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Magnetized deformable media in general relativity

by

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ABSTRACT. — In this article, we develop from a variational formulation the field equations, jump conditions across a discontinuity surface and nonlinear constitutive equations for a magnetized elastic medium with finite deformations in the frame of the general theory of relativity.

RÉSUMÉ. — Dans le présent article, nous généralisons dans le cadre de la relativité générale, la théorie des milieux continus déformables en interaction avec le champ magnétique donnée précédemment dans le cadre de la relativité restreinte [6]-[25]-[27]. Le milieu continu considéré est un milieu solide élastique sujet à des déformations finies et en interaction avec les champs gravifique et magnétique. Un principe variationnel qui suit la formulation que Taub [12] a donnée pour le schéma fluide parfait est employé. Toutes les équations du champ (équations d'Einstein, conservation de l'impulsion-énergie, équations de Maxwell dans un milieu matériel, conservation du flux d'entropie) en découlent ainsi que les conditions de saut à travers une surface de discontinuité. Comme dans le travail de Taub, il est montré que cette dernière ne peut être variée indépendamment du paramètre thermodynamique. Les lois non-linéaires de comportement sont également obtenues à partir d'un potentiel, l'énergie libre de Helmholtz qui est écrite sous forme invariante, ceci généralisant la contrainte habituellement imposée par le principe d'indifférence matérielle en mécanique classique des milieux continus.

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INTRODUCTION

Looking back at the early developments of relativistic continuum mechanics, we find two avenues along which attempts have been made to generalize classical concepts (those of small velocity physics): (a) the study of perfect fluids whose scheme is well-adapted to the applications of general relativity (GR) in the large *i. e.*, to cosmological problems and more recently to the study of more localized phenomena such as the gravitational collapse; (b) the study of elastic media in special relativity (SR) [1]-[4] which raised the question of the definition of a rigid body motion in SR (the latter question being posed as well in GR [5]). As far as the fluids are concerned.

from then on, we note further studies, to name a few: viscous fluids in SR [6]-[9], perfect fluids in GR [10]-[12], viscous fluids in GR [11], [13]-[14], general relativistic MHD [15]-[16], fluids with spin in SR [17].

General models of continuous materials in interaction with external fields and endowed with internal degrees of freedom have been proposed by Sedov *et al.* [18]-[19]. Yet, given the difficulty of obtaining large velocities for material media, the study of solids in SR and GR has been much less extensive. However, we note two recent approaches to elasticity in GR by Rayner [20] and Synge [21], the latter being, per se, a theory of hypoelasticity. Furthermore, a revival of interest came with the better understanding of classical continuum mechanics based on the consideration of finite deformations and non linear constitutive equations (*e. g.*, non-Hookean solids, non-Newtonian fluids). These latter concepts have been extended to SR and GR with a proper presentation of kinematics and deformation fields [22]-[24], [6] and applications to different continuous media: magnetized solids in SR [6], [25], magnetoviscoelasticity in GR [26], magnetized solids with electronic spin in SR [27], polar media in SR [27]-[28]; wave fronts have been studied in nonlinear elastic solids in SR [29] and GR [30].

In this article, we propose to extend the theory developed in Ref. [6] and [25] to GR. Therefore, we shall deal with a nonlinear elastic medium in interaction with the gravitational field and with electromagnetic fields. We use a variational principle whose formulation follows that of Taub for perfect fluids [12] and the general formulation of variational principle given in Ref. [27]. All the field equations (Einstein's equations, conservation of stress-energy-momentum, Maxwell's equations, conservation of entropy flux) as well as the corresponding jump conditions across a discontinuity surface are obtained. The nonlinear constitutive equations for this nondissipative medium are derived from the free energy since the latter is considered to be the potential relevant to the theory. This potential is written in an invariant form after introduction of *ad hoc* invariant arguments. The exposé as a whole follows the formulation of modern continuum mechanics.

The *raison d'être* of this article is twofold: (a) it is of course of pure theoretical interest to see if such an extension to GR is easily carried out; (b) it happens that some astrophysical objects whose study resorts to GR, such as neutron stars [31], present a thick solid crust of which the outer portion resembles terrestrial matter except that it is 10^{18} times more rigid than steel and much more incompressible. It is much easier to jiggle this matter than to compress it. However, the magnetic field is very intense

and somewhat 10^{12} times stronger than that of the earth and the smallest distortion of the crust has drastic effects as to this field and leads to real « neutron star-quakes ». A theoretical scheme as the one developed below, though involved, should apply to this category of phenomena.

1. PRELIMINARIES

1.1. Geometry.

Let V^4 be a four-dimensional riemannian manifold of normal hyperbolic metric $g_{\alpha\beta}$ ($\alpha, \beta = 1, 2, 3, 4$) of signature $(+, +, +, -)$. The square of the element of length reads:

$$(1.1) \quad ds^2 = g_{\alpha\beta}(x^\lambda) dx^\alpha dx^\beta$$

where x^α ($\alpha = 1, 2, 3, 4$) are the coordinates in V^4 . Locally Eq. (1.1) can be written in the pseudo-Euclidean form

$$(1.2) \quad ds^2 = (\omega^1)^2 + (\omega^2)^2 + (\omega^3)^2 - (\omega^4)^2$$

where the ω^i are a system of linearly independent Pfaff forms.

We call (\mathcal{D}) a closed domain of V^4 . Its frontier $(\partial\mathcal{D})$ is considered to be regular *i. e.*, smooth enough to allow the use of vector and tensor analysis. A 3-dimensional regular hypersurface (Σ) included in (\mathcal{D}) or intersecting (\mathcal{D}) will be considered later on. (Σ) is supposed to admit locally a Gaussian parametrization $x^\alpha = x^\alpha(a^i)$, ($i = 1, 2, 3$). Its positive normal is denoted by n_α .

On V^4 , the $g_{\alpha\beta}$ are C^1 , piecewise C^3 (cf. Ref. [11]). In the sequel all greek indices (small or capital) take the values 1, 2, 3, 4. Latin indices assume the values 1, 2, 3. The summation convention is used throughout. Parentheses around a set of indices denote symmetrization and brackets denote alternation. Commas or symbols ∂ are used to denote partial differentiation; semicolons or symbols ∇ are used for the covariant differentiation with respect to the metric $g_{\alpha\beta}$ *e. g.*, \tilde{A} being any tensorial quantity, we write:

$$(1.3) \quad \tilde{A}_{,\alpha} = \partial_\alpha \tilde{A} = \partial \tilde{A} / \partial x^\alpha \quad , \quad \tilde{A}_{;\alpha} = \nabla_\alpha \tilde{A}$$

$\Gamma_{\beta\gamma}^\alpha$ are the Christoffel symbols of the second kind constructed from $g_{\alpha\beta}$. g denotes the determinant of $g_{\alpha\beta}$, c is the velocity of light in vacuum and k is a constant proportional to the Newtonian gravitational constant k

(usually $\kappa = 8\pi k/c^4$). \mathbf{R} is the Ricci scalar curvature and $R^\alpha_{\beta\gamma\delta}$ is the fourth order curvature tensor. We have:

$$(1.4) \quad \Gamma^\alpha_{\beta\cdot\gamma} = \frac{1}{2} g^{\alpha\delta} (g_{\beta\delta,\gamma} + g_{\delta\gamma,\beta} - g_{\beta\gamma,\delta})$$

$$(1.5) \quad R^\alpha_{\beta\gamma\delta} = 2\Gamma^\alpha_{\beta\cdot[\delta,\gamma]} + 2\Gamma^\mu_{\beta\cdot[\delta}\Gamma^\alpha_{\gamma]\cdot\mu}$$

$$(1.6) \quad R_{\beta\delta} = R^\alpha_{\beta\gamma\delta} g^\alpha{}^\gamma, \quad \mathbf{R} = R_{\beta\delta} g^{\beta\delta}$$

The Einstein-Cartan tensor is written:

$$(1.7) \quad A_{\alpha\beta} \equiv R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} \mathbf{R}$$

1.2. The motion.

The motion of one particle or infinitesimally small element of continuous matter in V^4 is entirely described by the mapping

$$(1.8) \quad x^\alpha = x^\alpha(\bar{x}^\lambda)$$

which admits the particular representation

$$(1.9) \quad x^\alpha = x^\alpha(X^\Delta)$$

with

$$(1.10) \quad X^\Delta = (X^K, ic\tau), \quad i = \sqrt{-1}$$

where X^K are a set of Lagrangian coordinates in E^3 once known for each particle. τ is a monotonically increasing timelike parameter defined along the path (C_{X^K}) or world line of the particle labeled (X^K). It is chosen to be the proper time of (X^K). From (1.9) is defined the 4-velocity u^α of a particle. It is a 4-vector of constant magnitude such that

$$(1.11) \quad u^\alpha = \frac{dx^\alpha}{d\tau} = \left. \frac{\partial x^\alpha}{\partial \tau} \right|_{X^K}, \quad g_{\alpha\beta} u^\alpha u^\beta = -c^2$$

The operator $\frac{\partial}{\partial \tau} = u^\alpha \nabla_\alpha$ generalizes the notion of total time derivative of classical continuum mechanics.

The operator of projection or *projector* $P_{\alpha\beta}$ onto the hypersurface V^3_1 orthogonal to (C_{X^K}) at a point \underline{M} of (C_{X^K}) is defined according to:

$$(1.12) \quad P_{\alpha\beta}(\underline{M}) = g_{\alpha\beta} + \frac{1}{c^2} u_\alpha u_\beta, \quad P^\alpha{}_\alpha = 3$$

and satisfies the two properties

$$(1.13) \quad \left\{ \begin{array}{l} P_{\alpha\beta} P^{\beta\gamma} = P_{\alpha}{}^{\gamma}, \text{ idempotence} \\ P_{\alpha\beta} u^{\beta} = 0 \end{array} \right.$$

Given f^{α} a 4-vector field defined at \underline{M} , we have:

$$(1.14) \quad f^{\alpha} = \bar{f}^{\alpha} + f u^{\alpha} \quad \left\{ \begin{array}{l} \bar{f}^{\alpha} \equiv P_{\beta}{}^{\alpha} f^{\beta} \quad , \quad \bar{f}^{\alpha} \in V_1^3 \\ f \equiv -\frac{1}{c^2} f^{\alpha} u_{\alpha} \quad , \quad f u^{\alpha} \parallel u^{\alpha} \end{array} \right.$$

1.3. The deformation field

(see Ref. [6], [28] and [27]).

The *direct* and *inverse gradients* of the motion are respectively defined by:

$$(1.15) \quad x^{\alpha}{}_{,K} \equiv \overline{x^{\alpha}{}_{,K}} = P^{\alpha}{}_{\beta} x^{\beta}{}_{,K}, \quad \text{and} \quad X^K{}_{,\alpha} = \partial X^K / \partial x^{\alpha}$$

The last quantity is well-defined since (1.9) can be solved for X^K and τ , X^K and τ being independent. Hence,

$$(1.16) \quad X^K = X^K(x^{\alpha}) \quad , \quad \tau = \tau(x^{\alpha})$$

The tensors $x^{\alpha}{}_{,K}$ and $X^K{}_{,\alpha}$ are two-point tensor fields. In a rest frame, they reduce to their classical analogues since

$$(1.17) \quad x^{\alpha}{}_{,K} u_{\alpha} = 0 \quad , \quad X^K{}_{,\alpha} u^{\alpha} = 0$$

Then one can construct the relativistic Green deformation tensor C_{KL} and its reciprocal \bar{C}^{KL}

$$(1.18) \quad \left\{ \begin{array}{l} C_{KL} = g_{\alpha\beta} x^{\alpha}{}_{,K} x^{\beta}{}_{,L} = P_{\alpha\beta} x^{\alpha}{}_{,K} x^{\beta}{}_{,L} \\ \bar{C}^{KL} = g^{\alpha\beta} X^K{}_{,\alpha} X^L{}_{,\beta} \quad , \quad \bar{C}^{KL} C_{LM} = \delta_M^K \end{array} \right.$$

If δx^{α} is an infinitesimal increase in x^{α} corresponding to an increase δX^K of X^K , it is trivial to show that, with

$$\overline{\delta x^{\beta}} = P^{\beta}{}_{\alpha} \delta x^{\alpha}$$

one has:

$$(1.19) \quad (\delta s^*)^2 = g_{\alpha\beta} \overline{\delta x^{\alpha}} \overline{\delta x^{\beta}} = C_{KL} \delta X^K \delta X^L$$

(δs^*) is therefore a measure of an infinitesimal length element of V_1^3 . It is this quantity which must be kept constant when a rigid body motion « à la » Born is defined.

1.4. The electromagnetic field.

We call \underline{F} and \underline{M} respectively the *electromagnetic field two-form* and the *magnetization or polarization two-form*. If \tilde{dx}^α denotes the basic one-forms and \wedge indicates the exterior product, \underline{F} and \underline{M} are given by :

$$(1.20) \quad \underline{F} = \frac{1}{2} F_{\alpha\beta} \tilde{dx}^\alpha \wedge \tilde{dx}^\beta, \quad \underline{M} = \frac{1}{2} M_{\alpha\beta} \tilde{dx}^\alpha \wedge \tilde{dx}^\beta$$

The *electric displacement-magnetic field intensity 2-form* \underline{G} is then defined as:

$$(1.21) \quad \underline{G} \stackrel{\text{def}}{=} \underline{F} - \underline{M}$$

It is of interest to introduce duals of these forms by the formulas:

$$(1.22) \quad \widehat{F}^{\gamma\delta} = \frac{1}{2} \eta^{\alpha\beta\gamma\delta} F_{\alpha\beta} \quad \text{and} \quad \widehat{F}_{\alpha\beta} = \frac{1}{2} \eta_{\alpha\beta\gamma\delta} F^{\gamma\delta}$$

reciprocally

$$(1.23) \quad F^{\alpha\beta} = -\frac{1}{2} \eta^{\gamma\delta\alpha\beta} \widehat{F}_{\gamma\delta} \quad , \quad F_{\alpha\beta} = -\frac{1}{2} \eta_{\gamma\delta\alpha\beta} \widehat{F}^{\gamma\delta}$$

with

$$(1.24) \quad \eta_{\alpha\beta\gamma\delta} = \sqrt{|g|} \varepsilon_{\alpha\beta\gamma\delta} \quad \text{and} \quad \eta^{\alpha\beta\gamma\delta} = -\frac{1}{\sqrt{|g|}} \varepsilon^{\alpha\beta\gamma\delta}$$

where $\varepsilon_{\alpha\beta\gamma\delta}$ is the permutation symbol. The symbolism $\langle \dots, \dots \rangle$ indicating the inner (or contracted) product, it is interesting to consider the two following invariants:

$$(1.25) \quad \varphi = \frac{1}{2} \langle \underline{F}, \underline{F} \rangle = \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \quad , \quad \Phi = \langle \underline{M}, \underline{F} \rangle = \frac{1}{2} M_{\alpha\beta} F^{\alpha\beta}$$

They respectively represent the magnetic energy in free space per unit volume and the magnetic doublet energy per unit volume in matter up to a sign. Furthermore, we define the magnetization 2-form per unit of proper mass and the invariant \mathcal{F}^{KL} constructed from \underline{F} :

$$(1.26) \quad \pi_{\alpha\beta} = M_{\alpha\beta} / \rho$$

$$(1.27) \quad \mathcal{F}^{KL} = X^K_{,\alpha} F^{\alpha\beta} X^L_{,\beta}$$

In Eq. (1.26), ρ is the so-called invariant relativistic density of matter.

1.5. Thermodynamics.

We call η and ϵ respectively the *entropy* and the *internal energy* per unit of proper mass and θ the *thermodynamical temperature*. The foregoing quantities are measured by an observer following the element of matter in its motion. The *free energy density* ψ per unit of proper mass is thus defined by:

$$(1.28) \quad \psi = \epsilon - \theta\eta$$

In the sequel, we consider ψ to be the potential relevant to the theory and take it to be a function of \bar{C}^{KL} , \mathcal{F}^{KL} and θ , *i. e.*,

$$(1.29) \quad \psi = \psi(\bar{C}^{KL}, \mathcal{F}^{KL}, \theta)$$

This form assures that ψ is form-invariant under any coordinate transformation $\{x^\alpha \mapsto x^{\alpha*} : B^{\alpha}_{\beta} = \partial x^{\alpha*} / \partial x^\beta\}$. This statement generalizes the notion of objectivity as enunciated in classical continuum mechanics in Ref. [32] or in special relativistic continuum mechanics in Ref. [33]. Eq. (1.29) generalizes previous forms given in 3-dimensional approaches, *e. g.* in Ref. [34]. We say that (1.29) represents the free energy for a *nonlinear elastic magnetized homogeneous medium*.

1.6. Discontinuities.

We call $\widehat{\mathbf{1}}$ the dual of unity or element of volume in V^4 . It reads:

$$(1.30) \quad \widehat{\mathbf{1}} = d^4v = \sqrt{|g|} d^4x = \sqrt{|g|} \underset{\sim}{\mathbf{d}}x^1 \wedge \underset{\sim}{\mathbf{d}}x^2 \wedge \underset{\sim}{\mathbf{d}}x^3 \wedge \underset{\sim}{\mathbf{d}}x^4$$

The elements of surface on hypersurfaces embedded in V^4 are accordingly defined. For the consideration of discontinuity hypersurfaces such as (Σ) , we need a somewhat extended form of Stokes' theorem, easily provided for by the theory of distributions of L. Schwartz [35], [36] (though the hypotheses of continuity of this theory will not be used in the subsequent developments). If we take the divergence in the sense of distribution theory, we have:

$$(1.31) \quad \int_{(\mathcal{D}) \subset V^4} \{ \nabla \cdot \underset{\sim}{\mathbf{V}} \} d^4v = \int_{(\partial \mathcal{D})} \underset{\sim}{\mathbf{V}} \cdot \underset{\sim}{\mathbf{n}} d^3s$$

If there exists a discontinuity hypersurface (Σ) intersecting (\mathcal{D}) , across which the tensorial field \mathbf{V} suffers a jump $\sigma(\mathbf{V})$, we can write:

$$(1.32) \quad \{ \nabla \cdot \underset{\sim}{\mathbf{V}} \} = \nabla \cdot \underset{\sim}{\mathbf{V}} + \underset{\sim}{\mathbf{n}}_\Sigma \cdot \sigma(\underset{\sim}{\mathbf{V}}) \delta(l)$$

where the divergence in the $r - h - s$ of (1.32) is taken in the usual sense. We have $\sigma(\mathbf{V}) = [\mathbf{V}] = \mathbf{V}^+ - \mathbf{V}^-$; $\delta(l)$ is the one-dimensional delta distribution where l is the absciss defined along the unit local normal \mathbf{n}_Σ to (Σ) , oriented from the region $(-)$ to the region $(+)$. With the local parametrization of (Σ) , one can write:

$$(1.33) \quad \int_{(\mathcal{D})} \mathbf{n}_\Sigma \cdot \sigma[\mathbf{V}(a^i, l)] \delta(l) d^4v = \int_{(\Sigma)} \mathbf{n}_\Sigma \cdot \sigma[\mathbf{V}(\mathbf{M}, 0)] d^3\sigma$$

$$\mathbf{M} \in (\Sigma)$$

Thus, Eq. (1.31) can be written as:

$$(1.34) \quad \int_{(\mathcal{D}-\Sigma)} \nabla \cdot \mathbf{V} d^4v + \int_{(\Sigma)} [\mathbf{V}] \cdot \mathbf{n}_\Sigma d^3\sigma = \int_{(\partial\mathcal{D}-\Sigma)} \mathbf{V} \cdot \mathbf{n} d^3s$$

2. THE VARIATIONAL PRINCIPLE

As time goes on, the material body $(B) \subset E^3$, of boundary (∂B) $[(B_R)$ and (∂B_R) in the reference configuration or Lagrangian configuration described by the coordinates $X^K]$ sweeps out the region (\mathcal{D}) whose boundary is $(\partial\mathcal{D})$, in V^4 , in the interval of time $T = [t_1, t_2] \subset R$. With the domain (\mathcal{D}) of V^4 , we associate the following action:

$$(2.1) \quad A = \int_{(\mathcal{D})} \{ (2\kappa)^{-1} \mathbf{R} - \rho\psi \} \widehat{\mathbf{1}} - \int_{(\mathcal{D})} \varphi \widehat{\mathbf{1}}$$

2.1. What do we vary ?

The purpose of the variational principle given here in is to obtain the complete set of field equations *i. e.*, the Einstein's equations, the dynamical conservation laws (conservation of stress-energy-momentum), the thermodynamical conservation law (conservation of entropy flux since the medium considered is nondissipative), the Maxwell's equations in matter and the constitutive equations (for the stress-energy-momentum tensor, for the entropy and for \mathbf{G}). In the process, we expect to arrive at the jump conditions valid across the discontinuity hypersurface (Σ) . It is therefore clear that, in order to get these results, one must vary the gravitational poten-

tials $g_{\alpha\beta}$, the particle path $(C_{X\kappa})$ in V^4 , the thermodynamical parameter *i. e.*, θ , a quantity related to the electromagnetic field and, in some way, the discontinuity surface itself. Let us examine these different variations.

2.2. Variation of the particle path.

We set the infinitesimally small ε equal to zero along an unperturbed path $(C_{X\kappa})$ and consider the mapping

$$(2.2) \quad x^\alpha \mapsto \tilde{x}^\alpha = x^\alpha + \delta x^\alpha = x^\alpha + \varepsilon \xi^\alpha$$

Here \tilde{x}^α is the perturbed path $(\widetilde{C_{X\kappa}})$ infinitely close to $(C_{X\kappa})$. The resulting variation operator is thus immediately written $\delta = \varepsilon \frac{\xi}{\xi}$ where $\frac{\xi}{\xi}$ indicates the Lie derivative with respect to the 4-vector field ξ^α . Since $g_{\alpha\beta}$ depends upon x^λ , the variation of the path induces a variation of $g_{\alpha\beta}$ such that:

$$(2.3) \quad \delta g_{\alpha\beta} = \varepsilon \frac{\xi}{\xi} g_{\alpha\beta} = 2\varepsilon \xi_{(\alpha;\beta)} = 2(\delta x_{(\alpha);\beta})$$

If $\delta g_{\alpha\beta}$ is a proper variation of the $g_{\alpha\beta}$'s, the total variation of $g_{\alpha\beta}$ is given by:

$$(2.4) \quad \bar{\delta} g_{\alpha\beta} = \delta g_{\alpha\beta} + \delta \delta g_{\alpha\beta} = \delta g_{\alpha\beta} + 2(\delta x_{(\alpha);\beta})$$

2.3. Electromagnetic potentials.

Maxwell's equations in matter read [48]:

$$(2.5) \quad \underline{\mathbf{d}} \underline{\mathbf{F}} = 0 \quad \text{or} \quad (\eta^{\alpha\beta\gamma\delta} F_{\alpha\beta})_{;\delta} = 0$$

$$(2.6) \quad * \underline{\mathbf{d}} * \underline{\mathbf{G}} = \frac{1}{c} \underline{\mathbf{J}} \quad \text{or} \quad G^{\alpha\beta}{}_{;\beta} = \frac{1}{c} J^\alpha$$

where $\underline{\mathbf{J}}$ is the 4-vector current density. From (2.5), one deduces that $\underline{\mathbf{F}}$ can be expressed as:

$$(2.7) \quad \underline{\mathbf{F}} = \underline{\mathbf{d}} \underline{\mathbf{A}} \quad \text{or} \quad F_{\alpha\beta} = 2\nabla_{[\alpha} A_{\beta]} = A_{\beta;\alpha} - A_{\alpha;\beta}$$

where $\underline{\mathbf{A}}$ is the 1-form electromagnetic potential $\underline{\mathbf{A}} = A_\alpha \underline{\mathbf{d}}x^\alpha$. Then, Eq. (2.5) is identically satisfied from Poincaré's lemma. Taking the exterior derivative of (2.6), we get the equation of conservation of current:

$$(2.8) \quad * \underline{\mathbf{d}} * \underline{\mathbf{J}} = 0 \quad \text{or} \quad J^\alpha{}_{;\alpha} = 0$$

We define a time-like hypersurface $\Gamma(x^\alpha) = 0$ i. e.,

$$g^{\alpha\beta}\Gamma_{,\alpha}\Gamma_{,\beta} > 0$$

in all points of (Σ) . Its unit normal in four dimensions is defined by:

$$(2.9) \quad N_\gamma = \Gamma_{,\gamma}(|g^{\alpha\beta}\Gamma_{,\alpha}\Gamma_{,\beta}|)^{-\frac{1}{2}}, \quad g_{\alpha\beta}N^\alpha N^\beta = 1$$

Across such a hypersurface, the fields \underline{F} , \underline{G} and \underline{J} verify the jump relations (if there is no surface current):

$$(2.10) \quad [\tilde{\eta}^{\alpha\beta\gamma\delta}F_{\alpha\beta}]N_\delta = 0$$

$$(2.11) \quad [\tilde{G}^{\alpha\beta}\sqrt{|g|}]N_\beta = 0$$

$$(2.12) \quad [\tilde{J}^\alpha\sqrt{|g|}]N_\alpha = 0$$

In the variational process, the equation (2.5) is considered as a constraint imposed on the 4-potential A^α . To deal with it, we take over the method introduced by Weiss [37] (cf. Grot [25]). Let (\mathcal{F}) be an arbitrary 2-dimensional hypersurface of frontier $(\partial\mathcal{F})$. In integral form, Eq. (2.7) reads:

$$(2.13) \quad \int_{(\mathcal{F})} F_{\mu\lambda} \tilde{d}x^\mu \wedge \tilde{d}x^\lambda = \int_{(\partial\mathcal{F})} A_\mu \tilde{d}x^\mu$$

The variation of (2.13) yields the equation

$$(2.14) \quad \int_{(\mathcal{F})} [\delta F_{\alpha\beta} + F_{\beta\gamma}(\delta x^\gamma)_{;\alpha} - F_{\gamma\alpha}(\delta x^\gamma)_{;\beta}] \tilde{d}x^\alpha \wedge \tilde{d}x^\beta = \int_{(\partial\mathcal{F})} [\delta A_\alpha + A_\gamma(\delta x^\gamma)_{;\alpha}] \tilde{d}x^\alpha$$

but

$$(2.15) \quad \int_{(\partial\mathcal{F})} A_\gamma(\delta x^\gamma)_{;\alpha} \tilde{d}x^\alpha = - \int_{(\partial\mathcal{F})} A_{\gamma;\alpha} \delta x^\gamma \tilde{d}x^\alpha$$

since

$$\int_{(\partial\mathcal{F})} \tilde{d}(A_\gamma \delta x^\gamma) = \int_{(\mathcal{F})} * \tilde{d} * \tilde{d}(A_\gamma \delta x^\gamma) = 0$$

which follows from the application of Stokes' theorem and Poincaré's lemma. We now introduce the so-called *Weiss-gauge-invariant variation*

$$(2.16) \quad \widehat{\delta}A_\alpha = \delta A_\alpha - A_{\gamma;\alpha} \delta x^\gamma$$

and posit Eq. (2.14) to be valid for any 2-dimensional hypersurface (\mathcal{S}) in V^4 . Thus, we obtain the local variation of the field \underline{F} :

$$(2.17) \quad \delta F_{\alpha\beta} = 2(\widehat{\delta A}_{[\beta];\alpha]} - 2F_{\gamma[\beta}(\delta x^\gamma)_{;\alpha]})$$

In Eq. (2.16) we refer to $\widehat{\delta A}_\alpha$ as being the proper variation of the electromagnetic potential and to δA_α as its total variation. The latter involves the variation of the particle path (C_{x^k}) in V^4 .

2.4. Variation of the temperature.

We consider a proper variation $\widehat{\delta\theta}$ of the temperature and a variation $\widetilde{\delta\theta}$ due to the spacetime dependence. Thus, for the total variation of θ , we write:

$$(2.18) \quad \overline{\delta\theta} = \widehat{\delta\theta} + \widetilde{\delta\theta}$$

In order to express the last term, we introduce a new variable Θ in lieu of θ through the definition (cf. Taub [12] or Von Laue [38]):

$$(2.19) \quad \widetilde{\delta\theta} \stackrel{\text{def}}{=} \frac{\partial}{\partial \tau} (\delta\Theta) = u^\beta \nabla_\beta (\delta\Theta)$$

Thus,

$$\overline{\delta\theta} = \widehat{\delta\theta} + u^\beta \nabla_\beta (\delta\Theta)$$

Herein after, $\overline{\delta\theta}$ is used to denote this expression.

2.5. Variation of the discontinuity surface.

We recall the Gaussian equation of (Σ):

$$(2.20) \quad x^\mu = x^\mu(a_i) \quad , \quad \mu = 1, 2, 3, 4 \quad , \quad i = 1, 2, 3$$

A nearby discontinuity surface has the equation:

$$(2.21) \quad \widetilde{x}^\mu = x^\mu(a_i) + \varepsilon \zeta^\mu(a_i)$$

where ε is an infinitesimally small and ζ^μ is a 4-vector field. Given an integral

$I = \int_{(\mathcal{D})} \mathcal{F} \sqrt{|g|} d^4x$, its variation due to the variation of (Σ) is given by:

$$(2.22) \quad \delta_{\Sigma} I = \int_{(\mathcal{D})} \delta_{\Sigma} (\mathcal{F} \sqrt{|g|}) d^4x = \int_{-(\mathcal{D})}^{\mathcal{E}} \varepsilon \frac{\mathcal{E}}{\zeta} (\mathcal{F} \sqrt{|g|}) d^4x$$

$$(2.23) \quad \delta_{\Sigma} I = \int_{(\mathcal{D}) \rightarrow (\mathcal{D}_0)} \varepsilon \nabla^\alpha (\mathcal{F} \sqrt{|g|}) \zeta_\alpha d^4x = \int_{(\Sigma)} \varepsilon [\mathcal{F} \sqrt{|g|}] \zeta_\alpha n^\alpha d^3\sigma$$

In Eq. (2.23), we have applied the Green-Gauss theorem for a thin shell (\mathfrak{D}_0) enclosing (Σ) and taken account of the continuity of the 4-vector field ζ^α across (Σ) .

For convenience, we introduce the following notation:

$$(2.24) \quad f^*(X^\Delta) = f[x^\alpha(X^\Delta)]$$

then,

$$(2.25) \quad \sqrt{|g^*|} = J \sqrt{|g|}$$

where the jacobian J is defined as:

$$(2.26) \quad J \equiv \frac{1}{ic} \varepsilon_{\mu\nu\sigma\tau} \frac{\partial x^\mu}{\partial X^1} \frac{\partial x^\nu}{\partial X^2} \frac{\partial x^\sigma}{\partial X^3} u^\tau$$

For instance, we note $f_{,\alpha}^* = \partial f / \partial X^\alpha$, and tensorial equations posited valid in any system of coordinates can be indifferently written in the x^α -system or in the X^Δ -system.

2.6. Constraints.

The following constraints can be considered in the variational process:

(i) constant magnitude of the 4-velocity; we thus introduce a Lagrange multiplier \mathcal{M} the significance of which remains to be determined,

(ii) Eq. (2.13) which gives the variation of the field \underline{F} ,

(iii) continuity equation; with the domain (\mathfrak{D}) of V^4 , we associate the following integral of the motion:

$$(2.27) \quad \int_{(\mathfrak{D})} \rho \sqrt{|g|} d^4x = \text{const.}$$

where ρ is the *invariant density of matter*. The local differential form of (2.27) is known to be:

$$(2.28) \quad \nabla_\alpha(\rho u^\alpha) = 0 \quad \text{in } (\mathfrak{D} - \Sigma)$$

and does not need to be derived again. Upon use of the notation (2.24), the local integral of the motion can be written as:

$$(2.29) \quad \rho^* \sqrt{|g^*|} = \rho \sqrt{|g|}$$

By integration of Eq. (2.28) over a region consisting of a thin shell (\mathfrak{D}_0)

enclosing (Σ) and application of the Green-Gauss theorem, we obtain :

$$(2.30) \quad \int_{(\mathcal{D}) \rightarrow (\mathcal{D}_0)} \sqrt{|g|} \nabla_\alpha (\rho u^\alpha) d^4x = \int_{(\Sigma)} [\sqrt{|g|} \rho u^\alpha] n_\alpha d^3\sigma = 0$$

This relation is posited to be valid everywhere on (Σ) and thus yields the local form:

$$(2.31) \quad [\rho u^\alpha \sqrt{|g|}] n_\alpha = 0 \quad \text{across } (\Sigma)$$

In order to take account of Eq. (2.27) in the variation, we write:

$$(2.32) \quad \delta \int_{(\mathcal{D}-\Sigma)} \rho \sqrt{|g|} d^4x = \delta \int_{(\mathcal{D}^*-\Sigma^*)} \rho^* \sqrt{|g^*|} d^4X = 0$$

where, from here on, we shall denote (\mathcal{D}^*) , $(\partial\mathcal{D}^*)$ and (Σ^*) , the region or hypersurfaces (\mathcal{D}) , $(\partial\mathcal{D})$ and (Σ) referred to the X^Δ -system and we note:

$$(2.33) \quad \sqrt{|g^*|} d^4X \equiv ic \sqrt{|g^*|} d^3Xd\tau = ic \sqrt{|g^*|} dX^1 dX^2 dX^3 d\tau$$

the element of volume in this system. With (2.26), we have:

$$(2.34) \quad d^4x = J d^4X$$

$$d^4x = \varepsilon_{\mu\nu\sigma\tau} \frac{\partial x^\mu}{\partial X^1} \frac{\partial x^\nu}{\partial X^2} \frac{\partial x^\sigma}{\partial X^3} u^\tau d^3Xd\tau$$

Since (\mathcal{D}^*) is obviously not perturbed by the variation, we deduce, Eq. (2.32) being valid everywhere in $(\mathcal{D}^* - \Sigma^*)$, that:

$$(2.35) \quad \delta (\ln \rho^*) + \delta (\ln \sqrt{|g^*|}) = 0$$

We note that:

$$(2.36) \quad \bar{\delta} \sqrt{|g|} = -\frac{1}{2} \sqrt{|g|} g^{\alpha\beta} \bar{\delta} g_{\alpha\beta}$$

Hence, upon use of (2.4), we transform (2.35) to:

$$(2.37) \quad \frac{\bar{\delta} \rho^*}{\rho^*} - \frac{1}{2} g^{\mu\nu} [\bar{\delta} g_{\mu\nu} + 2(\bar{\delta} x_{(\mu};_{\nu)})] = 0$$

(iv) incompressibility: for the sake of completeness, we may impose

a condition of incompressibility on the material. Such a condition is:

$$(2.38) \quad P^{\alpha\beta}[\delta g_{\alpha\beta} + 2(\delta x_{(\alpha};\beta)] = 0$$

This follows from the relation $\delta(\ln \sqrt{|g|}) = 0$. The latter relation is the equivalent of the condition $\text{div } \underline{v} = 0$ encountered in classical continuum mechanics. To take account of (2.38), the introduction of the Lagrange multiplier p referred to as the *mechanical pressure* is necessitated.

(v) on the frontier ($\partial\mathcal{D}$) we shall assume that:

$$(2.39) \quad \delta x_\alpha = 0 \quad , \quad \delta g_{\alpha\beta} = \delta g_{\alpha\beta,\gamma} = 0 \quad , \quad \widehat{\delta}A_\alpha = 0 \quad , \quad \delta \Theta = 0$$

and on the singular surface (Σ)

$$(2.40) \quad [\delta x_\alpha] = [\widehat{\delta}A_\alpha] = [\delta g_{\alpha\beta}] = [\delta g_{\alpha\beta,\gamma}] = 0$$

At this point, we do not set $[\delta \Theta] = 0$.

2.7. Form of the variational principle.

Following the general scheme of variational principles in continuum physics as enunciated in Ref. [27], we write the desired variational principle in the form:

$$(2.41) \quad \Delta A + \delta W + \delta \overline{W} = 0$$

where A is given by (2.1) and we have set:

$$(2.42) \quad \delta W = - \int_{(\mathcal{D})} \frac{1}{2} \rho \mathcal{M}(g_{\alpha\beta} u^\alpha u^\beta + c^2) \sqrt{|g|} d^4 x$$

$$(2.43) \quad \delta \overline{W} = - \int_{(\mathcal{D}-\Sigma)} \left(\frac{1}{c} J^\alpha \widehat{\delta}A_\alpha + \rho \eta \widehat{\delta}\theta \right) d^4 x - \int_{(\Sigma)} \frac{1}{c} K^{\alpha\beta} \widehat{\delta}A_\beta n_\alpha d^3 \sigma$$

The expression (2.42) represents the only constraint introduced in integral form. We discard the case of incompressibility for the time being. In Eq. (2.43), following the tradition established by Lagrange and Piola, we have introduced, in a selective manner, indeterminate multipliers for the basic arguments varied in the Lagrangian density in ($\mathcal{D} - \Sigma$) and on (Σ). The terminology associates a physical significance to these quantities: J^α is the *4-vector current*, η is the *entropy density* per unit of proper mass

and $K^{\alpha\beta}$ is the *surface current bivector* prescribed on (Σ) . We have by definition:

$$(2.44) \quad K^\beta = \frac{1}{2} (K^{\alpha\beta} - K^{\beta\alpha}) n_\alpha$$

Note that we did not introduce indeterminate multipliers on $(\partial\mathcal{D} - \Sigma)$ because of (2.39) nor did we introduce such a quantity for δx_α in $(\mathcal{D} - \Sigma)$. It would correspond to the 4-body force that we need not consider here; the only forces appearing in the treatment are due to the deformation field, the flux of matter and the electromagnetic field and can be expressed as divergences of second order tensors, thus included in the total stress-energy-momentum tensor.

The symbol Δ appearing in Eq. (2.41) stands for:

$$(2.45) \quad \Delta \Rightarrow (\delta g_{\alpha\beta}, \delta x_\alpha, \widehat{\delta\theta}, \delta\Theta, \widehat{\delta A}_\alpha, \delta_\Sigma)$$

The variation is understood to be carried out in the « undeformed » X^Δ -system of coordinates which is unperturbed in the variation. Hence all tensor fields appearing in (2.41) are written with the help of the formalism (2.24). Note that the δ -variation of the coordinates commutes with the derivatives with respect to the generalized Lagrangian coordinates

$$X^\Delta = (X^K, ic\tau).$$

3. THE VARIATION

3.1. Intermediate results.

We shall only give some important steps and the results of the variation (2.41). First, we note the elementary variations [here we omit the notation (2.24)]:

$$(3.1) \quad \delta X^{K, \alpha} = -X^{K, \beta} (\delta x^\beta)_{, \alpha}$$

$$(3.2) \quad \delta u^\alpha = \delta \left(\frac{\partial x^\alpha}{\partial \tau} \right) = u^\beta (\delta x^\alpha)_{, \beta}$$

$$\delta \sqrt{|g|} = -\sqrt{|g|} g^{\alpha\beta} (\delta x_\alpha)_{, \beta}$$

$$(3.3) \quad \bar{\delta} \sqrt{|g|} = -\frac{1}{2} \sqrt{|g|} g^{\alpha\beta} \bar{\delta} g_{\alpha\beta}$$

$$(3.4) \quad \sqrt{|g|} \delta \rho + \rho \bar{\delta} \sqrt{|g|} = 0$$

The latter is none other than Eq. (2.37). With the definitions (1.18)₂ and (1.27) and on account of (2.17), it is easily found that:

$$(3.5) \quad \bar{\delta} C^{-1\text{KL}} = -X^{K,\alpha} X^{L,\beta} \bar{\delta} g_{\alpha\beta}$$

$$(3.6) \quad \bar{\delta} \mathcal{F}^{\text{KL}} = -2X^{K,\alpha} X^{L,\rho} F^{\beta\rho} \bar{\delta} g_{\alpha\beta} + 2X^{K,\beta} X^{L,\alpha} (\widehat{\delta A}_{[\alpha];\beta]}$$

Here we have used the fact that:

$$\bar{\delta} g^{\alpha\beta} = -g^{\alpha\gamma} g^{\beta\delta} \bar{\delta} g_{\gamma\delta}$$

Hence, carrying the results (3.5), (3.6) and the variation (2.18) in the variation of ψ , we get:

$$(3.7) \quad \bar{\delta} \psi = \frac{\partial \psi}{\partial C^{-1\text{KL}}} \bar{\delta} C^{-1\text{KL}} + \frac{\partial \psi}{\partial \mathcal{F}^{\text{KL}}} \bar{\delta} \mathcal{F}^{\text{KL}} + \frac{\partial \psi}{\partial \theta} \bar{\delta} \theta$$

that is:

$$(3.8) \quad \begin{aligned} \rho \bar{\delta} \psi = & -\frac{1}{2} (t^{\beta\alpha} - M^{\beta\sigma} F^{\alpha}_{\sigma}) \bar{\delta} g_{\alpha\beta} - M^{\beta\alpha} (\widehat{\delta A}_{\alpha};\beta] \\ & - \rho \bar{\eta} \widehat{\delta} \theta - \rho \bar{\eta} (\delta \Theta)_{;\beta} u^{\beta} \end{aligned}$$

where we have defined the *stress-energy-momentum tensor* $t^{\beta\alpha}$ due to the deformation field, the *magnetization tensor* $M^{\gamma\sigma}$ per unit of proper volume and the quantity $\bar{\eta}$ by:

$$(3.9) \quad t^{\beta\alpha} \equiv -2\rho \left(\frac{\partial \psi}{\partial C^{-1\text{KL}}} X^{L,\beta} + \frac{\partial \psi}{\partial \mathcal{F}^{\text{KL}}} X^{L,\rho} F^{\beta\rho} \right) X^{K,\alpha}$$

$$(3.10) \quad M^{\gamma\sigma} \equiv -2\rho \frac{\partial \psi}{\partial \mathcal{F}^{\text{KL}}} X^{K,[\gamma} X^{L,\sigma]}$$

$$(3.11) \quad \bar{\eta} \equiv -\frac{\partial \psi}{\partial \theta}$$

In the sequel, we also need the following results:

(a)

$$(3.12) \quad \bar{\delta} (g_{\alpha\beta} u^{\alpha} u^{\beta}) = u^{\alpha} u^{\beta} \bar{\delta} g_{\alpha\beta}$$

since

$$\bar{\delta} g_{\alpha\beta} = g_{\alpha\beta;\gamma} \delta x^{\gamma} = 0$$

from Ricci's lemma.

(b) Upon use of (2.17) and (3.3), the variation of the second term of (2.1) is [excluding the variation due to the variation of (Σ)]:

$$\begin{aligned} \bar{\delta} \left\{ - \int_{(\mathbb{D}^*)} \varphi^* \sqrt{|g^*|} d^4 X \right\} \\ = - \int_{(\mathbb{D}^*)} \bar{\delta} \varphi^* \sqrt{|g^*|} d^4 X + \int_{(\mathbb{D}^*)} \frac{1}{2} \varphi^* \sqrt{|g^*|} g^{\alpha\beta} \bar{\delta} g_{\alpha\beta}^* d^4 X \end{aligned}$$

i. e.,

(3.13)

$$- \bar{\delta} \int_{(\mathbb{D}^*)} \varphi^* \sqrt{|g^*|} d^4 X = - \int_{(\mathbb{D}^*)} \left\{ \frac{1}{2} T_{(\text{em. v})}^{\alpha\beta} \bar{\delta} g_{\alpha\beta}^* + F^{\alpha\beta} (\widehat{\delta} A_a)^* \right\} \sqrt{|g^*|} d^4 X$$

where we have defined the *stress-energy-momentum* $T_{(\text{em. v})}^{\alpha\beta}$ of the *electromagnetic field in vacuum* by:

$$(3.14) \quad T_{(\text{em. v})}^{\alpha\beta} \equiv - F^{\alpha}{}_{\gamma} F^{\gamma\beta} + \frac{1}{4} F^{\rho\sigma} F_{\sigma\rho} g^{\alpha\beta}$$

(c) Upon using (2.23) we find the variation due to the variation of (Σ) :

$$(3.15) \quad \delta_{\Sigma} A = \int_{(\Sigma)} \varepsilon \left[\left\{ (2\kappa)^{-1} \mathbf{R} - \rho\psi - \varphi \right\} \sqrt{|g|} \right]_{\alpha}^* n^{\alpha} d^3 \sigma^*$$

(d) The variation of the term involving \mathbf{R} in (2.1), due to the combined variations of the metric and of the particle path closely follows the classical procedure (cf. Taub [12], Landau and Lifshitz [39], or Weber [40]). With (3.3), we obtain:

$$(3.16) \quad \bar{\delta}(\mathbf{R}\sqrt{|g|}) = \sqrt{|g|} g^{\alpha\beta} \bar{\delta} R_{\alpha\beta} + \left(R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} \mathbf{R} \right) \sqrt{|g|} \bar{\delta} g_{\alpha\beta}$$

of which the first term in the $r - h - s$ can be rearranged. Using the definitions (1.5-6) then (1.4), we have [47]:

$$\begin{aligned} g^{\alpha\beta} \bar{\delta} R_{\alpha\beta} &= g^{\alpha\beta} \left\{ (\delta \Gamma_{\alpha}{}^{\gamma})_{;\beta} - (\delta \Gamma_{\alpha}{}^{\gamma})_{;\gamma} \right\} \\ &= (g^{\alpha\gamma} \delta \Gamma_{\alpha}{}^{\beta} - g^{\alpha\beta} \delta \Gamma_{\alpha}{}^{\gamma})_{;\gamma} \\ (3.17) \quad &= \left\{ (g^{\beta\delta} g^{\alpha\gamma} - g^{\alpha\beta} g^{\gamma\delta}) \delta g_{\alpha\beta} \right\}_{;\delta} \end{aligned}$$

We now integrate this result over $(\mathbb{D})^*$ and, applying (1.34) while taking account of (2.39)₂, we get:

$$(3.18) \quad \int_{(\mathbb{D})^*} g^{\alpha\beta} \delta R_{\alpha\beta} \sqrt{|g|} d^4X = \int_{(\Sigma)^*} [(g^{\alpha\beta} g^{\gamma\delta} - g^{\beta\delta} g^{\alpha\gamma}) \sqrt{|g|} \delta g_{\alpha\beta;\gamma}]^* n_\delta d^3\sigma$$

We thus gather the contributions (3.8), (3.12), (3.13), (3.15), (3.16) and (2.43) to yield:

$$(3.19) \quad \begin{aligned} & \frac{1}{2} \int_{(\mathbb{D}-\Sigma)^*} (\kappa^{-1} A^{\alpha\beta} - T^{\alpha\beta}) \delta g_{\alpha\beta} \sqrt{|g|} d^4X - \int_{(\mathbb{D})^*} T^{\alpha\beta} (\delta x_\alpha)_\beta \sqrt{|g|} d^4X \\ & - \int_{(\mathbb{D})^*} G^{\alpha\beta} (\widehat{\delta A}_\alpha)_\beta \sqrt{|g|} d^4X - \int_{(\mathbb{D}-\Sigma)^*} \frac{1}{c} J^\alpha \widehat{\delta A}_\alpha \sqrt{|g|} d^4X \\ & - \int_{(\Sigma)^*} \frac{1}{c} K^{\alpha\beta} \widehat{\delta A}_\beta n_\alpha d^3\sigma \\ & + \int_{(\mathbb{D})^*} \rho (\widehat{\eta} - \eta) \widehat{\delta \theta} \sqrt{|g|} d^4X + \int_{(\mathbb{D})^*} \rho \widehat{\eta} (\delta \Theta)_{;\beta} u^\beta \sqrt{|g|} d^4X \\ & + \int_{(\Sigma)^*} (2\kappa)^{-1} [(g^{\alpha\beta} g^{\gamma\delta} - g^{\beta\delta} g^{\alpha\gamma}) \sqrt{|g|} \delta g_{\alpha\beta;\gamma}]^* n_\delta d^3\sigma \\ & + \int_{(\Sigma)^*} \varepsilon [\{ (2\kappa)^{-1} \mathbf{R} - \rho \psi - \varphi \} \sqrt{|g|}]^* \zeta_\alpha n^\alpha d^3\sigma = 0 \end{aligned}$$

where the stars superposed to letters stand for the notation (2.24). In Eq. (3.19) we have defined the *Einstein-Cartan tensor* $A^{\alpha\beta}$ and the *electric displacement-magnetic field intensity bivector* $G^{\alpha\beta}$ according to (1.7) and (1.21) respectively. We also introduced the *total stress-energy-momentum tensor* $T^{\alpha\beta}$ and the *stress-energy-momentum tensor* $T_{(em.m)}^{\alpha\beta}$ of the electromagnetic field in matter by:

$$(3.20) \quad T^{\alpha\beta} \equiv \rho \mathcal{M} u^\alpha u^\beta - t^{\beta\alpha} + T_{(em.m)}^{\alpha\beta}$$

$$(3.21) \quad T_{(em.m)}^{\alpha\beta} \equiv -F^\alpha{}_\gamma G^{\gamma\beta} + \frac{1}{4} F^{\rho\sigma} F_{\rho\sigma} g^{\alpha\beta}$$

The latter definition is different from those of Sedov [18], Abraham [41] or Minkowski [42] but is similar to that used by Grot and Eringen [6] and Maugin [27] in SR, and is closely related to that of De Groot and Suttorp [44]. We believe, following the arguments given in Ref. [44] that it assumes the most satisfactory form though, in the present case, $T_{(em.m)}^{\alpha\beta}$ is not a symmetric tensor. Nevertheless, note that the total energy-momentum tensor given by Eq. (3.20) will be posited *symmetric* since, as no spin occurs in the treatment, we shall require the *angular momentum density* $m_{\alpha\beta\gamma}$ defined by

$$m_{\alpha\beta\gamma} \equiv T_{\alpha\beta}x_\gamma - T_{\alpha\gamma}x_\beta = 2T_{\alpha[\beta}x_{\gamma]}$$

to satisfy the conservation law

$$(3.22) \quad m^{\alpha}_{\beta\gamma;\alpha} = 0$$

or, with the equation (3.28) obtained below

$$(3.23) \quad T^{[\alpha\beta]} = 0$$

The nonsymmetry of $T_{(em.m)}^{\alpha\beta}$ is therefore counterbalanced by that of the tensor $t^{\beta\alpha}$ which has no reason to be symmetric. We note that $T^{\alpha\beta}$ is none other that:

$$T^{\alpha\beta} = 2 \frac{\partial}{\partial g_{\alpha\beta}} \left[\frac{1}{2} \rho \mathcal{M}(g_{\alpha\beta} u^\alpha u^\beta + c^2) + \rho\psi + \varphi \right]$$

If ψ depended upon derivatives of $g_{\alpha\beta}$, we should replace $\partial/\partial g_{\alpha\beta}$ by the Euler-Lagrange derivative $\delta/\delta g_{\alpha\beta}$.

We can now apply for each term of the form $\underset{\sim}{C} \cdot (\delta\underset{\sim}{D})_{;\alpha}$ the rule:

$$\underset{\sim}{C} \cdot (\delta\underset{\sim}{D})_{;\alpha} = (\underset{\sim}{C} \cdot \delta\underset{\sim}{D})_{;\alpha} - \underset{\sim}{C}_{;\alpha} \cdot \delta\underset{\sim}{D}$$

and use the theorem (1.34) for the divergence terms. Taking account of (2.39), we obtain:

$$(3.24) \quad \int_{(\mathbb{D}-\Sigma)^*} \overset{*}{T}^{\alpha\beta}_{;\beta} \sqrt{|g^*|} \delta x_\alpha d^4X + \int_{(\Sigma)^*} \left[\overset{*}{T}^{\alpha\beta} \sqrt{|g^*|} \right] n_\beta \delta x_\alpha d^3\sigma$$

$$+ \frac{1}{2} \int_{(\mathbb{D}-\Sigma)^*} (\kappa^{-1} \overset{*}{A}^{\alpha\beta} - \overset{*}{T}^{\alpha\beta}) \sqrt{|g^*|} \delta g_{\alpha\beta} d^4X$$

$$+ \int_{(\mathbb{D}-\Sigma)^*} \left\{ (\overset{*}{G}^{\alpha\beta} \sqrt{|g^*|})_{;\beta} - \frac{1}{c} \overset{*}{J}^\alpha \sqrt{|g^*|} \right\} \widehat{\delta A}_\alpha d^4X + \int_{(\mathbb{D}-\Sigma)^*} \rho (\overset{*}{\eta} - \overset{*}{\eta}) \widehat{\delta \theta} \sqrt{|g^*|} d^4X$$

continued

(3.24) continued

$$\begin{aligned}
 & + \int_{(\Sigma)}^* \left\{ \left[\tilde{G}^{\alpha\beta} \sqrt{|g|} \right]^* n_\beta - \frac{1}{c} K^{\beta\alpha} n_\beta \right\} \widehat{\delta} A_\alpha d^3\sigma - \int_{(\Sigma)}^* \left[\tilde{\rho} \tilde{\eta} u^\alpha \sqrt{|g|} \delta\Theta \right]^* n_\alpha d^3\sigma \\
 & - \int_{(\mathcal{D}-\Sigma)}^* \left(\tilde{\rho} \tilde{\eta} u^\beta \sqrt{|g|} \right)_{;\beta} \delta\Theta d^4X \\
 & + \int_{(\Sigma)}^* (2\kappa)^{-1} \left[(g^{\alpha\beta} g^{\gamma\delta} - g^{\beta\delta} g^{\alpha\gamma}) \sqrt{|g|} \delta g_{\alpha\beta;\gamma} \right]^* n_\delta d^3\sigma \\
 & + \int_{(\Sigma)}^* \varepsilon \left[\{ (2\kappa)^{-1} \mathbf{R} - \rho\psi - \varphi \} \sqrt{|g|} \right]^* \zeta_\alpha n^\alpha d^3\sigma = 0
 \end{aligned}$$

3.2. Field equations.

If Eq. (3.24) is posited to be valid for any volume in $(\mathcal{D} - \Sigma)$ and any hypersurface (Σ) and for any variations $\delta x_\alpha, \delta g_{\alpha\beta}, \widehat{\delta} A_\alpha, \delta\Theta, \widehat{\delta}\theta$ satisfying the constraints (2.39) and (2.40), then we deduce from (3.24) the local field equations in $(\mathcal{D} - \Sigma)$ and across (Σ) . Furthermore, the field equations can be written in tensor form in any system of coordinates. Therefore, we abandon the formalism (2.24) and simply write the equations in the x^α -system. We obtain:

(i) in $(\mathcal{D} - \Sigma)$

$$(3.25) \quad \eta = \bar{\eta} \equiv - \frac{\partial\psi}{\partial\theta}$$

$$(3.26) \quad (\rho\eta u^\alpha)_{;\alpha} = 0$$

$$(3.27) \quad A^{\alpha\beta} = \kappa T^{\alpha\beta}$$

$$(3.28) \quad T^{\alpha\beta}_{;\beta} = 0$$

$$(3.29) \quad G^{\alpha\beta}_{;\beta} = \frac{1}{c} J^\alpha$$

(ii) across (Σ)

$$(3.30) \quad [\sqrt{|g|} T^{\alpha\beta}] n_\beta = 0$$

$$(3.31) \quad [\sqrt{|g|} G^{\alpha\beta}] n_\beta = \frac{1}{c} K^\alpha$$

$$(3.32) \quad \underline{\underline{[\sqrt{|g|} (g^{\alpha\beta} g^{\gamma\delta} - g^{\beta\delta} g^{\alpha\gamma})]}]} = 0$$

$$(3.33) \quad \underline{\underline{[\sqrt{|g|} (g^{\alpha\beta} g^{\gamma\delta} - g^{\beta\delta} g^{\alpha\gamma}) (\delta_{\alpha}^{\sigma} \Gamma_{\beta, \gamma}^{\lambda} + \delta_{\beta}^{\sigma} \Gamma_{\alpha, \gamma}^{\lambda})]}]} n_{\delta} = 0$$

$$(3.34) \quad \underline{\underline{[\{ (2\kappa)^{-1} \mathbf{R} - \rho\psi - \varphi \} \sqrt{|g|} [e_{\alpha}^{\zeta} n^{\alpha} - [\rho\eta u^{\alpha} \sqrt{|g|} \delta\Theta] n_{\alpha}]}]} = 0$$

Here Eqs. (3.32) and (3.33) are obtained by setting the coefficients of $\delta g_{\alpha\beta}$ and $\delta g_{\alpha\beta, \gamma}$ separately equal to zero (in the starred system where derivatives and variations commute) after having explicitly expressed the covariant derivatives. Finally we have omitted the stars.

Eq. (3.25) constitutes the definition of the entropy density as a function of the free energy while Eq. (3.26) represents the conservation of entropy flux since the process considered is nondissipative. The ten equations (3.27) [in fact only six of them are independent as a consequence of Bianchi's identities] are the Einstein's equations. Eq. (3.28) is the conservation law for the energy-momentum. Eq. (3.29) is the group of Maxwell's equations that supplements Eqs. (2.5) and (2.8). Eqs. (3.30) and (3.31) are the jump relations which supplement Eqs. (3.28) and (3.29) respectively. The jump relations (3.32) and (3.33) concerning the geometry were already extensively discussed by Taub [12], O'Brien and Synge [44] and Lichnerowicz [11]. We shall not discuss them here. However, we point out that, with Lichnerowicz's conditions of continuity for $g_{\alpha\beta}$ (cf. § 1.1.), these two jump relations are satisfied with:

$$\underline{\underline{[g_{\alpha\beta}]}]} = \underline{\underline{[g_{\alpha\beta, \gamma}]}]} = 0 \quad \text{across } (\Sigma)$$

It remains to analyze the last jump condition (3.34) which, because of the presence of the deformation field and of the electromagnetic field in matter, slightly differs from the relation found by Taub for a perfect fluid [12]. For comments, the latter author refers to the classical analogue (cf. Taub [45]). We shall examine this jump relation in section 5. Before we need determine the expression of the unknown \mathcal{M} .

4. DETERMINATION OF THE LAGRANGE MULTIPLIER

Here we follow the procedure used by Taub for perfect fluids in GR [12] and extended to solids in SR [27]. In order to determine \mathcal{M} , we perform the differentiation indicated in Eq. (3.28) and contract the result with

u_α *i. e.*, we project the resulting 4-vector along the world velocity. We note that, by using (3.29) and (2.5), the divergence of the electromagnetic stress-energy-momentum tensor $T_{(em, m)}^{\alpha\beta}$ can be written as a volume force $f_{(em)}^\alpha$ according to the relation:

$$(4.1) \quad T_{(em, m)}^{\alpha\beta}{}_{;\beta} = -f_{(em)}^\alpha$$

with

$$(4.2) \quad f_{(em)}^\alpha = f_{(L)}^\alpha + f_{(S-G)}^\alpha$$

$$(4.3) \quad f_{(L)}^\alpha = \frac{1}{c} J^\gamma F^\alpha{}_\gamma, \quad f_{(S-G)}^\alpha = \frac{1}{2} M^{\sigma\rho} F_{\sigma\rho}{}^{;\alpha}$$

of which the former is the *Lorentz force* and the latter is the *Stern-Gerlach force* in a magnetized medium. In the subsequent developments, we shall assume that the current is only due to convection *i. e.*, we shall take:

$$(4.4) \quad J^\gamma P_{\alpha\gamma} = 0, \quad J^\gamma \propto u^\gamma$$

thus,

$$(4.5) \quad J_\gamma F^{\alpha\gamma} u_\alpha = 0$$

from the skewsymmetry of $F^{\alpha\gamma}$.

Performing the differentiation in (3.28) while taking account of (3.20), we obtain:

$$(4.6) \quad \rho \mathcal{M}_{;\beta} u^\beta u^\alpha + \rho \mathcal{M} u^\alpha{}_{;\beta} u^\beta - t^{\beta\alpha}{}_{;\beta} = f_{(L)}^\alpha + f_{(S-G)}^\alpha$$

where we used the continuity equation (2.28). Upon contracting Eq. (4.6) with u_α and using (4.5), (1.11)₂ and the fact that:

$$(4.7) \quad u^\alpha{}_{;\beta} u_\alpha = 0$$

which follows from (1.11)₂, we get:

$$(4.8) \quad -\rho c^2 \dot{\mathcal{M}} - t^{\beta\alpha}{}_{;\beta} u_\alpha = \frac{1}{2} M^{\sigma\rho} \dot{F}_{\sigma\rho}$$

where we used the notation

$$(4.9) \quad \frac{\partial}{\partial \tau} \underset{\sim}{A} = \dot{\underset{\sim}{A}} = \underset{\sim}{A}_{;\alpha} u^\alpha$$

With (1.17)₂, (3.9) and (3.10), we remark that we can write

$$(4.10) \quad t^{\beta\alpha}{}_{;\beta} u^\alpha + \frac{1}{2} M^{\sigma\rho} \dot{F}_{\sigma\rho} = -2\rho \left[\frac{\partial \psi}{\partial \mathcal{C}^{KL}} X^{L,\beta} + \frac{\partial \psi}{\partial \mathcal{F}^{KL}} X^L{}_{,\rho} F^{\beta\rho} \right] (X^{K,\alpha})_{;\beta} u_\alpha - \rho \frac{\partial \psi}{\partial \mathcal{F}^{KL}} X^{K,\sigma} X^{L,\rho} \dot{F}_{\sigma\rho}$$

If we note that:

$$(4.11) \quad (X^{K,\sigma})_{;\beta} u_\alpha = (X^{K,\sigma})^{;\alpha} u_\alpha \equiv \dot{\bar{X}}^{K,\beta}$$

$$(4.12) \quad \dot{\bar{C}}^{KL} = 2X^{K,\beta} \dot{\bar{X}}^{L,\beta}$$

$$(4.13) \quad \dot{\bar{\mathcal{F}}}^{KL} = 2X^{L,\rho} F^{\beta\rho} \dot{\bar{X}}^{K,\beta} + X^{K,\sigma} \dot{\bar{F}}_{\sigma\rho} X^{L,\rho}$$

then, the *r.h.s* of Eq. (4.10) is nothing else than:

$$-\rho \left[\frac{\partial\psi}{\partial\dot{\bar{C}}^{KL}} \dot{\bar{C}}^{KL} + \frac{\partial\psi}{\partial\dot{\bar{\mathcal{F}}}^{KL}} \dot{\bar{\mathcal{F}}}^{KL} \right]$$

But from (1.29) we have:

$$\rho\dot{\psi} = \rho \left[\frac{\partial\psi}{\partial\dot{\bar{C}}^{KL}} \dot{\bar{C}}^{KL} + \frac{\partial\psi}{\partial\dot{\bar{\mathcal{F}}}^{KL}} \dot{\bar{\mathcal{F}}}^{KL} + \frac{\partial\psi}{\partial\theta} \dot{\theta} \right]$$

or using (1.28), (3.25) and the conservation of entropy flux (3.26), this gives:

$$(4.14) \quad \rho\dot{\epsilon} = \rho \left[\frac{\partial\psi}{\partial\dot{\bar{C}}^{KL}} \dot{\bar{C}}^{KL} + \frac{\partial\psi}{\partial\dot{\bar{\mathcal{F}}}^{KL}} \dot{\bar{\mathcal{F}}}^{KL} \right]$$

Thus, we can write Eq. (4.8) as:

$$(4.15) \quad c^2 \dot{\mathcal{M}} = \dot{\epsilon}$$

Integrating over proper time and introducing the constant of integration c^2 (the rest energy per unit of proper mass), we obtain:

$$(4.16) \quad \mathcal{M} = 1 + \frac{\epsilon}{c^2}$$

Therefore, the final form of the total stress-energy-momentum tensor (3.20) reads:

$$(4.17) \quad T^{\alpha\beta} = \rho \left(1 + \frac{\epsilon}{c^2} \right) u^\alpha u^\beta - t^{\beta\alpha} + T^{\alpha\beta}_{(em,m)}$$

where

$$(4.18) \quad \epsilon = \psi - \frac{\partial\psi}{\partial\theta} \theta = \epsilon(\dot{\bar{C}}^{KL}, \dot{\bar{\mathcal{F}}}^{KL}, \theta)$$

is the so-called *internal energy*, a quantity which reduces to the classical internal energy of 3-dimensional continuum mechanics in a local rest frame.

As far as this reduction is concerned, we must emphasize here that all dynamical equations and electromagnetic equations given above reduce in a satisfactory manner to their special relativistic form as given, for instance, by Grot and Eringen [6] in a local inertial frame. Moreover, in the limit $c \mapsto \infty$, these reduce to those of the classical theory of finite deformations in electromagnetoelasticity such as given by Dixon and Eringen [46] (if we neglect the quadrupole terms in the latter theory). In these limits, the constitutive equations (3.9) and (3.10) go to constitutive equations satisfying the principle of objectivity of Söderholm [33] for the special relativistic case and to constitutive equations satisfying the classical principle of objectivity [32] for the classical case.

Remark. — If we consider an incompressible medium then, it is not difficult to show that the condition (2.38) leads to adding a term $-pF^{\alpha\beta}$ to the expression (4.17) where, we emphasize, p is an *unknown* to be determined upon solution of a peculiar well-posed problem.

Let us finally note that, with the known value of \mathcal{M} given by Eq. (4.16), remarking that we have:

$$(4.19) \quad \begin{aligned} t^{\beta\alpha}{}_{;\beta}u_\alpha &= (t^{\beta\alpha}u_\alpha)_{;\beta} - t^{\beta\alpha}u_{\alpha;\beta} \\ &= -t^{\beta\alpha}u_{\alpha;\beta} \end{aligned}$$

since

$$t^{\beta\alpha}u_\alpha = 0$$

from (3.9) and (1.17)₂, Eq. (4.8) can be written as:

$$(4.20) \quad \rho\dot{\epsilon} = t^{\beta\alpha}u_{\alpha;\beta} - \frac{1}{2}M^{\sigma\rho}\dot{\overline{F}}_{\sigma\rho}$$

This equation which gives the proper time rate of the internal energy ϵ may be considered as the conservation law corresponding to this quantity for a nondissipative process, in $(\mathcal{D} - \Sigma)$. Remark however that it is not independent of Eq. (3.28). Indeed the equations (3.28) can be replaced by Eq. (4.20) and the three independent equations obtained by projecting Eq. (3.28) onto V_1^3 . These latter are the Cauchy's equations of motion. Upon use of (3.20), (1.12), (4.5) and (4.19), they read:

$$(4.21) \quad \begin{aligned} \rho\dot{u}^\gamma + \frac{1}{c^2} \left(\rho\epsilon\dot{u}^\gamma + t^{\beta\alpha}u_{\alpha;\beta}u^\gamma - \frac{1}{2}M_{\sigma\mu}\dot{\overline{F}}^{\sigma\mu}u^\gamma \right) \\ = t^{\beta\gamma}{}_{;\beta} + \frac{1}{c}J_\sigma F^{\gamma\sigma} + \frac{1}{2}M_{\sigma\mu}F^{\sigma\mu;\gamma} \end{aligned}$$

where we have collected the terms whose contribution vanishes when we take the classical limit $c \mapsto \infty$.

5. THE JUMP RELATION (3.34)

First we shall transform the second term of Eq. (3.34). Instead of the infinitesimal quantity $\delta\Theta$, we introduce the finite scalar Ξ whose dimension is (temperature \times time), by:

$$(5.1) \quad \delta\Theta = \varepsilon\Xi$$

Then, setting

$$(5.2) \quad u_{(n)} = u^\alpha n_\alpha, \quad \text{similarly} \quad \zeta_{(n)} = \zeta_\alpha n^\alpha$$

we note that, with (5.1) and (2.31), we can write:

$$(5.3) \quad [\rho\eta u^\alpha \sqrt{|g|} \delta\Theta] n_\alpha \equiv \varepsilon\rho\sqrt{|g|} u_{(n)} [\eta\Xi]$$

The first term of Eq. (3.34) is transformed as follows. Take the trace (tr = trace) of Eq. (3.27) and using (4.17) write

$$(5.4) \quad \mathbf{R} = -\kappa \text{tr}(\mathbf{T}^\alpha_\beta) = \rho\mathcal{M}c^2 + \text{tr}(t^\beta_\alpha) + 2\Phi$$

since $\mathbf{T}^{\alpha\beta}_{(\text{em.v})}$ is a traceless tensor. Thus we have

$$(5.5) \quad \begin{aligned} & \llbracket \{ (2\kappa)^{-1} \mathbf{R} - \rho\psi - \varphi \} \sqrt{|g|} \rrbracket \varepsilon\zeta_\alpha n^\alpha = \varepsilon\rho\sqrt{|g|} u_{(n)} \zeta_{(n)} \\ & \llbracket (2u_{(n)})^{-1} (c^2 + \varepsilon + \rho^{-1} \text{tr}(t^\beta_\alpha) - \rho^{-1} G_{\alpha\beta} F^{\alpha\beta} - \rho^{-1} F_{\alpha\beta} F^{\alpha\beta}) - u_{(n)}^{-1} \psi \rrbracket \end{aligned}$$

since

$$\varphi - \Phi = \frac{1}{2} G_{\alpha\beta} F^{\alpha\beta} + \frac{1}{2} F_{\alpha\beta} F^{\alpha\beta}$$

With $\varepsilon\rho\sqrt{|g|} u_{(n)} \neq 0$, collecting the results (5.3) and (5.5), we can write Eq. (3.34) as:

$$(5.6) \quad \llbracket u_{(n)}^{-1} \{ c^2 + \varepsilon + \rho^{-1} \text{tr}(t^\beta_\alpha) - (2\rho)^{-1} (G_{\alpha\beta} + F_{\alpha\beta}) F^{\alpha\beta} - 2\psi \} \rrbracket = 2 \llbracket \eta\Xi \rrbracket$$

across (Σ)

This equation is a constraint imposed on the variational process. The variation of the discontinuity surface as defined by (2.21) cannot be arbitrary; it must be linked to the variation of the temperature through Eq. (5.6).

To end with, we give a more explicit form for the jump relation (3.30). With (2.31) and the notation (5.2), Eq. (3.30) can be written:

$$(5.7) \quad \rho\sqrt{|g|} u_{(n)} \llbracket \left(1 + \frac{\varepsilon}{c^2} \right) u^\alpha - \rho^{-1} u_{(n)}^{-1} (t^\alpha_{(n)} - \mathbf{T}^\alpha_{(\text{em.m})_{(n)}}) \rrbracket = 0$$

where we have set

$$(5.8) \quad t_{(n)}^\alpha = t^{\beta\alpha} n_\beta, \quad T_{(\text{em.m})n}^{\alpha\beta} = T_{(\text{em.m})n}^{\alpha\beta}$$

Eq. (5.7) corresponds to the jump condition for the energy flux familiar to us in hydrodynamics and more generally in classical continuum mechanics.

6. CONCLUSION

We have obtained the field equations for a *nonlinear elastic magnetized homogeneous solid* in the frame of general relativity. In the continuous region ($\mathcal{D} - \Sigma$) they consist of Eqs. (3.27), (3.28) [or equivalently (4.20) and (4.21)], (2.28), (2.5), (2.6) and (2.8). Across the discontinuity surface (Σ), the jump equations (3.32), (3.33), (3.30), (2.31), (2.10), (3.31) and (2.12) must hold, Eq. (5.6) being a constraint imposed on the variation of (Σ). The constitutive equations are given by Eqs. (3.25), (3.9) and (3.10).

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- [47] For this term, we need not consider the total variation $\delta \bar{g}_{\alpha\beta}$ as was pointed out by TAUB, cf. Eq. (4.9) of Ref. [12].
- [48] (\underline{d}) is the operator of *codifferentiation*; a star indicating that we take the dual of the quantity at its right, e. g., $\underline{d} * \underline{G} = \underline{\widehat{dG}}$.

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