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Section A :
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Hidden Variables and 2-dimensional Hilbert Space

by

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ABSTRACT. — On the basis of Gleason's theorem it has been proved that a hidden-variables model of standard quantum mechanics is impossible except for the case of quantum systems describable in 2-dimensional Hilbert space. Two actually existing models of hidden parameters for a spin $\frac{1}{2}$ particle have been analyzed. It has been demonstrated that " hidden pure states " of this models are nothing but probability measures on the logic associated with 2-dimensional Hilbert space. The unusual richness of the set of probability measures in question enables us to construct many similar models, one example of such a construction is given.

1. INTRODUCTION

Since the advent of the quantum theory it has been accompanied by the hidden variables hypothesis. The latter expresses the dissatisfaction of some theoreticians with the actual interpretation of the quantum mechanics. However, the term " hidden-variables interpretation of quantum mechanics " does not mean the same in the pronouncements of adherents of the actual form of quantum mechanics and those of supportes of the hidden variables idea. We have abstracted in section 2 a rigorous formulation of what opponents of " deterministic " interpretation of quantum mechanics mean by a hidden-variables model of usual quantum theory. We are able to prove in a simple and direct way that it is impossible to construct such a model for systems requiring description in a Hilbert space of dimension greater than two. The proof is based on the Gleason theorem [3] which is not valid for the case of 2-dimensional Hilbert space. This case is investigated and we demonstrate that the set $\mathcal{M}_2$ of all probability measures on the logic $\mathcal{L}_2$ of

2-dimensional Hilbert space $\mathcal{H}_2$ is rich in non-regular measures, i. e. non-describable in terms of density matrices in $\mathcal{H}_2$. Two actually known models of hidden variables for the spin $\frac{1}{2}$ particle (Bell [1], Kochen and Specker [5]) are analysed in detail and proved to be based on this unusual richness of $\mathcal{M}_2$. Moreover, we have constructed a new model of this type and suggested a possibility of constructing a large number of similar ones. It seems to indicate such constructions to be vain. In $\mathcal{M}_2$ one may find also other types of measures. For instance, we prove that states of the " drop of non-hilbertian quantum liquid " — the fictitious " physical system " invented by Mielnik [6], may be regarded as some probability measures on $\mathcal{L}_2$.

2. THE HIDDEN VARIABLES PROBLEM

The " logical " approach to quantum mechanics is particularly useful to discuss the hidden variable problem. A fundamental notion of this approach is the logic $\mathcal{L}$ of the system, i. e. the set of all elementary measurements (yes-no measurements) feasible on the system. It is commonly accepted that the logic is an orthomodular complete ortholattice isomorphic to the lattice of all orthogonal projectors in a complex Hilbert space $\mathcal{H}$ (we have neglected purely technical peculiarities related to superselection rules). Elements of the logic $\mathcal{L}$ will be denoted : $a, b, c, \ldots$ and corresponding projectors in $\mathcal{H}$: $Q_a, Q_b, Q_c, \ldots$.

" Logical " relations in $\mathcal{L}$ can be translated to familiar relations in the set of all projectors in $\mathcal{H}$. Thus the partial ordering $a \leq b$ in $\mathcal{L}$ means $Q_a Q_b = Q_b Q_a = Q_a$, i. e. the closed linear manifold corresponding to Q_a is contained in the closed linear manifold corresponding to Q_b. The orthocomplementation $a \rightarrow a'$ in $\mathcal{L}$ is realised in $\mathcal{H}$ by the involutive mapping $Q_a \rightarrow Q_{a'} = I - Q_a$, with I, the unit operator in $\mathcal{H}$. The lattice meet $a \wedge b$ of $a, b \in \mathcal{L}$ corresponds to the projector $Q_{a \wedge b}$ with the proper subspace equal to the common part of closed manifolds associated to Q_a and Q_b. The lattice join $a \vee b = (a' \wedge b')'$ thus $Q_{a \vee b} = I - Q_{a' \wedge b'}$. The closed linear manifold corresponding to $Q_{a \vee b}$ is the one spanned by proper subspaces of Q_a and Q_b. The lattice $\mathcal{L}$ is complete, in particular : $\bigwedge_{a \in \mathcal{L}} a = 0$, $\bigvee_{a \in \mathcal{L}} a = e$, with $Q_0 = 0$, $Q_e = I$. Atoms of $\mathcal{L}$ are in correspondence to projectors on 1-dimensional subspaces of $\mathcal{H}$.

The second basic constituent of the quantum theory is the set of states $\mathcal{S}$ with elements $\alpha, \beta, \gamma, \ldots$. Any state is a probability measure on the logic $\mathcal{L}$. The Gleason's theorem states that any such measure on a logic of projectors in Hilbert space $\mathcal{H}$ with dimension greater than 2 is describable by means of density matrix, i. e. is a state in the sense of von Neuman. Thus the set of all probability measures on the logic $\mathcal{L}$ associated with the Hilbert space of dimension greater than 2 is identified with the set $\mathcal{S}$ of states, and pure states are in 1-to-1-correspondence with 1-dimensional subspaces of $\mathcal{H}$ (we have neglected again superselection rules). The set $\mathcal{S}$ possesses many regular properties, for instance it is σ-convex, order-determining and separating.

Quantum mechanics is a statistical theory, it provides us only with statistical predictions about possible outcomes of measurements. If $\alpha \in \mathcal{S}$ and A — a quantum observable, then the expectation value for A-type measurement on the system in state α is denoted by α (A). This number is in general different from any proper value of A even if α is a pure state, and one cannot predict with certainty the result of a single measurement. This " indeterministic " feature of the quantum mechanics leads to the concept of hidden variables. Generally, neither opponents nor adherents of the idea of hidden variables state precisely the exact meaning of this concept. Now we should like to formulate what the term " hypothesis of hidden variables " means in pronouncements of opponents of this idea. We follow implicit or explicit assumptions contained in papers of Jauch and Piron [4], Kochen and Specker [5] and others.

A successful introduction of hidden variables to quantum mechanics means that :

(i) There exists a classical statistical theory with the phase space Ω. Elements of Ω are regarded as hidden pure states of the considered quantum system.

(ii) Any hidden pure state $\omega \in \Omega$ is " dispersion-free ", i. e. it has the property that an expectation value ω (A) for any observable A is equal to some of proper values of A.

(iii) " Usual " quantum states α are described as some distributions of density of probability on the space Ω (classical mixed states).

(iv) " Usual " quantum observables are described as some Borel functions $\Omega \rightarrow$ R with R — the real line (classical observables).

(v) " Usual " quantum expectation values are identical with corresponding values calculated in the classical (hidden variables) formalism.

(vi) The above mentioned replacement of quantum observables by Borel functions on Ω is of such a type that corresponding imbedding of the logic $\mathcal{L}$ into the Boole' an logic associated with Ω preserves partial ordering, orthocomplementation and joins of compatible elements in $\mathcal{L}$.

These requirements have their justifications. The first five reflect the common meaning of the " classical re-interpretation " of quantum mechanics, proclaimed by the hidden variables hypothesis. The last requirement mirrors a widespread opinion that such properties of the logic $\mathcal{L}$ as partial ordering, orthocomplementation and joins of compatible elements are directly supported by experiments and must be preserved in any form of quantum theory.

There exist some refutations of the hidden variables hypothesis in the above (or stronger) sense. However, all those impossibility proofs may be criticized (*see* e. g. [1]), therefore we propose a new one. The main advantage of our demonstration is simplicity and evidence.

Let us consider a hidden variables model of a quantum system describable in a countably-dimensional complex Hilbert space and let ω denote a hidden pure state in this model. This state is a probability measure on the classical Boole' an logic associated with the phase space Ω. By (vi) the state ω determines also a probability measure ω on $\mathcal{L}$. The measure ω associates to any element of $\mathcal{L}$ the value 0 or 1 (by (ii)). Let a_1 denote an atom of $\mathcal{L}$ such that $\omega(a_1) = 1$. In the set $\mathcal{A}$ of all atoms of $\mathcal{L}$ there exists a maximal subset of pairwise orthogonal atoms $a_1, a_2, a_3, \ldots$. Obviously $\omega(a_i) = 0$, $i = 2, 3, \ldots$. If a dimension of the space $\mathcal{H}$ is greater than 2, then the Gleason's theorem forces the measure ω to be described by means of some density matrix ρ_ω in $\mathcal{H}$. It is easy to see that the only density matrix ρ with property

$$\mathrm{Tr}(\rho\, a_1) = 1, \qquad \mathrm{Tr}(\rho\, a_i) = 0 \qquad (i = 2, 3, \ldots) \quad \text{is } Q_{a_1}.$$

Thus $\rho_\omega = Q_{a_1}$ but evidently there exists in $\mathcal{H}$ an atomic projector Q such that $\mathrm{Tr}(Q_{a_1} Q) \neq 0, 1$. The conclusion is : the possibility of hidden variables model (in the above sense) is contrary to the Gleason's theorem.

In the above proof we have neglected superselection rules. In a general case we conclude that the existence of the hidden variables model forces the set of atoms $\mathcal{A}$ to be equal to the set $\{a_1, a_2, a_3, \ldots\}$, i. e. the considered system must be a " classical " one (but with a countable set of pure states).

Some adherents of the deterministic re-interpretation of quantum mechanics understand the term of " hidden variables model " in a different way. That is why there exists such a model [2] contrary to our result.

The only possibility of a successful realisation of the hidden variables model (as defined above) is in the case of the system described in 2-dimensional Hilbert space $\left(\text{e. g. spin observables and spin states of a spin } \frac{1}{2} \text{ particle}\right)$. Now we shall analyse this interesting case.

The logic $\mathcal{L}_2$ associated to the 2-dimensional Hilbert space $\mathcal{H}_2$ is composed only of the set of atoms $\mathcal{A}_2$ and two elements $0, e$. The set $\mathcal{M}_2$ of all probability measures on $\mathcal{L}_2$ consists of all functions α : $\mathcal{L}_2 \rightarrow [0, 1]$ such that $\alpha(0) = 0$, $\alpha(e) = 1$ and

$$\alpha(a) + \alpha(a') = 1 \tag{1}$$

for every atom a of $\mathcal{L}_2$. There exist in $\mathcal{M}_2$ measures non-describable in terms of density matrices in $\mathcal{H}_2$, e. g. the measure assigning values 0, 1 to some pair of orthogonal atoms and value $\frac{1}{2}$ to the other elements of $\mathcal{A}_2$. Among these non-regular measures there are some with properties corresponding to hidden pure states. Namely any function $\omega : \mathcal{L}_2 \rightarrow \{0, 1\}$ with property (1) satisfied for every pair of orthogonal elements in $\mathcal{L}_2$ may be interpreted as dispersion-free state on $\mathcal{L}_2$ and conversely. Functions of this kind will be called 0, 1-functions (or 0, 1-measures) on $\mathcal{L}_2$.

The mentioned features of $\mathcal{L}_2$ and $\mathcal{M}_2$ suggest the possibility of construction of the hidden variables model for a spin $\frac{1}{2}$ particle. We shall examine two actually existing models of this type ([1], [5]) and prove that they are essentially based on the unusual properties of $\mathcal{M}_2$.

3. HIDDEN VARIABLES MODELS FOR SPIN $\frac{1}{2}$ PARTICLE

The first model (satisfying conditions formulated in section 2) of hidden variables for spin observables and spin states of a spin $\frac{1}{2}$ particle was proposed by Bell [1].

" Usual " quantum states are in this case represented by spinors ψ forming the 2-dimensional Hilbert space $\mathcal{H}_2$. Quantum observables are represented by Hermitian 2×2 matrices :

$$A = u\,\varepsilon + \sum_{i=1}^{3} x_i\,\sigma_i \tag{2}$$

with ε, the unit 2×2 matrix; σ_i $(i = 1, 2, 3)$, Pauli matrices; u, x_i $(i = 1, 2, 3)$, real numbers. Hidden states in Bell's model are labelled by spinors ψ and a real parameter λ, $-\frac{1}{2} \leq \lambda \leq \frac{1}{2}$. If $\psi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ then the measurement of observable A in the hidden state $\omega_{\psi,\lambda}$ gives with certainty the proper value of A

$$\omega_{\psi,\lambda}(A) = u + |\mathbf{x}|\,\mathrm{sign}\left(\lambda\,|\mathbf{x}| + \frac{1}{2}|x_3|\right)\mathrm{sign}\,X \tag{3}$$

with $|\mathbf{x}| = (x_1^2 + x_2^2 + x_0^2)^{\frac{1}{2}}$, sign X $= +1$ if $X \geq 0$ and -1 if $X < 0$,

$$X = \begin{cases} x_3 & \text{if } x_3 \neq 0, \\ x_1 & \text{if } x_3 = 0, \quad x_1 \neq 0, \\ x_2 & \text{if } x_3 = 0, \quad x_1 = 0. \end{cases}$$

The " usual " quantum state determined by spinor ψ is described as uniform (over λ) mixture of hidden states $\omega_{\psi,\lambda}$ with $-\frac{1}{2} \leq \lambda \leq \frac{1}{2}$. In the case of $\psi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ we have obtained the familiar quantum expectation value

$$(\psi, A\,\psi) = \int_{-\frac{1}{2}}^{\frac{1}{2}} d\lambda \left\{ u + |\mathbf{x}| \operatorname{sign}\left(\lambda\,|\mathbf{x}| + \frac{1}{2}|x_3|\right) \operatorname{sign} X \right\} = u + x_3.$$

We shall demonstrate that hidden states of Bell's model are probability measures on $\mathcal{L}_2$. The 2×2 Hermitian matrix (2) is a projector on a 1-dimensional subspace of $\mathcal{H}_2$ if $u = \frac{1}{2}$, $|\mathbf{x}| = \frac{1}{2}$. Thus the set $\mathcal{A}_2$ is in 1-to-1-correspondence with the set of all points of the sphere S of radius $\frac{1}{2}$ in the 3-dimensional Euclidean space (x_1, x_2, x_3). Orthogonal atoms are represented by opposite points of S.

An application of the formula (3) to atoms of $\mathcal{L}_2$ $\left(\text{i. e. to observables satisfying the condition } u = \frac{1}{2}, |\mathbf{x}| = \frac{1}{2}\right)$ leads to

$$\omega_{\psi,\lambda}(a) = \frac{1}{2} + \frac{1}{2} \operatorname{sign}(\lambda + |x_3|) \operatorname{sign} X,$$

$$\omega_{\psi,\lambda}(a') = \frac{1}{2} - \frac{1}{2} \operatorname{sign}(\lambda + |x_3|) \operatorname{sign} X$$

with $a \in \mathcal{A}_2$. Thus

$$\omega_{\psi,\lambda}(a) + \omega_{\psi,\lambda}(a') = 1$$

and the condition (1) is satisfied by Bell's hidden states. This means that states $\omega_{\psi,\lambda}$ are 0, 1-measures on $\mathcal{L}_2$ and belongs to $\mathcal{M}_2$.

Thus Bell's hidden states are probability measures on $\mathcal{L}_2$ and differ from " usual " quantum states in the property that they are not represented by density matrices in $\mathcal{H}_2$. The existence of such type of meausres is compatible with Gleason's theorem, because this theorem holds for Hilbert spaces of dimension greater than 2.

The second hidden variables model for spin $\frac{1}{2}$ was proposed by Kochen and Specker [5]. Now we shall examine this construction.

Matrices of trace zero in $\mathcal{H}_2$ form 3-dimensional Euclidean space with the unit sphere S^2 corresponding to matrices with proper values ± 1. If A denotes a 2×2 Hermitian matrice with proper values v_1, v_2 ($v_1 \neq v_2$), then

$$\sigma(A) = \frac{2}{v_1 - v_2} A - \frac{v_1 + v_2}{v_1 - v_2} \varepsilon \tag{4}$$

is a matrix with proper values ± 1 corresponding to A. In this manner we associate with A a point $P_{\sigma(A)}$ on S^2.

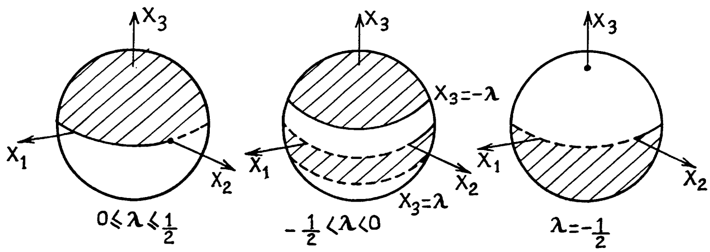

Fig. 1. — 0, 1-function $\omega_{\psi,\lambda}(a)$, $a \in S$ with $\psi = \begin{pmatrix}1\\0\end{pmatrix}$.
Shadowed regions are mapped on 1.

The phase space Ω of hidden states is chosen as S^2. A quantum observable A with two different proper values v_1, v_2 is represented by function

$$f_A(\omega) = \begin{cases} v_1 & \text{for } \omega \in S^+_{P_{\sigma(A)}}, \\ v_2 & \text{otherwise,} \end{cases} \tag{5}$$

with $S^+_{P_{\sigma(A)}}$ the "north" hemisphere of S^2 with the "North Pole" at $P_{\sigma(A)}$. If $v_1 = v_2 = v$ for some observable A, then $f_A(\omega) = v$ for every point $\omega \in S^2$.

The quantum state $\psi = \begin{pmatrix}1\\0\end{pmatrix}$ is described by the density of probability function

$$\rho(\omega) = \begin{cases} \frac{1}{\pi} x_3 & \text{for} \quad 0 \leqq x_3 \leqq 1, \\ 0 & \text{for} \quad -1 \leqq x_3 < 0 \end{cases} \tag{6}$$

on the phase space S^2, with x_3 the third coordinate of a point ω.

We shall analyse the Kochen and Specker model and demonstrate that hidden pure states ω of this are also probability measures on $\mathcal{L}_2$.

Let us observe that the function f_A given by (5) is not correctly defined. Indeed, if we consider for example the case of $A = a \in \mathcal{A}_2$, then

$$f_a(\omega) = \begin{cases} 1 & \text{for} \quad \omega \in S^{+}_{P_{\sigma(A)}}, \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

If $S^{+}_{P_{\sigma(A)}}$ denotes the closed (open) hemisphere of S^2, then

$$f_a^{-1}(1) \cap f_{a'}^{-1}(1) \neq \emptyset \qquad [f_a^{-1}(1) \cup f_{a'}^{-1}(1) \neq S^2]$$

with a' the orthocomplement of a in $\mathcal{L}_2$, contrary to the condition (vi) of the previous section. Note that this condition is accepted by Kochen

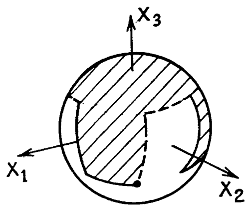

Fig. 2. — 0, 1-function $\tilde{\omega}_{\psi,\lambda}(a)$ with $\psi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $0 < \lambda < \frac{1}{2}$.

and Specker in a somewhat different formulation. We must improve the definition (7) of $f_a(\omega)$. Thus for instance, if $a = \sigma_3$, then we define

$$f_a(\omega) = \begin{cases} 1 & \text{for} \quad \omega \in S^{+}_{P_{\sigma(A)}}, \\ 1 & \text{for} \quad x_3 = 0, \quad x_1 > 0, \\ 1 & \text{for} \quad x_3 = 0, \quad x_1 = 0, \quad x_2 > 0, \\ 0 & \text{otherwise,} \end{cases} \tag{7'}$$

with $S^{+}_{P_{\sigma(A)}}$ the open hemisphere of S^2; x_1, x_2, x_3, coordinates of $\omega \in S^2$. In a general case we define $f_a(\omega)$ in an analogous way, with the previous change of basis in the linear space of trace-less Hermitian matrices on $\mathcal{H}_2$.

It is easy to see that the function $\omega(a) = f_a(\omega)$ defined by means of such improved functions $f_a(\omega)$ is a 0, 1-measure on $\mathcal{L}_2$ and is identical with the Bell's hidden state $\omega_{\psi,\lambda}$ for $\lambda = 0$ and some ψ.

Thus we have demonstrated the set of 0, 1-measures on $\mathcal{L}_2$ used in the Kochen and Specker model to be contained in the analogous set associated to Bell's model. The more numerous set of hidden pure states in Bell's model enables a description of quantum states by more simple mixtures than in model of Kochen and Specker.

The above discussed models are not the only possible ones. The set of 0, 1-functions on $\mathcal{L}_2$ is sufficiently large to enable us to construct plenty of similar models. For example, let us define the hidden pure states as 0, 1-functions on $\mathcal{L}_2$ labelled by spinors ψ and a real parameter λ, $-\frac{1}{2} < \lambda < \frac{1}{2}$, such that for $\psi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$:

$$
(8) \quad \tilde{\omega}_{\psi,\lambda}(a) = \begin{cases} 1 & \text{for } \; |\lambda| < x_3 \leq \frac{1}{2}, \\ & \quad 0 \leq \varphi \leq 2\pi, \\ 0 & \text{for } \; -\frac{1}{2} \leq x_3 < -|\lambda|, \\ & \quad 0 \leq \varphi \leq 2\pi, \\ \frac{1}{2}(1 + \operatorname{sign}\lambda) & \text{for } \; -|\lambda| \leq x_3 \leq |\lambda|, \\ & \quad 2n\frac{\pi}{3} \leq \varphi < (2n+1)\frac{\pi}{3} \\ & \quad (n = 0, 1, 2), \\ \frac{1}{2}(1 - \operatorname{sign}\lambda) & \text{for } \; -|\lambda| \leq x_3 \leq |\lambda|, \\ & \quad (2n+1)\frac{\pi}{3} \leq \varphi < (2n+2)\frac{\pi}{3}, \end{cases}
$$

with x_3, φ, the third coordinate and the azimuthal angle of thepoint a on the sphere S (the sphere S was being introduced in discussion of the Bell's model). A simple calculation suffices to test that a quantum states ψ corresponds to the uniform (over λ) density of probability distribution on hidden states $\tilde{\omega}_{\psi,\lambda}$, $-\frac{1}{2} < \lambda < \frac{1}{2}$.

It is evident, that one can invent many of similiar models [at least by further splitting of the interval $[0, 2\pi]$ in (8)]. The existence of such a great number of possibilities suggests that similiar constructions are futile and far from physical reality. This opinion is supported by the impossibility of generalization of this models on quantum systems described in Hilbert spaces of dimension greater than 2, as we have proved in the previous section. The described constructions are possible owing to the fact that the Gleason's theorem does not hold for the 2-dimensional Hilbert space.

The set $\mathfrak{M}_2$ of probability measures on $\mathcal{L}_2$ is much more diversified than an analogous set for the case of a higher dimensional Hilbert space. This variety is caused by the existence of the set of non-regular measures in $\mathfrak{M}_2$. It is interesting to see that in $\mathfrak{M}_2$ one can find not only 0, 1-measures, corresponding to the hidden pure states, but also other " exotic " measures. As an example, we shall demonstrate that states of the " drop of non-Hilbertian quantum liquid ", the fictitious " physical " system invented by Mielnik [6], are represented by some measures in $\mathfrak{M}_2$.

Let us consider again the " phase space " Ω introduced in the model of Kochen and Specker. We define the measure μ_a on $\mathcal{L}_2$ as the uniform (over Ω) mixture of all measures $\omega_{\psi,0}$ with property that the atom a_ψ representing ψ is placed on the " north " hemisphere of S with the ,North Pole " at a. A simple calculation shows that the transition probability between two such measures μ_{a_1} and μ_{a_2} is equal to $1 - \frac{\theta}{\pi}$ with θ, the angle between directions defined by points a_1 and a_2 on S and the center of S. But that is the transition probability characteristic for Mielnik's probability space T (2, 3). Thus this space is contained in $\mathfrak{M}_2$. Measures μ_a have a very interesting interpretation as states of " drop of non-Hilbertian quantum liquid "; *see* Mielnik's paper [6] for details. Probably the set $\mathfrak{M}_2$ contains numerous examples of similiar " non-Hilbertian " spaces.

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