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On a class of infinite products occurring in quantum statistical mechanics


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On a class of infinite products occurring in quantum statistical mechanics

by

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ABSTRACT. — We study the class of infinite products

\[ f_c^L(\lambda) = \sum_{n=0}^{\infty} a_n^L \lambda^n = \prod_{j=1}^{\infty} \left\{ 1 + \lambda c(k_j)^2 \right\}, \quad k_j = j \frac{2\pi}{L} \quad \text{and} \quad L \in \mathbb{R}^+, \]

where \( c(k) = \frac{A}{k^m}, \quad k \geq 0, \quad A > 0, \quad m > \frac{1}{2} \), which occurs naturally in quantum statistical mechanics. In particular, we compute the limits

\[ \lim_{n \to \infty} \frac{a_{n+1}^L}{a_n^L} \quad \text{and} \quad \lim_{n \to \infty} \frac{1}{L} \log a_n^L \]

which are relevant in the problem of the thermodynamic limit of the BCS superconducting state. By the same way, we get new results concerning the infinite products of the form

\[ g_\rho(\mu) = \sum_{n=0}^{\infty} b_n \mu^n = \prod_{j=1}^{\infty} \left\{ 1 + \mu \frac{A}{j^{1/\rho}} \right\}, \quad A > 0, \quad 0 < \rho < 1. \]
In particular we are able to compute the limits

\[
\lim_{n \to \infty} n^{1/\rho} \frac{b_{n+1}}{b_n} \quad \text{and} \quad \lim_{n \to \infty} \frac{1}{2n} \log n^\rho b_n^{1/\rho}
\]

RÉSUMÉ. — Nous étudions la classe de produits infinis

\[
f_c^L(\lambda) = \sum_{n=0}^{\infty} a_n^L \lambda^n = \prod_{j=1}^{\infty} \left\{ 1 + \lambda c(k)^2 \right\}, \quad k_j = j \frac{2\pi}{L} \quad \text{et} \quad L \in \mathbb{R}^+,
\]

où \( c(k) = \frac{A}{k^m} \), \( k \geq 0 \), \( A > 0 \), \( m > \frac{1}{2} \), qui se présentent naturellement en mécanique statistique quantique. En particulier nous calculons les limites

\[
\lim_{n \to \infty} \frac{a_n^{L+1}}{a_n^L} \quad \text{et} \quad \lim_{n \to \infty} \frac{1}{L} \log a_n^L
\]

qui ont leur importance dans la limite thermodynamique de l'état de la supraconductivité de Bardeen-Cooper-Schriefer. Simultanément nous obtenons des résultats nouveaux relatifs aux produits infinis de la forme

\[
g_\rho(\mu) = \sum_{n=0}^{\infty} b_n \mu^n = \prod_{j=1}^{\infty} \left\{ 1 + \mu \frac{A}{j^{1/\rho}} \right\}, \quad A > 0, \ 0 < \rho < 1
\]

En particulier nous sommes en mesure de calculer les limites

\[
\lim_{n \to \infty} n^{1/\rho} \frac{b_{n+1}}{b_n} \quad \text{et} \quad \lim_{n \to \infty} \frac{1}{2n} \log n^\rho b_n^{1/\rho}
\]

I. INTRODUCTION

It is now well known that algebras of observables are useful in the kinematical description, from a quantum mechanical point of view, of systems of interacting particles. For that purpose, we constructed, in a preceeding paper [7], a family of states (positive linear functionals of norm 1) over a Clifford-C*-algebra, each of which characterized by a real function \( c \) of real argument and describing a « condensate » of pairs of fermions with density \( d \).
In that work, we studied a class of entire functions of the complex variables \( \lambda \), depending on the function \( c \) and on a length \( L \), defined by the infinite products

\[
f^L_c(\lambda) = \prod_{j=1}^{\infty} \left( 1 + \lambda c(k_j)^2 \right), \quad L \in \mathbb{R}^+, \quad k_j = j \frac{2\pi}{L}
\]

More precisely, if we write

\[
f^L_c(\lambda) = \sum_{n=0}^{\infty} a_n^L \lambda^n
\]

where

\[
a_n^L = \sum_{0 < j_1 < j_2 < \ldots < j_n} c(k_{j_1})^2 c(k_{j_2})^2 \ldots c(k_{j_n})^2
\]

the main point was the existence of the limit

\[
\gamma(d) = \lim_{n \to \infty, L \to \infty} \frac{d_{n+1}^L}{d_n^L}, \quad d \in \mathbb{R}^+
\]

and we were able to prove the following theorem:

**Theorem.** — Let \( c \) be a real, bounded, decreasing, square integrable function of a real positive variable \( k \), tending to zero, when \( k \) tends to infinity, faster than \( k^{-1/2} \). Let \( d \) and \( L \) be positive reals.

Then the limit (4) exists. Moreover, defining the function \( g(d) \) as

\[
g(d) = \lim_{n \to \infty, L \to \infty} \frac{1}{L} \log d_n^L,
\]

then \( g(d) \) exists, is convex and differentiable, and

\[
\gamma(d) = e^{2\pi g(d)}
\]

Finally one has the following integral formula

\[
d = \frac{1}{\pi} \int_0^\infty \frac{c^2(k)}{\gamma(d) + c^2(k)} \, dk
\]

The methods employed are rather involved and do not fully exploit the analyticity properties of \( f^L_c \). On the other hand, the theorem quoted
above is only an existence theorem, so that except for very special choices for \(c ([1], (109))\), it does not allow to compute explicitly the quantities \(g(d)\) and \(\gamma(d)\) as functions of \(d\).

Our purpose is to restate that theorem, using now entire functions technics. To that end, we have to make a different choice for our class of functions \(c\), giving up definiteness at the origin (and so boundedness and square integrability) but requiring an homogeneity condition. The quantities \(a_m^L\), \(g(d)\) and \(\gamma(d)\) are now explicitly given.

By the same way, we study the class of infinite products

\[
 g_\rho(\mu) = \prod_{j=1}^{\infty} \left\{ 1 + \mu \frac{A^2}{j^{1/\rho}} \right\}, \quad A > 0, \quad 0 < \rho < 1
\]

obtaining some new results for \(\rho \neq \frac{1}{2}\).

**II. CHOICE OF THE CLASS OF FUNCTIONS \(c\)**

Let us consider, for the moment, positive functions \(c(k)\), defined for \(k > 0\), and vanishing at infinity faster than \(k^{-1/2}\). The infinite products (1) are then convergent and define entire functions \(f^L_c\) whose zeros lie on the real negative axis and are given by

\[
 \lambda_j = -\frac{1}{c(k_j)^2}
\]

By definition ([4], I, 4), ([5], 2.5.2), the convergence exponent of the sequence (9) is the greatest lower bound \(\rho\) of the reals \(\alpha\) such that

\[
 \sum_{j=1}^{\infty} \left\{ \frac{1}{c(k_j)^2} \right\}^{-\alpha} = \sum_{j=1}^{\infty} c(k_j)^{2\alpha} < +\infty
\]

If, by now, we restrict ourselves to functions such that

\[
 c(k) \sim \frac{A}{k^m}, \quad k \to \infty, \quad m > \frac{1}{2}, \quad A > 0
\]

a necessary and sufficient condition for the convergence of (10) is

\[
 2m\alpha = 1 + \varepsilon, \quad \varepsilon > 0
\]
so that
\begin{equation}
\rho = \inf_{\varepsilon > 0} \left\{ \alpha : \frac{1}{2m} + \frac{\varepsilon}{2m} = \frac{1}{2m} < 1 \right\}
\end{equation}

On the other hand, the $f_{\mathcal{L}}$ are infinite canonical products of order $\rho$ and genus $p$ ([4], I, Th. 7), ([5], 2.6.5), the genus $p$ being related to the order $\rho$ according to
\begin{equation}
p \leq \rho \leq p + 1, \quad p \text{ integer.}
\end{equation}

In the case of (11), we can then conclude that the entire functions $f_{\mathcal{L}}$ are characterised by
\begin{equation}
p = 0, \quad 0 < \rho = \frac{1}{2m} < 1, \quad \frac{1}{2} < m + \infty
\end{equation}

Furthermore, let $n(r), r \in \mathbb{R}^+$, the function giving the number of zeros of $f_{\mathcal{L}}$ with modulus less than or equal to $r$. One can show ([4], I, lemma 1), ([5], 2.5.8) that
\begin{equation}
\rho = \lim_{r \to \infty} \frac{\log n(r)}{\log r}
\end{equation}
and one defines the density $\Delta$ of the sequence (9) as
\begin{equation}
\Delta = \lim_{r \to \infty} \frac{n(r)}{r^\rho}
\end{equation}

We are going to estimate these quantities, restricting now to functions $c$ such that (11) holds, which are monotonic and differentiable at least for $k$ large and such that
\begin{equation}
c'(k) \sim -\frac{B}{k^{m+1}}, \quad k \to \infty, \quad B > 0
\end{equation}

Turning back to the definition of $n(r)$, we can write that
\begin{equation}
n(r) = \max \left\{ j : \frac{1}{c(k_j)^2} \leq r \right\} = \max \left\{ j : k_j \leq \nu \left( \frac{1}{\sqrt{r}} \right) \right\} = \left[ \frac{L}{2\pi} \nu \left( \frac{1}{\sqrt{r}} \right) \right]
\end{equation}

where $\nu$ is the inverse function of $c$ (defined at least for $r$ large enough) and where the squared brackets mean « the largest integer contained in ». Thanks to our hypothesis (18), it is an easy task to show that
\begin{equation}
n(r) \sim \frac{L}{2\pi} A^{2\rho} r^\rho, \quad r \to \infty
\end{equation}
(a result which agrees with (16)) and that

\begin{equation}
\Delta = \frac{L}{2\pi} A^{2\rho}
\end{equation}

From these estimations, it is actually possible to deduce that ([4], I, Th. 25), ([5], 4.1.1)

\begin{equation}
\log f_c^L(re^{i\theta}) \sim e^{i\rho \theta} \frac{L}{2} A^{2\rho} (\csc \pi \rho) r^\rho, \quad -\pi < \theta < +\pi, \quad r \to \infty
\end{equation}
or, taking the real part, that

\begin{equation}
\log |f_c^L(re^{i\theta})| \sim \frac{L}{2} A^{2\rho} (\csc \pi \rho) \cos \rho \theta \cdot r^\rho, \quad -\pi < \theta < +\pi, \quad r \to \infty
\end{equation}

uniformly with respect to $\theta$ if $-\pi + \varepsilon \leq \theta \leq \pi - \varepsilon$.

Then, the indicator function $h_c^L(\theta)$ of $f_c^L$ is given by ([4], I, 15), ([5], 2.1.8)

\begin{equation}
h_c^L(\theta) = \lim_{n \to \infty} \frac{\log |f_c^L(re^{i\theta})|}{r^\rho} = \frac{L}{2} A^{2\rho} (\csc \pi \rho) \cos \rho \theta, \quad -\pi < \theta < +\pi
\end{equation}
and also for $-\pi \leq \theta \leq +\pi$ as $h_c^L$ is a continuous function, defined by periodicity for other values of $\theta$.

From now on, we are able to compute the type of $f_c^L$ according to the formula ([4], I, 1 and Th. 29), ([5], 2.1.4)

\begin{equation}
\tau(L) = \lim_{r \to \infty} \frac{\log \max_{|\lambda| = r} |f_c^L(\lambda)|}{r^\rho} = \max_\theta |h(\theta)| = \frac{L}{2} A^{2\rho} \csc \pi \rho
\end{equation}
as well as, by ([4], I, Th. 2), ([5], 2.2.10)

\begin{equation}
\tau(L) = \frac{1}{\rho} \lim_{n \to \infty} n(a_n^L)^{\rho/n}
\end{equation}

Comparing formulas (20) and (16), we see that the $\lim$ occurring in (16) is in fact a limit, as well as the ones in (17), (24), (25) and consequently in (26).

Unfortunately, the comparison of formulas (25) and (26) does not allow to estimate the limit (4) because of a lack of uniformity with respect to $L$.

A simple case where uniformity can be recovered is given by

\begin{equation}
a_n^L = \left( \frac{L}{2\pi} \right)^{n/\rho} b_n \quad b_n \text{ independent of } L
\end{equation}
The following proposition, the proof of which is trivial, shows that (27) holds if and only if \( c(k) \) is a homogeneous function of degree \(-\frac{1}{2\rho}\):

**Proposition.** — If

\[
 f^L_c(\lambda) = \prod_{j=1}^{\infty} \left( 1 + \lambda c(k_j)^2 \right) = \sum_{n=0}^{\infty} a_n^{L} \lambda^n
\]

where \( k_j = j \frac{2\pi}{L} \), then \( a_n^{L} = \left( \frac{L}{2\pi} \right)^{n/\rho} b_m b_n \) independent of \( L \), if and only if

\[
 c(k) = \frac{A}{k^{1/2\rho}} = \frac{A}{k^m},
\]

\( \rho = \frac{1}{2m} \) being the order of \( f^L_c \).

So our conclusion is that entire functions technics can be easily applied to our problem if we restrict ourselves to the class of functions

\[
 c(k) = \frac{A}{k^m}, \quad \frac{1}{2} < m < \infty, \quad A > 0, \quad 0 < k < \infty
\]

for which conditions (11) and (18) are evidently fulfilled.

### III. THE MAIN THEOREM

From now on, we can write, thanks to (25),

\[
 f^L_c(\lambda) = \sum_{n=0}^{\infty} \left( \frac{L}{2\pi} \right)^{n/\rho} b_n \lambda^n = \sum_{n=0}^{\infty} \frac{(\tau(L)^{1/\rho} \lambda)^n}{\frac{A^{2\rho \pi}}{(\sin \pi \rho)} b_n^{n/\rho}}
\]

and

\[
 g_\rho(\mu) = \prod_{j=1}^{\infty} \left\{ 1 + \mu \frac{A^2}{j^{1/\rho}} \right\} = \sum_{n=0}^{\infty} b_n \mu^n
\]

Moreover, the expressions

\[
 \frac{a_n^{L}}{\left( \frac{\tau(L) \rho}{n} \right)^{n/\rho}} = \frac{b_n}{\left( \frac{A^{2\rho \pi}}{(\sin \pi \rho)} \right)^{n/\rho} \left( \frac{\rho}{n} \right)^{n/\rho}}
\]
and (where $\Gamma$ means the usual gamma function)

\[
\frac{a_{n+1}^L}{a_n^L} \left\{ \frac{\tau(L)e^{\rho}}{n+1} \right\}^{n+1 \rho} \frac{b_{n+1}}{b_n} = \frac{\left( \frac{A^{2\rho\pi}}{n+1} \right)^{n+1 \rho} \left( \frac{e^{\rho}}{n+1} \right)^{n+1 \rho}}{\sin \pi \rho} \frac{1}{b_n}
\]

\[
\lim_{n \to \infty} \frac{\left( \frac{A^{2\rho\pi}}{n+1} \right)^{n+1 \rho}}{\sin \pi \rho} \frac{1}{\Gamma \left( \frac{n+1}{\rho} + 1 \right)} = \frac{\chi(n+1; \rho)}{\chi(n; \rho)}
\]

are now independent of $L$. Therefore the result we are aiming at is equivalent to

\[
\lim_{n \to \infty} \frac{\chi(n+1; \rho)}{\chi(n; \rho)} = 1
\]

or

\[
b_n = \left( \frac{A^{2\rho\pi}}{\sin \pi \rho} \right)^{n \rho} \frac{\chi(n; \rho)}{\Gamma \left( \frac{n}{\rho} + 1 \right)} \quad \text{where} \quad \lim_{n \to \infty} \frac{\chi(n+1; \rho)}{\chi(n; \rho)} = 1
\]

Incidentally it is interesting to remark that

\[
g_{\rho}(\mu) = \sum_{n=0}^{\infty} \left( \frac{A^{2\rho\pi}}{\sin \pi \rho} \right)^{n \rho} \frac{\chi(n; \rho)}{\Gamma \left( \frac{n}{\rho} + 1 \right)} \mu^n
\]

is of order $\rho$ and type

\[
\tau = \frac{A^{2\rho\pi}}{\sin \pi \rho} = \tau(2\pi)
\]

as is also the Mittag-Leffler function [6]

\[
E_{\rho}(\mu) = \sum_{n=0}^{\infty} \left( \frac{A^{2\rho\pi}}{\sin \pi \rho} \right)^{n \rho} \frac{\chi(n; \rho)}{\Gamma \left( \frac{n}{\rho} + 1 \right)} \mu^n
\]

which shows the close relation between $E_{\rho}$ and $g_{\rho}$ and asserts the well
known fact that $E_\rho$ is, in some sense [7], the simplest entire function of a given order and type.

Another way of writing (33) is the following (provided the limits exists):

$$
\lim_{n \to \infty} \frac{a_{n+1}^L}{a_n} = \lim_{n \to \infty} \left( \frac{\tau(L)e^\rho}{n+1} \right)^{n+1} \left( \frac{\tau(L)e^\rho}{n} \right)^{n/n^\rho}
$$

$$
= \lim_{n \to \infty} \left( \frac{LA^{2\rho e^\rho \csc \pi \rho}}{2(n+1)} \right)^{n+1} \left( \frac{LA^{2\rho e^\rho \csc \pi \rho}}{2n} \right)^{n/n^\rho}
$$

$$
= \lim_{n \to \infty} \left( \frac{A^{2\rho e^\rho \csc \pi \rho}}{d} \right)^{1/n^\rho} \frac{1}{\left( 1 + \frac{1}{d} \right) \left( 1 + \frac{1}{n} \right)^{1/n^\rho}} = \left( \frac{A^{2\rho e^\rho \csc \pi \rho}}{d} \right)^{1/n^\rho}
$$

So, once the existence of the limit is proved, then necessarily

$$
\gamma(d) = \left( \frac{A^{2\rho e^\rho \csc \pi \rho}}{d} \right)^{1/n^\rho}
$$

and hence

$$
\lim_{n \to \infty} n^{1/n^\rho} \frac{b_{n+1}}{b_n} = (\pi A^{2\rho e^\rho \csc \pi \rho})^{1/n^\rho}
$$

Before proving our main theorem, let us give a lemma:

**LEMMA.** — We have the inequalities:

$$
\frac{a_n^L}{a_{n-1}^L} > \frac{a_{n+1}^L}{a_n^L}
$$

and

$$
\frac{b_n}{b_{n-1}} > \frac{b_{n+1}}{b_n}
$$

These inequalities are well known in the theory of entire functions ([5], 2.8.2) but we restate the proof by sake of completeness.

One has:

$$
\frac{f'(\lambda)}{f(\lambda)} = \sum_{j=1}^{\infty} \frac{c(k_j)^2}{1 + \lambda c(k_j)^2}
$$

and

$$
\left\{ \frac{f''(\lambda)}{f(\lambda)} \right\}' = \sum_{j=1}^{\infty} \frac{-c(k_j)^4}{(1 + \lambda c(k_j)^2)^2}
$$

So $f(\lambda)f''(\lambda) < F'(\lambda)^2$ if $\lambda \in \mathbb{R}$ or, else, $f^{(n-1)}(\lambda)f^{(n+1)}(\lambda) < f^{(n)}(\lambda)^2$ by appli-
'cation of the same inequality to $f^{(n-1)}(\lambda)$, which is also an entire function of the same order, type and genus as $f$. It follows that

$$\frac{a_{n+1}^L}{a_n^L} < \frac{n}{n+1} \frac{a_{n+1}^L}{a_{n-1}^L} = \frac{a_n^L}{a_{n-1}^L}.$$ 

It is interesting to remark that, by adapting to our case the proof of ([1 bis], lemma 3, (80)), one has conversely

$$\frac{n}{b_{n-1}} \frac{b_n}{b_{n+1}} \leq \frac{n+1}{n} \frac{a_{n+1}^L}{a_n^L} + \frac{A^2}{\pi d}$$

(43)

We are now able to prove our theorem.

**Theorem.** — Let $c(k) = \frac{A}{k^n}$ with $k > 0$, $A > 0$, $m > \frac{1}{2}$, $L$ and $d$ be positive, and

$$f_c^L(\lambda) = \sum_{n=0}^{\infty} a_n^{L} \lambda^n = \prod_{j=1}^{\infty} \left(1 + \lambda c(k_j)^2\right), \quad k_j = j \frac{2\pi}{L}$$

(44)

$$g_\rho(\mu) = \sum_{n=0}^{\infty} b_n \mu^n = \prod_{j=1}^{\infty} \left(1 + \mu \frac{A^2}{j^{1/\rho}}\right), \quad \rho = \frac{1}{2m}, \quad 0 < \rho < 1$$

Then

$$\lim_{n \to \infty} \frac{1}{2\pi} \sum_{L \to \infty} L \cdot d \cdot \frac{\log d}{2\rho} \log \left(\frac{A^2 \rho \csc \pi \rho}{d}\right)$$

and

$$\lim_{n \to \infty} \frac{1}{2\pi} \log n^{1/\rho} b_n^{1/n} = \frac{1}{2\pi \rho} \log (\pi A^2 \rho \csc \pi \rho) = \frac{1}{\pi}$$

Moreover

$$\lim_{n \to \infty} \frac{a_{n+1}^L}{a_n^L} = \gamma(d) = \left(\frac{A^2 \rho \csc \pi \rho}{d}\right)^{1/\rho}$$

(47)

$$\lim_{n \to \infty} n^{1/\rho} \frac{b_{n+1}}{b_n} = \gamma = (\pi A^2 \rho \csc \pi \rho)^{1/\rho} = \frac{1}{\pi}$$

and

$$\begin{cases} 
\gamma(d) = e^{2\pi (1/d)} \\
\gamma = e^{2\pi (1/\pi)}
\end{cases}$$

(49)
Finally one has the following integral formulas

\[
\left\{ \begin{array}{l}
d = \frac{1}{\pi} \int_0^\infty \frac{c(k)^2}{c(k)^2 + \gamma(d)} \, dk = \frac{1}{\pi} \int_0^\infty \frac{dk}{1 + \frac{\gamma(d)}{A^2} k^{2m}} \\
1 = \int_0^\infty \frac{c(k)^2}{c(k)^2 + \gamma} \, dk = \int_0^\infty \frac{dk}{1 + \frac{\gamma}{A^2} k^{2m}}
\end{array} \right.
\]

(50)

Proof (our proof is similar to that used recently by Dobrushin and Min-los \[9\]). — Formulas (26) and (36) tell us that

\[\forall \varepsilon > 0, \exists N(\varepsilon) : n > N(\varepsilon) \Rightarrow \left(1 - \frac{\varepsilon}{\tau \rho}\right)^{n/\rho} < \frac{b_n}{\left(\frac{\tau \rho}{n}\right)^{n/\rho}} < \left(1 + \frac{\varepsilon}{\tau \rho}\right)^{n/\rho}\]

which gives rise to the equivalent formulas

\[
\left| \frac{1}{n} \log b_n - \frac{1}{\rho} \log \frac{\tau \rho}{n} \right| < \frac{\varepsilon}{\tau \rho^2}
\]

or

\[
\left| \frac{\log b_n}{\rho} - 1 \right| < \frac{\varepsilon}{\tau \rho} \left| \frac{1}{\log \frac{\tau \rho}{n}} \right| < \varepsilon' \quad \text{for } n \text{ large enough}
\]

Consequently,

\[
\lim_{n \to \infty} \frac{1}{L} \log a_{L_n}^L = \lim_{L \to \infty} \frac{d}{2\rho} \left\{ \log \frac{L}{2\pi} + \frac{\rho}{n} \log b_n \right\}
\]

\[
= \lim_{L \to \infty} \frac{d}{2\rho} \left\{ \log \frac{L}{2\pi} + \log \frac{\tau \rho}{n} \right\} = \frac{d}{2\rho} \log \frac{\tau \rho}{\pi d} = g(d)
\]

and we get (45) or, in the same way, (46).

On the other hand, we have, from (27),

\[
\log \frac{a_{L_{n+1}}^L}{a_{L_n}^L} = \frac{1}{\rho} \log \frac{L}{2\pi} + \log \frac{b_{n+1}}{b_n}
\]

It is sufficient to prove (47) to get also (48) and (49). We shall proceed in two steps, proving successively that

a) \[
\lim_{n \to \infty} \frac{a_{L_{n+1}}^L}{a_{L_n}^L} \geq e^{2g'(d)}
\]
a) Inequality (41) allows to write, for \( m \) positive integer,
\[
\lim_{n \to \infty} \frac{a_{n+1}^L}{a_n^L} \leq e^{2g(d)}
\]
or else
\[
\frac{a_{n+1}^L}{a_n^L} = \frac{a_{n+1}^L}{a_{n+1}^L} \cdot \frac{a_{n+1}^L}{a_{n+1}^L} \cdot \frac{a_{n+1}^L}{a_{n+1}^L} \cdots \frac{a_{n+1}^L}{a_{n+1}^L} \leq \left( \frac{a_{n+1}^L}{a_b^L} \right)^{m+1}
\]
Taking the limit of both sides for \( n \to \infty, L \to \infty, 2n/L = \delta, m \to \infty \), we get
\[
\lim_{n \to \infty} \frac{a_{n+1}^L}{a_n^L} \geq e^{2Lg(d)}
\]
which proves the desired result thanks to the arbitrariness of \( \delta \).

b) Inequality (41) allows to write, for \( m \) positive integer,
\[
\lim_{n \to \infty} \frac{a_{n+1}^L}{a_n^L} \leq e^{2g(d) - g(d)}
\]
or else
\[
\frac{a_{n+1}^L}{a_n^L} = \frac{a_{n+1}^L}{a_{n+1}^L} \cdot \frac{a_{n+1}^L}{a_{n+1}^L} \cdot \frac{a_{n+1}^L}{a_{n+1}^L} \cdots \frac{a_{n+1}^L}{a_{n+1}^L} \leq \left( \frac{a_{n+1}^L}{a_b^L} \right)^{m+1}
\]
Taking the limit of both sides for \( n \to \infty, L \to \infty, 2n/L = \delta, m \to \infty \), we get
\[
\lim_{n \to \infty} \frac{a_{n+1}^L}{a_n^L} \geq e^{2Lg(d) - g(d)}
\]
which again proves the desired result (b) thanks to the arbitrariness of \( \delta \).

Finally, formulas (50) can be easily deduced from the identity ([10], 3.241, 2):
\[
\int_0^\infty \frac{dk}{1 + k^{2m}} = \frac{\pi}{2m} \csc \frac{\pi}{2m}, \quad m > \frac{1}{2}
\]
To end this part, we want to mention that it is possible to compute the
quantities $b_n$ and consequently $a_n$, with the help of the $\zeta$ function of Riemann, using methods patterned from the Fredholm theory of integral equations [11]. This result cannot be considered new. It is contained, for instance, into the formulas ([10], 8.334, 1, 8.321, 2)

$$g_\rho(\mu) = \prod_{j=1}^{\infty} \left\{ 1 + \frac{\mu A^2}{j^{2m}} \right\} = \frac{1}{\mu A^2} \prod_{k=1}^{2m} \frac{1}{\Gamma \left[ -\left( -\mu A^2 \right)^{2m} \exp \frac{2nki}{2m} \right]}
$$

and

$$\frac{1}{\Gamma(z + 1)} = \sum_{k=0}^{\infty} d_k z^k$$

where

$$d_0 = 1, \quad d_{n+1} = \frac{\sum_{k=0}^{n} (-1)^k s_{k+1} d_{n-k}}{n+1}, \quad s_1 = C, \quad s_n = \zeta(n) \quad \text{for} \quad n \geq 2$$

and $C = $ Euler's constant $= 0.57721 \ldots$

However, for sake of completeness, we give here a direct proof of it. From (30) we can deduce that

$$g'_\rho(\mu) = \sum_{j=1}^{\infty} \frac{c(j)^2}{1 + \mu c(j)^2} = \sum_{j=1}^{\infty} \sum_{n=0}^{\infty} (-1)^n c(j)^{2n+2} \mu^n = \sum_{n=0}^{\infty} (-1)^n s_{n+1} \mu^n$$

provided that

$$\left| \mu c(j)^2 \right| < 1, \quad j = 1, 2, \ldots \quad \text{i.e.} \quad |\mu| < \frac{1}{c(1)^2} = \frac{1}{A^2}$$

if we define

$$\sigma_p = \sum_{j=1}^{\infty} c(j)^{2p} = \sum_{j=1}^{\infty} A^{2p} \frac{1}{j^{p/\rho}} = A^{2p} \zeta(p/\rho)$$

But, on the other hand,

$$g_\rho(\mu) = \sum_{n=0}^{\infty} b_n \mu^n, \quad g'_\rho(\mu) = \sum_{n=0}^{\infty} (n+1) b_{n+1} \mu^n$$

so that formula (53) gives rise to the relation

$$\sum_{n=0}^{\infty} (n+1) b_{n+1} \mu^n = \left\{ \sum_{m=0}^{\infty} b_m \mu^m \right\} \left\{ \sum_{q=0}^{\infty} (-1)^q \sigma_{q+1} \mu^q \right\}$$
from which we deduce that

\[ b_{n+1} = \sum_{m+q=n}^{\infty} (-1)^q b_m \sigma_{q+1}; \quad b_0 = 1 \]

Solving this system of equations, we get the following formula, which is quite familiar in the theory of integral equations (see for instance [11])

\[
\begin{pmatrix}
\sigma_1 & 1 & 0 & \cdots & 0 \\
\sigma_2 & \sigma_1 & 2 & 0 & \cdots & 0 \\
\sigma_3 & \sigma_2 & \sigma_1 & 3 & 0 & \cdots & 0 \\
& & & & & \ddots & \ddots & \ddots & 0 \\
& & & & & \ddots & \ddots & \ddots & 0 \\
& & & & & \ddots & \ddots & \ddots & 0 \\
\sigma_n & \sigma_{n-1} & \sigma_{n-2} & \cdots & \cdots & \sigma_1
\end{pmatrix}
\]

(59)

\[ b_n = \frac{1}{n!} \]

For instance

\[
\begin{align*}
b_1 &= A^2 \zeta(1) \\
b_2 &= \frac{A^2}{2} \left[ \zeta(1) - \zeta(2) \right]
\end{align*}
\]

(60)

\[
\text{etc.}
\]

**IV. AN EXAMPLE**

As an illustration of the preceedings results, let us now study the well known case where

\[ c(k) = \frac{1}{k}, \quad A = 1, \quad m = 1, \quad \tau(L) = \frac{L}{2}, \quad \tau = \pi, \quad \rho = \frac{1}{2} \]

(61)

We have then:

\[
\begin{align*}
f_{1/4}(k) &= \prod_{j=1}^{\infty} \left\{ 1 + \lambda \left( \frac{L}{2\pi} \right)^2 \frac{1}{j^2} \right\} = \frac{\sin \frac{iL}{2} \sqrt{\lambda}}{iL \sqrt{\lambda}} \\
g_{1/2}(\mu) &= \prod_{j=1}^{\infty} \left\{ 1 + \mu \frac{1}{j^2} \right\} = \frac{\sin i\pi \sqrt{\lambda}}{i\pi \sqrt{\lambda}}
\end{align*}
\]

(62)
and consequently,

\[
\tag{63}
\alpha_n^L = \left( \frac{L}{2\pi} \right)^{2n} \frac{\pi^{2n}}{(2n+1)!}; \quad \beta_n = \frac{\pi^{2n}}{(2n+1)!}
\]

We can then immediately see that the ratio (31) is independent of \( L \) and that

\[
\chi\left(n; \frac{1}{2}\right) = \frac{1}{2n+1}; \quad \lim_{n \to \infty} \frac{\chi\left(n + 1; \frac{1}{2}\right)}{\chi\left(n; \frac{1}{2}\right)} = 1
\]

Moreover, one gets directly that

\[
\tag{64}
\lim_{n \to \infty} \frac{\alpha_{n+1}^L}{\alpha_n^L} = \left( \frac{1}{2d} \right)^2 \quad \text{and} \quad \lim_{n \to \infty} n^2 \frac{\beta_{n+1}}{\beta_n} = \frac{\pi^2}{4}
\]

in agreement with formulas (47) and (48).

In the same way, we can compute the limits

\[
\tag{65}
\lim_{n \to \infty} \frac{1}{L} \log \alpha_n^L = d(1 - \log 2d)
\]

\[
\lim_{n \to \infty} \frac{1}{2\pi} \log n^2 \beta_n^L = \frac{1}{\pi} \left( 1 - \log \frac{2}{\pi} \right)
\]

These results are consistent with formulas (45) and (46).

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