_\mu(p)$. They are reducible non-unitary but $\Gamma$-unitary representations which occur immediately in the quantization, by correspondence principle, of the classical electromagnetic field. In order to obtain the unitary representations corresponding to free physical photons with helicities $\pm 1$ one projects out the two redundant components of the vector $\varphi_\mu(p)$. We may achieve this by imposing the Lorentz condition

\[ p^\mu \varphi_\mu(p) = 0 \]

and the gauge condition

\[ \varphi_\mu(p) \rightarrow \varphi_\mu(p) + p_\mu \lambda(p), \]

where $\lambda(p)$ is a scalar function. Let $H_1$ be the subspace of $H$ consisting of functions $\varphi_\mu(p)$ satisfying (8). Functions of the form $p_\mu \lambda(p)$ do satisfy (10). Let us introduce the equivalence relation in $H$ defined by

\[ \varphi_\mu(p) \sim \tilde{\varphi}_\mu(p) \quad \text{if} \quad \varphi_\mu(p) - \tilde{\varphi}_\mu(p) = p_\mu \lambda(p) + f_\mu(p), \]

where $p_\mu f_\mu(p) = 0$. Then in the quotient space $H/H_1$, a unitary representation of the Poincaré group is realized [1].

In general, in order to separate out unitary irreducible representations corresponding to a massless particle of spin $J$, we may also utilize the following connection between the canonical wave function $\chi^\lambda(p)$ of the massless particle of helicity $\lambda$, and the wave function $\Psi_{\alpha \beta}(p)$ which transforms according to a finite-dimensional irreducible representation $(\alpha, \beta)$ of $SL(2, \mathbb{C})$:

\[ \chi^\lambda(p) = a_\lambda \sum_{\alpha, \beta} D^{a \alpha \beta}_{\alpha_0 \beta_0 \alpha \beta} [h_\mathbf{0}(p)^{-1}] \psi_{\alpha \beta}(p), \quad \lambda = \alpha + \beta, \]
where \( \{ (A)_{j} \} \) is the matrix of the irreducible representation \((a, b)\) of \(SL(2, \mathbb{C})\) and the Lorentz transformation \(h_{0}(p)\) satisfies \(h_{0}(p)a = p\), where \(a = (1, 0, 0, 1)\). We have the invariant scalar product

\[
(\chi^{\lambda}, \chi^{\lambda}) = \int d\mu(p) \overline{\chi^{\lambda}}(p) \chi^{\lambda}(p);
\]

hence the irreducible unitary representation \([0, \lambda]\) of the Poincaré group in the space \(H\) of wave functions \(\chi^{\lambda}(p)\) given by (11). In order now to apply equation (11) it remains to decompose the tensor product \(\bigotimes_{i=1}^{N} (D_{00}^{i} \oplus D_{10}^{i})\) into the irreducible representations \((a, b)\) of \(SL(2, \mathbb{C})\).

We remark that the form of vector-valued representations of \(P\) on the space \(H\) of functions \(\varphi_{\mu}(p)\) arises from the quantization of the classical coupling \(\varphi_{\mu} J^{\mu}\) of the electro-magnetic field to a matter current \(J^{\mu}\). If we want a theory of physical photons we may obtain directly the unitary representations of \(P\) realized in the space of two-dimensional vector functions \(\varphi_{\alpha}(p), \alpha = \pm 1\). This may be achieved by taking the representation \(\tilde{k} \rightarrow D_{\tilde{r}}\) of \(\tilde{E}_{2}\) in the form

\[
(12) \quad \tilde{k} = (\tilde{a}, r) \rightarrow D_{\tilde{r}} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{i\theta'} \end{pmatrix}, \quad \tilde{a} \in T^{i}, \quad r \in U(1).
\]

(We have taken this two-dimensional reducible representation because of the parity doubling in order to have both of the helicities \(\pm 1\).) However, it is not known how to write the coupling of the physical photons to particles and the form of the Coulomb field, for example, in the space of the unitary representations of \(P\).

3° The representations of the Poincaré group for massless spin 2 particles (e.g., gravitons) will have the same properties. We may obtain the irreducible unitary representations for massless spin 2 particles starting from the one-dimensional representation of the subgroup \(\tilde{K}\) of the little group \(K\):

\[
\tilde{k} = (\tilde{a}, r) \rightarrow D_{\tilde{k}} = e^{i\theta r}, \quad \tilde{a} \in T^{i}, \quad r \in U(1).
\]

However, again it is not known how to write the coupling of gravitons to matter using this Hilbert space of states. Hence, from classical considerations one starts from a reducible representation \(\bigotimes_{i=1}^{3} (D^{00} \oplus D^{10})\) of \(SL(2, \mathbb{C})\) restricts it to \(\tilde{E}(2)\) and induces to the Poincaré group \(P\). The representation so obtained will be only \(\Gamma\)-unitary and reducible. To get rid of redundant components we can use again the technique of projection operators given in equation (11).
Our formulation suggests the following generalizations:

(a) For the space-like representations of the Poincaré group, \( m^2 < 0 \), the little group \( K \) is \( T' \otimes SU(1,1) \). The reduction of the finite-dimensional representations of \( SL(2, C) \) with respect to SU(1,1) yields non-unitary faithful representations of \( K \), hence the induced representations of \( P \) are also non-unitary but \( T' \)-unitary.

(b) For the null-representations of \( P \),

\[
p_\mu = 0 \quad (m^2 = 0), \quad K = T' \otimes SL(2, C),
\]

the induced representations are again non-unitary (\( T' \)-unitary) for finite dimensional representations of \( SL(2, C) \).

(c) We can also immediately extend the theorem to infinite-component fields. Here one starts with unitary infinite-dimensional representations of \( SL(2, C) \), for example. Thus, we consider the space of tensor valued functions \( \phi_A(p) \), where \( A \) now has an infinite range determined by the representation of \( SL(2, C) \). Again we distinguish various little groups. For \( m^2 > 0 \), we have an infinitely reducible unitary representation of \( P \). For \( m = 0 \), \( \tilde{K} = \tilde{E}(2) \), the unitary representations of \( SL(2, C) \) restricted to \( \tilde{E}(2) \) are now unitary and infinite-dimensional. Hence the induced representations of \( P \) will be unitary by virtue of equations (1) and (2). These are the so-called "continuous spin" representations of the Poincaré group. It is interesting that finite-dimensional tensors of \( SL(2, C) \) give rise to zero-mass particles with a single value of helicity, whereas the infinite-dimensional tensors of \( SL(2, C) \) give rise to zero-mass "particles" with all values of the helicity \( 0, \pm 1, \pm 2, \ldots \).

It is evident that this technique can be readily applied to other stability subgroups, as well as to non-unitary infinite-dimensional representations of \( SL(2, C) \).

We would like to thank Professors M. Flato and D. Sternheimer for valuable discussions and suggestions.

REFERENCES


*(Manuscrit reçu le 24 avril 1972.)*