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The theorem on Unitary Equivalence of Fock Representations

by

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ABSTRACT. — We prove that two Fock states ω_J and ω_K (not necessarily gauge invariant) on the CAR-algebra are unitarily equivalent if and only if $|J - K|$ is a Hilbert-Schmidt operator. We calculate explicitly the norm difference $\|\omega_J - \omega_K\|$.

Let (H, s) be a separable Euclidean space and J and K complex structures on (H, s) , i. e.

$$\begin{aligned} J^+ &= -J; & J^2 &= -1, \\ K^+ &= -K; & K^2 &= -1. \end{aligned}$$

Consider the operators

$$P = [J, K]_+; \quad Q = [J, K]_-$$

and let $P = U|P|$, $Q = V|Q|$ be their polar decompositions, $|Q|$, $|P|$ and U commute with J and K ; consequently the dimension of $\text{Ker } P$ is even or infinite; Q is a normal operator, therefore V can be chosen such that $V^+ = -V$, $V^2 = 1$. The same notations as in [1] are used : $\mathfrak{A} = \overline{\mathfrak{A}}(H, s)$ is the CAR-algebra and ω_J is any pure quasi-free state on $\mathfrak{A}$; J satisfies : $J^+ = -J$, $J^2 = -1$.

THEOREM 1. — *Let the operator P be diagonalizable [i. e. $(\psi_i)_{i \in \mathbb{N}}$ orthonormal basis of H such that $P\psi_i = \mu_i\psi_i$, $\mu_i \in \mathbb{R}$ (reals)], then there exists a family of subspaces $(H_n)_{n \in \mathbb{N}}$ of H invariant under J and K such that :*

$$(i) \quad H = \bigoplus_{n=0}^{\infty} H_n;$$

- (ii) $\dim H_0$ and $\dim H_1$ is even or infinite, $\dim H_n = 4$ for $n \geq 2$;
 (iii) $P = \sum_n \lambda_n p_n$, where $P_n H = H_n$; $\lambda_0 = -2$, $\lambda_1 = 2$ and $-2 < \lambda_n < 2$ for $n \geq 2$.

Proof. — Let $F = \text{Ker } Q$; F and $F^\perp$ (orthogonal complement of F for s) are invariant for J and K .

(a) Suppose $F^\perp = \{0\}$; then $JK = \frac{P}{2}$ is unitary and Hermitian, there exists a decomposition $F = H_0 + H_1$ such that $P = -P_0 + P_1$, where P_0 and P_1 are the orthogonal projection operators on H_0 respectively H_1 , which are invariant under J and K and therefore $\dim H_0$ and $\dim H_1$ is even or infinite.

(b) Suppose $F = \{0\}$, let H_α be subspaces of H such that $PH_\alpha = \lambda_\alpha H_\alpha$. Because $[P, J]_- = [P, K]_- = 0$, the subspaces H_α are invariant for J and K . Remark that $P^2 + Q^+ Q = 4$, $Q^+ Q = |Q|^2$; therefore $|Q|$ has the same proper subspaces H_α as $|P|$. Let $|Q| H_\alpha = \mu_\alpha H_\alpha$, then $\lambda_\alpha^2 + \mu_\alpha^2 = 4$ for all α . Take any $\psi_\lambda \in H_\lambda$ and consider the subspaces H_{ψ_λ} generated by the real orthogonal set $\{\psi_\lambda, V\psi_\lambda, J\psi_\lambda, JV\psi_\lambda\}$. It is clear that H_{ψ_λ} is a real subspace invariant under J and K of dimension four.

In general $H = F + F^\perp$ the results of (a) and (b) prove the theorem.

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LEMMA. — Let π_J and π_K be the Fock representations associated with J respectively K . If π_J and π_K are unitarily equivalent then $[J, K]_+$ has -2 as the only accumulation point of its spectrum.

Proof. — Let $\{\psi_j\}_{j \in \mathbb{N}}$ be any infinite orthonormal set of H and

$$L_n = \frac{-i}{n} \sum_{j=1}^n B(\psi_j) B(J\psi_j),$$

then

$$(\Omega_J, \pi_J(L_n) \Omega_J) = \omega_J(L_n) = 1.$$

Using Schwartz's inequality, we have

$$\|\pi_J(L_n) \Omega_J\| = 1 \quad \text{furthermore} \quad \left\| \left[\prod_{i=1}^k B(\psi_i), L_n \right]_- \right\| \leq \frac{k}{n}$$

proving

$$1 - \frac{k}{n} \leq \left\| \pi_J(L_n) \prod_{i=1}^k \pi_J(B(\psi_i) \Omega_J) \right\| \leq 1 + \frac{k}{n},$$

i. e. $\pi_J(L_n)$ tends strongly to one for n tending to infinity. Because π_J and π_K are unitarily equivalent $\pi_K(L_n)$ tends strongly to one on $\mathcal{H}_K$ and therefore weakly.

Further the expression

$$\omega_K(L_n) = (\Omega_K, \pi_K(L_n) \Omega_K) = -\frac{1}{2n} \sum_{i=1}^n s(P \psi_i, \psi_i)$$

must tend to one for all orthonormal sets $(\psi_i)_{i \in \mathbb{N}}$ which is possible if P has no accumulation points in its spectrum different from -2 .

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THEOREM 2. — *If ω_J and ω_K are pure quasi-free states, then π_J and π_K are unitarily equivalent iff $|J - K|$ is a Hilbert-Schmidt operator.*

Proof. — By Theorem 1,

$$H = \bigoplus_{n=0}^{\infty} H_n; \quad P = \sum_{n=0}^{\infty} \lambda_n P_n; \quad P_n H = H_n,$$

where $\dim H_n = 4$ for $n \geq 2$. By the lemma, $\dim H_1 < \infty$. Let $\{\Phi_1, \dots, \Phi_r; J\Phi_1, \dots, J\Phi_r\}$ be an orthonormal basis of H_1 and

$$u_1 = \prod_{k=1}^r B(\Phi_k).$$

In each H_n ($n \geq 2$) we choose the following orthonormal basis $(\psi_n, V\psi_n, J\psi_n, JV\psi_n)$, where ψ_n is any normalized vector of H_n and let

$$u_n = B(J\psi_n) B(\psi'_n),$$

where

$$\psi'_n = \frac{1}{(2 - \lambda_n)^{\frac{1}{2}}} (J\psi_n + K\psi_n).$$

If u_0 is the unit of $\overline{\mathfrak{A}(H_0, s)}$, then for all $n \geq 0$ and all $x \in \overline{\mathfrak{A}(H_n, s)}$,

$$\omega_K(x) = \omega_J(u_n^* x u_n).$$

In order that $U = \bigotimes_{n=0}^{\infty} \pi_{J_n}(u_n)$ is a unitary operator on $\mathcal{H}_J = \bigotimes_{n=0}^{\infty} \mathcal{H}_{J_n}$ (J_n is the restriction of J to H_n) it is necessary and sufficient that

$$U \Omega_J \in \mathcal{H}_J \text{ i. e. } \prod_{n=2}^{\infty} (\Omega_{J_n}, \pi_{J_n}(u_n) \Omega_{J_n}) = \prod_{n=2}^{\infty} \frac{1}{2} (2 - \lambda_n)^{\frac{1}{2}}$$

does not vanish. But

$$\begin{aligned} \prod_{n=2}^{\infty} \frac{1}{2} (2 - \lambda_n)^{\frac{1}{2}} \neq 0 &\Leftrightarrow \prod_{n=2}^{\infty} \left(\frac{1}{2} - \frac{\lambda_n}{4} \right) \neq 0 \\ &\Leftrightarrow \frac{1}{4} \sum_{n=2}^{\infty} (2 + \lambda_n) < \infty \quad \Leftrightarrow \operatorname{Tr} (2 + P) < \infty. \end{aligned}$$

Otherwise $(J - K)^+ (J - K) = 2 + P$, therefore π_J and π_K are unitarily equivalent if $|J - K|$ is a Hilbert-Schmidt operator.

Conversely, suppose that $|J - K|$ is not a Hilbert-Schmidt operator,

hence $\prod_{i=2}^{\infty} \left(\frac{1}{2} - \frac{\lambda_i}{4} \right) = 0$. Let $E_{n,m} = \bigoplus_{i=n}^m H_i$; the restrictions of ω_J

and ω_K to $\mathfrak{A}(E_{n,m}, s)$ remain pure states unitarily equivalent because

if $U_{n,m} = \prod_{i=n}^m u_i$, then

$$\forall x \in \mathfrak{A}(E_{n,m}, s), \quad \omega_J(x) = \omega_K(u_{n,m} x u_{n,m}^*) \quad [1].$$

Hence by Lemma 2.4 of [2]

$$\begin{aligned} \|(\omega_J - \omega_K) | \mathfrak{A}(E_{n,m}, s)\| &= 2(1 - |\omega_J(u_{n,m})|^2)^{\frac{1}{2}} \\ &= 2 \left(1 - \prod_{i=n}^m \left(\frac{1}{2} - \frac{\lambda_i}{4} \right) \right)^{\frac{1}{2}}. \end{aligned}$$

Denote by $\mathfrak{A}(E_n, s)^c$ the commutant of $\mathfrak{A}(E_n, s)$ in $\mathfrak{A}$. By lemma 2.3 of [2],

$$\|(\omega_J - \omega_K) | \mathfrak{A}(E_n, s)^c\| = \|(\omega_J - \omega_K) | \overline{\mathfrak{A}(E_n^{\perp}, s)}\|.$$

Since $\overline{\mathfrak{A}(E_n^{\perp}, s)}$ is the inductive limit of $\mathfrak{A}(E_{n,m}, s)$ when $m \rightarrow \infty$, we have

$$\|(\omega_J - \omega_K) | \mathfrak{A}(E_n, s)^c\| = \lim_{m \rightarrow \infty} \|(\omega_J - \omega_K) | \mathfrak{A}(E_{n,m}, s)\| = 2.$$

By lemma 2.1 of [2] π_J and π_K are not unitarily equivalent.

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COROLLARY. — *The representations π_J and π_K are unitarily equivalent if $\|\omega_J - \omega_K\| < 2$, and*

$$\|\omega_J - \omega_K\| = 2 \left(1 - \prod_{i=1}^{\infty} \left(\frac{1}{2} - \frac{\lambda_i}{4} \right) \right)^{\frac{1}{2}}.$$

Proof. — Lemma 2.1 of [2] proves that if π_J is not unitarily equivalent with π_K , then $\|\omega_J - \omega_K\| = 2$. Otherwise if π_J and π_K are equivalent, it follows from the calculations done in Theorem 2, that

$$\|\omega_J - \omega_K\| = 2 \left(1 - \prod_{i=1}^{\infty} \left(\frac{1}{2} - \frac{\lambda_i}{4} \right) \right)^{\frac{1}{2}}.$$

Q. E. D.

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