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The limit of massive electrodynamics

by

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ABSTRACT. — A simple derivation of the limit to Maxwell theory of massive electrodynamics is given, exhibiting the transformation of the longitudinal mode into a scalar field decoupled from the current. It is noted that this field remains coupled to gravitation, however.

RÉSUMÉ. — On démontre d'une façon simple que la limite de l'électrodynamique massive est la théorie de Maxwell, la partie longitudinale du champ massif devenant un champ scalaire découplé du courant, mais conservant un couplage normal avec la gravitation.

INTRODUCTION

Of the two basic differences between massive and massless fields, the finite range of the propagators and potentials presents no problem insofar as limiting behaviour is concerned. The difficulties lie in the presence of lower helicity states for a given spin (> 0) in the massive case as compared to the two degrees of freedom (helicity $\pm s$) in massless theory. What happens to these additional degrees of freedom in the limit ?

A smooth transition is clearly desirable in electrodynamics [1], and has been demonstrated field theoretically by many authors [2], [3].

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It is by now well-known, then, that the Proca field interacting with a conserved current tends smoothly to Maxwell theory as $m \rightarrow 0$. In view of renewed interest in possible experimental detection of an eventual photon mass (see Ref. [4] for a recent review) as well as in the corresponding (but more complicated) question for the Yang-Mills [5] and gravitational [5], [6] fields, a simple derivation of the electrodynamics limit may be found useful. Our approach will use the transverse-longitudinal decomposition of the fields; we will see explicitly how the helicity one and zero modes behave, the former reproducing the Maxwell field, the latter decoupling from the current and becoming a free scalar field. This scalar will be seen, however, to retain a normal coupling to gravitation through its non-vanishing stress-tensor.

TRANSVERSE-LONGITUDINAL DECOMPOSITION

The first-order action describing the Proca field coupled to a (prescribed, for simplicity) conserved source j^μ ,

$$(1 \ a) \quad I = \int d^4 x \left[\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} F^{\mu\nu} (\partial_\mu A_\nu - \partial_\nu A_\mu) - \frac{1}{2} m^2 A_\mu A^\mu + j^\mu A_\mu \right]$$

takes the form ($E^i \equiv F^{0i}$)

$$(1 \ b) \quad I = \int d^4 x \left[-\mathbf{E} \cdot \dot{\mathbf{A}} - \frac{1}{2} \{ \mathbf{E}^2 + (\nabla \times \mathbf{A})^2 \} + \mathbf{j} \cdot \mathbf{A} - \frac{1}{2} m^2 \mathbf{A}^2 - \frac{1}{2} m^{-2} (\nabla \cdot \mathbf{E} - j^0)^2 \right]$$

once the constraint variable $A_0 = m^{-2} (\nabla \cdot \mathbf{E} - j^0)$ is removed. This action clearly involves three independent pairs of conjugate variables ($-\mathbf{E}, \mathbf{A}$), which describe the two helicity-one and one helicity-zero modes⁽¹⁾. This is exhibited by decomposing the variables into transverse ($\mathbf{E}^T, \mathbf{A}^T$) and longitudinal ($\mathbf{E}^L, \mathbf{A}^L \equiv \nabla E^L, \nabla A^L$) parts, and using their

⁽¹⁾ The presence of three degrees of freedom here, instead of two for $m \equiv 0$, is due formally to the fact that A_0 is no longer a linear Lagrange multiplier as in the Maxwell action, but appear also quadratically in the mass term. Thus variation with respect to A_0 gives an equation determining the latter, while in Maxwell theory it leads to the Gauss equation which relates $\nabla \cdot \mathbf{E}$ and j^0 , leaving A_0 undetermined (and irrelevant). In the massive case, the Lorentz gauge condition is necessary for consistency between the constraint and the time-development equations; for $m \equiv 0$, this is already ensured by the identity $\partial_{\mu\nu}^2 F^{\mu\nu} \equiv 0$.

orthogonality under the integral to write the action as

$$(2 a) \quad I = I_T + I_L + I_0;$$

$$(2 b) \quad I_T \equiv \int d^4 x \left[-\mathbf{E}^T \cdot \dot{\mathbf{A}}^T - \frac{1}{2} \{ (\mathbf{E}^T)^2 + (\nabla \times \mathbf{A}^T)^2 + m^2 (\mathbf{A}^T)^2 \} + \mathbf{j}^T \cdot \mathbf{A}^T \right];$$

$$(2 c) \quad I_L \equiv \int d^4 x \left[E^L \nabla^2 \dot{A}^L - \frac{1}{2} E^L (-\nabla^2) \left(1 - \frac{\nabla^2}{m^2} \right) E^L - \frac{1}{2} m^2 A^2 (-\nabla^2) A^L - \nabla \cdot \mathbf{j} A^L + j^0 \frac{\nabla^2}{m^2} E^L \right];$$

$$(2 d) \quad I_0 \equiv -\frac{1}{2} m^{-2} \int d^4 x (j^0)^2.$$

We note first the decoupling of the T and L modes along with the presence of a contact term I_0 . The action I_T differs from that of the Maxwell field ($m \equiv 0$) not only through the obvious mass term, but through the absence of the instantaneous Coulomb interaction,

$$(3) \quad I_C = -\frac{1}{2} \int d^4 x j^0 (-\nabla^{-2}) j^0$$

Otherwise, I_T is in the canonical (gauge-invariant) form of Maxwell theory, with the same conjugate variables ($-\mathbf{E}^T, \mathbf{A}^T$), Hamiltonian and interaction. We shall recover I_C shortly, and note here that the mass term vanishes smoothly with m .

The longitudinal action I_L is not yet in canonical ($p\dot{q} - H$) form because of the various functions of ∇^2 both in the kinetic and Hamiltonian terms. We therefore introduce variables

$$(4) \quad \begin{cases} \bar{\chi} \equiv E^L (-\tilde{\nabla}^2)^{\frac{1}{2}} \omega^{\frac{1}{2}}, & \varphi \equiv A^L (-\tilde{\nabla}^2)^{\frac{1}{2}} \omega^{-\frac{1}{2}} m^2; \\ \omega \equiv (m^2 - \nabla^2), & \tilde{\nabla}^2 \equiv \frac{\nabla^2}{m^2} \end{cases}$$

rescaled by the appropriate numerical functions of the positive quantity $(-\nabla^2)$. At this point, we have

$$(5) \quad \left\{ \begin{aligned} I_L + I_0 = & - \int d^4 x \left[\bar{\chi} \dot{\varphi} + \frac{1}{2} \{ \bar{\chi}^2 + (\nabla \varphi)^2 + m^2 \varphi^2 \} \right] \\ & - \int d^4 x j^0 (-\tilde{\nabla}^2)^{\frac{1}{2}} \omega^{-\frac{1}{2}} \bar{\chi} + \int d^4 x j^L (-\tilde{\nabla}^2)^{\frac{1}{2}} \omega^{\frac{1}{2}} \varphi \\ & - \frac{m^{-2}}{2} \int d^4 x (j^0)^2. \end{aligned} \right.$$

The first integral is just the action of a free massive scalar field in first order form with variables $(\bar{\chi}, \varphi)$, but the coupling involves both $\bar{\chi}$ and φ . We remove the coupling to the momentum variable $\bar{\chi}$ by the translation

$$(6) \quad \bar{\chi} = \chi - j^0 (-\tilde{\nabla}^2)^{\frac{1}{2}} \omega^{-\frac{1}{2}}$$

which yields

$$(7) \quad \left\{ \begin{aligned} I_L + I_C &= - \int d^4 x \left[\chi \dot{\varphi} + \frac{1}{2} \{ \chi^2 + m^2 \varphi^2 + (\nabla \varphi)^2 \} \right] \\ &+ \int d^4 x \left[-j^0 (-\tilde{\nabla}^2)^{\frac{1}{2}} \omega^{-\frac{1}{2}} + j^L (-\tilde{\nabla}^2)^{\frac{1}{2}} \omega^{\frac{1}{2}} \right] \varphi \\ &- \frac{1}{2} \int d^4 x j^0 [m^{-2} + \tilde{\nabla}^2 \omega^{-1}] j^0. \end{aligned} \right.$$

For a prescribed current (or a point charge), current-conservation ⁽²⁾ may be used ($\dot{j}^0 = -\nabla^2 j^L$) to re-write the coupling term as

$$(8) \quad I_{\text{INT}} = + m \int d^4 x \varphi (-\nabla^2)^{\frac{1}{2}} \omega^{-\frac{1}{2}} j^L.$$

while the matter-matter part combines to

$$(9) \quad I_C = - \frac{1}{2} \int d^4 x j^0 \omega^{-1} j^0.$$

This direct interaction is of course the Yukawa analog of the Coulomb law and limits smoothly, i. e. $\omega^{-1} \rightarrow (-\nabla^2)$. Together with the limit of I_T discussed previously, I_C gives precisely Maxwell theory in its gauge-invariant form ⁽³⁾, namely that obtained from the usual one by use of current conservation and the constraint to eliminate A^0 and A^L .

The longitudinal action consists of a part having the structure of a free Lorentz scalar field, even though (E^L, A^L) were the helicity-zero parts of a vector originally ⁽⁴⁾, together with the coupling Equation (8) to the longitudinal current j^L . The crucial factor in the limit is of

⁽²⁾ It is well-known that no limit need exist if the source is non-conserved since the scalar part then contained by the current will act as a finite source of the helicity-zero field.

⁽³⁾ Despite the "radiation gauge-like" form of the helicity-one part, we have not imposed any gauge condition. Indeed, for $m \neq 0$ the Lorentz condition $\partial_\mu A^\mu = 0$ holds. Conversely, the fact that A^μ obeys the Lorentz condition for finite m does not mean that we obtain the Lorentz gauge form in the limit! We note also that the point $\mathbf{k} = 0$ (where ∇^2 vanishes) presents no difficulty.

⁽⁴⁾ The canonical commutation relations between χ and φ implied by $\chi \dot{\varphi}$ in Equation (7) lead to the correct Poincaré algebra for the generators $P^\mu, J^{\mu\nu}$ constructed in the usual way from χ and φ .

course the coefficient m in I_{INT}^k , which leads to its vanishing :
 $I_{\text{INT}}^k \sim m \int d^4x \varphi j^L \rightarrow 0$.

The effective weakening of the longitudinal coupling may also be seen in a different way by examining which photon helicities are effectively exchanged near the photon pole. One finds, using conservation, that

$$(10) \quad \frac{j^\mu j_\mu}{p^2 + m^2} \sim \frac{(\mathbf{j}^T)^2 + \left(\frac{m^2}{p_0^2}\right) (\mathbf{j}^L)^2}{p^2 + m^2}$$

so that longitudinal photon exchange is depressed by a factor m^2 relative to transverse exchange. Looking at the linear field equations alone does not yield this information, since they merely read $(\square - m^2) A_\mu = j_\mu$ for all components.

Thus each longitudinal photon emission or absorption is damped by a factor m , and the helicity-zero field contains only whatever "in" excitations it may have had initially.

GRAVITATIONAL COUPLING

Although matter is now transparent to longitudinal photons, there is one interaction — with gravitation — which remains undamped. The stress-tensor in terms of the original variables is

$$(11) \quad T_{\mu\nu} = \left(F_{\mu\alpha} F^{\nu\alpha} - \frac{1}{4} \eta_{\mu\nu} F^2 \right) + m^2 \left(A_\mu A_\nu - \frac{1}{2} \eta_{\mu\nu} A^2 \right)$$

so that the energy density, for example, is

$$(12) \quad \left\{ \begin{aligned} T_{00} &= \frac{1}{2} [(\mathbf{E}^T)^2 + (\nabla \times \mathbf{A}^T)^2 + m^2 (\mathbf{A}^T)^2] + \frac{1}{2} [(\mathbf{E}^L)^2 + m^2 (\mathbf{A}^L)^2] \\ &\quad + \mathbf{E}^T \cdot \mathbf{E}^L + m^2 \mathbf{A}^T \cdot \mathbf{A}^L + (2m^2)^{-1} (\nabla \cdot \mathbf{E} - j^0)^2. \end{aligned} \right.$$

We are not interested in the transverse parts, which limit correctly to the Maxwell form. The remainder must be expressed in terms of the correctly scaled variables (as is clear also from the presence of m^{-2}). It then follows that $m^2 \mathbf{A}^T \cdot \mathbf{A}^L$ vanishes ($m^2 \mathbf{A}^L \sim m \varphi \rightarrow 0$). Since

$$(13) \quad \left\{ \begin{aligned} \mathbf{E}^L &\sim -m \nabla^{-2} \bar{\chi} \sim -m \nabla^{-2} \chi + \nabla^{-2} j^0 \rightarrow + \nabla^{-2} j^0, \\ \nabla \cdot \mathbf{E} - j^0 &\rightarrow -m \chi, \end{aligned} \right.$$

the $\mathbf{E}^T \cdot \mathbf{E}^L$ term $\sim \mathbf{E}^T \cdot \nabla \nabla^{-2} j^0$ which is part of the usual Maxwell energy density ($\mathbf{E}^T \cdot \mathbf{E}^L \equiv \mathbf{E}^T \cdot \nabla \nabla^{-2} j^0$), and similarly $\frac{1}{2} (\mathbf{E}^L)^2$ limits to the Coulomb energy density. We are thus left with the non-Maxwell

terms $\frac{1}{2}m^2 (\mathbf{A}^L)^2 + \frac{1}{2}m^{-2} (\nabla \cdot \mathbf{E} - j^0)^2$. The first term limits to $\frac{1}{2}(\nabla\varphi)^2$; from $\nabla \cdot \mathbf{E} - j^0 \sim -m\chi$ the second becomes $\frac{1}{2}\chi^2$. Thus the total energy density is (not surprisingly) the sum of the Maxwell contribution together with the re-scaled free field's energy density

$$(14) \quad T_{00}[\varphi] = \frac{1}{2}(\chi^2 + (\nabla\varphi)^2)$$

in the limit. Thus, whatever initial primordial longitudinal photons may be present will have a corresponding "weight", and can, in principle, be detected by gravitational experiments ⁽⁵⁾.

The finiteness of the gravitational coupling does not affect any of the usual conclusions concerning longitudinal photons in which gravitation is negligible. Nor is there any problem with energy equipartition, since equilibrium in black bodies (unlike black holes) does not involve gravitation. Totally negligible as these contributions may be, they find a parallel in neutrino theory. There, the difference between the two-component theory and a four-component form could also be distinguished in principle through the gravitational interaction of the two otherwise uncoupled components. In both systems, only the strictly massless case permits a "two component" formulation, while a mass, however small, is necessarily accompanied by an additional, gravitationally coupled, field.

APPENDIX

The choice of longitudinal field variables in text is of course not unique : one may equivalently eliminate E^L in favour of A^0 . The constraint is then to be read as $\nabla \cdot \mathbf{E} = j^0 - m^2 A^0$, which yields the following alternate form of Equation (2) :

$$(A.1) \quad \left\{ \begin{array}{l} I = I_T + \bar{I}_L + I_C, \\ \bar{I}_L \equiv -m^2 \int d^4x \left[A^0 \dot{A}^L + \frac{1}{2}m^2 A^0 (-\nabla^{-2}) A^0 \right. \\ \quad \left. - A^0 (-\nabla^{-2}) j^0 + \frac{1}{2}(A^0)^2 + \frac{1}{2}(\nabla A^L)^2 \right] \\ \quad - \int d^4x (\partial_\mu j^\mu) A^L, \\ I_C \equiv -\frac{1}{2} \int d^4x j^0 (-\nabla^{-2}) j^0 \end{array} \right.$$

⁽⁵⁾ The non-vanishing of $T_{\chi\chi}^z \sim \frac{1}{2}[\chi^2 - (\nabla\varphi)^2]$ is not surprising since it is due to the scalar field alone; if desired, it can be "improved away" by adding the usual factor $\sim (\partial_{\mu\nu}^2 - \tau_{\mu\nu} \square) \varphi^2$ to $T_{\mu\nu}$.

where I_T is as in Equation (2 b), and the longitudinal variables are (A^0, A^L) . Here the Coulomb interaction I_C is already in its final form. We then rescale with

$$\begin{cases} A^0 = m^{-1} (1 - m^2 \nabla^{-2})^{-\frac{1}{2}} \pi \sim m^{-1} \pi \\ A^L = m^{-1} (1 - m^2 \nabla^{-2})^{\frac{1}{2}} \varphi \sim m^{-1} \varphi \end{cases}$$

to obtain the same free scalar field limit of $\bar{I}_L$ as in text. The fact that $\partial_\mu j^\mu$ must vanish faster than m to obtain the desired limit is evident from the $A^L \partial_\mu j^\mu$ term.

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