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On a modified Classification Scheme for elementary Particles

by

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ABSTRACT. — A modified classification scheme for the different known elementary particle families (baryons, mesons, baryon resonances, etc.) is proposed. Contrary to the familiar symmetry-classifications with higher unitary symmetries this classification model admits nontrivial mixings between the inhomogeneous Lorentz group and some internal interaction symmetry. One thus obtains multiplets with different masses and consequently nontrivial mass-formulae.

SOMMAIRE. — Un nouveau schème de classification des différentes familles de particules élémentaires (baryons, mesons, etc.) est proposé. A l'encontre de la classification usuelle de symétries à l'aide de groupes de symétrie unitaire, ce modèle de classification admet des couplages non triviaux du groupe de Lorentz inhomogène et de certaines symétries internes d'interaction. De ce fait, on obtient des multiplets possédant des masses différentes et, par conséquent, des formules de masse non triviales.

I. INTRODUCTION

In the framework of a phenomenological discussion of elementary particles the starting point is a symmetry classification by so-called higher unitary symmetry groups SU(n) (especially $n = 3$). The basic results of such a classification are the multiplets themselves and the way in which they split as the symmetry is broken. Confidence in these multiplet assignments came from the Gell-Mann-Okubo mass formula, which, for the baryons, is amazingly accurate. However, the possibility of obtaining mass formulae for particles belonging to the same irreducible representations of the interaction group is related to the problem of combining interaction symmetries and relativistic invariance in a nontrivial way. The impossibility of such combinations has been pointed out by McGlinn [*1*] whose negative result refers to semi-simple internal symmetry groups. A great number of further investigations of this subject ([*2*] and references quoted therein) have confirmed this no-go-theorem.

The object of this paper is to exhibit a symmetry scheme which gives rise to a mixing of the connected inhomogeneous Lorentz group $P_+^\uparrow$ with some internal noncompact and non-semi-simple symmetry group G and which allows a classification of particle families with fixed spin J and parity P but *different* masses m_j, $j = 1, \ldots, n$.

II. DESCRIPTION OF NONTRIVIAL MIXINGS BETWEEN SPACE-TIME AND INTERNAL SYMMETRY

Within a multiplet of an internal symmetry group G all particles possess the same spin and parity which means for these quantum numbers to be invariant under internal symmetry transformations:

$$[M_{\mu\nu}, X_k] = 0 \qquad k = 1, \ldots, \dim G \tag{1}$$

where $M_{\mu\nu} = - M_{\nu\mu}$ denote the covariant components of the angular momentum tensor and X_k the infinitesimal generators of the Lie algebra $\mathscr{G}$ of G. But the internal quantum numbers may not be translation invariant, since the particles in a multiplet have different masses. Therefore:

$$[P_\mu, X_k] \neq 0 \tag{2}$$

McGlinn [*1*] has proved that if (1) holds $\forall X_k \in \mathscr{G}$, the semi-simple Lie algebra of G, then all these infinitesimal operators commute with every infinitesimal operator of the translations $\{ (a, I) \}$. If $\mathscr{G}(P_+^\uparrow)$ (Lie algebra of $P_+^\uparrow$) and $\mathscr{G}$ are by assumption subalgebras of some Lie algebra $\mathscr{G}(E)$, this means then:

$$\mathscr{G}(E) = \mathscr{G}(P_+^\uparrow) \oplus \mathscr{G} \qquad \text{(direct sum)} \tag{3}$$

or equivalently:

(3′) $E = P_+^\uparrow \times G$ (direct product)

which yields a degenerate mass spectrum and thus invalidates (2). One is therefore led to consider such a symmetry scheme which generalizes the symmetry structure (3′), i. e. which provides couplings of the type $E = P_+^\uparrow \perp G$, where E denotes some still unspecified mixing between $P_+^\uparrow$ and G ($\perp$ symbolizes this nontrivial mixing). These couplings entail in the general case $\Delta m_{ik} = m_i - m_k \neq 0$ [4] (m_i, m_k: any masses of the multiplet) $\Delta m_{ik} = 0$, $\forall i, k$ as a special case corresponding to that mixing which represents the direct product (3′). The latter may be characterized, according to [3] by the following short exact sequence:

(3″) $$1 \dashrightarrow G \overset{i}{\dashrightarrow} E \underset{u}{\overset{\Phi}{\rightleftarrows}} P_+^\uparrow \dashrightarrow 1$$

where i denotes an injective mapping, $\Phi \circ u = 1_{P_+^\uparrow}$ is the identity automorphism of $P_+^\uparrow$, u a homomorphism and $P_+^\uparrow$, G normal subgroups of E.

One obtains the required generalization by referring to the following

THEOREM 1 [4]. — Nontrivial mixings between some internal symmetry and the relativistic invariance are determined by elements of the set of extensions of $P_+^\uparrow$ by G: Ext ($P_+^\uparrow$, G).

Indeed, let $\Delta m_{ij} = m_i - m_j \neq 0$ be the nonvanishing mass differences between particles m_i, m_j, ... of some multiplet. Then the proof of Theorem 1 can be achieved, as shown in detail in [4], by considering the following correspondences:

(4) $$\Delta m_{ij} \dashrightarrow K_{xy}^{ij} := K_{xy}^{i} - K_{xy}^{j}$$

K_{xy}^{i} and K_{xy}^{j} are positive Kerneldistributions in 2 variables which are, according to the Schwartz Kernel Theorem [5], in bijective correspondence with the single particle-Hilbert spaces $\mathfrak{H}(m_i, s)$, s: spin of the multiplet,

(4′) $$K_{xy}^{i} \longleftrightarrow \mathfrak{H}(m_i, s) \quad \forall i$$

(5) $$K_{xy}^{ij}\mathscr{D} \longleftrightarrow \{ f \}^{ij} \in H^2(P_+^\uparrow, C(G))$$ ($\mathscr{D}$: space of testing functions)

(6) $$H^2(P_+^\uparrow, C(G)) \longleftrightarrow \text{Ext}\,(P_+^\uparrow, G)$$

i. e.

(7) $$\Delta m_{ij} \neq 0 \longleftrightarrow E_{ij} \in \text{Ext}\,(P_+^\uparrow, G)$$

$H^2(P_+^\uparrow, C(G))$ denotes the second cohomology group of $P_+^\uparrow$ over the centre of G (see ref. [6]).

The assignment (5) is proved, according to [4] by making use of the following definition

(8) $$\mathscr{U}(f(L_r, L_s))\Phi_0 = \psi^{ij}_{(L_r,L_s)}(\varphi_0) = \Phi_1$$

with

$$f(L_r, L_s) = g \in G$$

$$\Phi_0, \Phi_1 \in \mathfrak{H}_{G_1} \quad \text{(representation space of } \mathscr{C}(G) = G_1)$$

$$(L_r, L_s) \in P^{\uparrow}_{+} \times P^{\uparrow}_{+}$$

(9) $$\psi^{ij} \in K^{ij}\mathscr{D} \subset \mathscr{D}'(\mathbb{R}^4, \mathfrak{H}_{G_1})$$

(space of $\mathfrak{H}_{G_1}$-valued distributions on $\mathbb{R}^4$).

Remarks:

1. — Since Galindo [9] has shown that extensions corresponding to Theorem 1 can be of nontrivial type, provided G be neither semi-simple nor compact, the correspondences (4)-(7) exhibit the existence of extensions belonging to nontrivial mixings which are associated with non-vanishing mass-differences. Thus Theorem 1 provides a possibility of a particle classification with nondegenerate mass-spectrum as shown in our subsequent discussion.

2. — Referring to the short exact sequence (3″) the mapping

(10) $$f\colon P^{\uparrow}_{+} \times P^{\uparrow}_{+} \dashrightarrow C(G) \qquad f \in H^2(P^{\uparrow}_{+}, C(G))$$

accounts for the deviation of the homomorphism-law, i. e.

(11) $$u(L_r)u(L_s) = f(L_r, L_s)u(L_rL_s)$$

which means formula (3″) is neither a direct nor a semi-direct product (otherwise stated: the exact sequence (3″) does not split).

3. — In the case of abelian symmetries G, C(G) = G and Ext $(P^{\uparrow}_{+}, G)$ thus becomes a group [6].

In order to obtain a mass formula associated with the framework of Theorem 1 we set the following

Postulate

There exists a selfadjoint mass-breaking (or symmetry-breaking) operator

(12) $$A \in \mathscr{L}(\mathscr{F}, \mathscr{F}) \qquad \mathscr{F} = \bigoplus_{n=0}^{\infty} \mathfrak{H}^{\otimes n}\text{: Fock space}$$

such that

a) $\Delta m = (\psi(\varphi), A\psi(\varphi))$ and

b) $D_A \overset{\text{dense}}{\subseteq} \mathscr{F}$.

Remark 4. — The Fock space of formula (12) is specified as follows:

$$(13)\quad \begin{cases} \bigoplus_{n=0}^{\infty} \mathfrak{H}^{\otimes n} = \left\{ \Phi = (\varphi_0, \varphi_1, \ldots, \varphi_k, \ldots) \mid \varphi_k \in \mathfrak{H}^{\otimes k} : \sum_{k=0}^{\infty} \|\varphi_k\|^2 < \infty \right\} \\ \mathfrak{H}^{\otimes 0} = \langle \Omega \rangle = \{ C\Omega \mid C \in \mathbb{C} \} = \text{vacuum} \\ \mathfrak{H}^{\otimes 1} \subset \mathscr{D}'(\mathbb{R}^4, \mathfrak{H}_G)\text{: single-particle-space} \\ \mathfrak{H}^{\otimes k} = \mathfrak{H}^{\otimes 1} \underbrace{\otimes \ldots \otimes}_{k} \mathfrak{H}^{\otimes 1} \end{cases}$$

Remark 5. — Condition (12 *a*) is to be understood in the sense of relationship (9), i. e.:

$$\psi^{ij} = K^{ij}\varphi \in \mathscr{D}'(\mathbb{R}^4, \mathfrak{H}_G) \Rightarrow \psi(\varphi') \in \mathfrak{H}_G \subset \mathfrak{H}^{\otimes 1} \quad \varphi, \varphi' \in \mathscr{D}$$

The mass-breaking operator A is defined on these vectors, since

$$\mathfrak{H}_G \subset \mathfrak{H}^{\otimes 1}$$

The mathematical argument to support this postulate is the following

THEOREM 2. — A sufficient condition for the existence of a mass-formula within the group theoretical framework of Theorem 1 is the existence of some mass-breaking operator A whose eigenstates are the vectorvalued distributions (9), that is

$$(14)\qquad A\psi^{ij}(\varphi) = \Delta m \psi^{ij}(\varphi) \qquad \Delta m := \Delta m_{ij} = m_i - m_j$$

$$(14')\qquad \psi^{ij} \in \mathscr{D}'(\mathbb{R}^4, \mathfrak{H}_G) \qquad \text{(masses of some multiplet)}$$

Proof. — Starting with Theorem 1 one can specify the structure of the mass breaking operator A. Indeed, according to (6) and (7) one has:

$$(7')\qquad \Delta m_{ij} \neq 0 \longleftrightarrow \{ f \}_{ij} \neq \{ 0 \} \in H^2(P_+^\uparrow, C(G))$$

That is: to each mass difference corresponds uniquely a cohomology class and *vice versa*. Therefore the expectation values $\Delta m = (\Phi, A\Phi)$, that is, the postulated mass breaking operator A must be constant on each cohomology class $\{ f \} \in H^2(P_+^\uparrow, C(G))$. Thus the mass breaking operator may be symbolized by $A = C(f)$ (C stands for « constant » and f denotes the representative $\in \{ f \} \neq \{ 0 \}$), where

$$(15)\qquad C\colon H^2(P_+^\uparrow, C(G)) \xrightarrow{\text{into}} \mathscr{L}(\mathscr{F}, \mathscr{F})$$

denotes this constant map. Therefore obviously the following relationship must hold:

(16) $$f_1 \not\sim f_2 \bmod B^2(P_+^\uparrow, C(G)) \Leftrightarrow C(f_1) \neq C(f_2)$$

In particular

(17) $$C(0) = 0 \Leftrightarrow \{ f \} = \{ 0 \}$$

if and only if the map (15) is linear as can be readily seen. In order that (14) be consistent with the existence of the vectorvalued distributions (9), the following condition must obviously hold:

(18) $$\psi^{ij} \in \mathscr{D}'(\mathbb{R}^4, \mathfrak{H}_G) \text{ is eigenstate of } A = C(f) \Leftrightarrow \psi^{ij} \notin \mathfrak{H}_{P_+^\uparrow} \otimes \mathfrak{H}_G$$

Statement (18) is equivalent with statement (17), i. e. $C(f) \neq 0$ has vector-valued distribution-eigenstates which belong to non-vanishing cohomology classes. Thus (18) clearly exhibits the fact that the mass-degeneracy of the multiplet spectrum is related to $K^{ij} = 0$, $\forall i, j$. Otherwise stated, this turns out again, that mass-degeneracy can only be associated with direct-product coupling. Theorem 2 thus guarantees the existence of a mass formula of the type

(19) $$m = m_0 + \Delta m(X_1, \ldots, X_n)$$

i. e. the Δm's will depend on the internal quantum numbers as will be specified in section IV.

Remark 6. — It turns out that in our framework the conjecture of Böhm [7], i. e. that the eigenstates of $M^2 = P_\mu \cdot P^\mu$ be « generalized » ones in order to obtain a mass-formula, does not hold.

Remark 7. — The « internal-parameter-independent » model, which refers to a tensor-product-single-particle Hilbert space $\mathfrak{H}^{\otimes 1} = \mathfrak{H}_{P_+^\uparrow} \otimes \mathfrak{H}_G$ i. e. which decouples the internal symmetry from the space-time-symmetry, results in the following multiplet-structure:

(20) $$\mathfrak{H}^{\otimes 1} = \bigoplus_{k=1}^{n} \mathfrak{H}_k^{\otimes 1} \quad ; \quad \mathfrak{H}_1^{\otimes 1} = \ldots = \mathfrak{H}_n^{\otimes 1}$$

Thus the discrete mass-eigenvalue can obviously never be written in the form

(21) $$E = F(C_1, C_2, \ldots)$$

i. e. such that the energy is a simple function of the Casimiroperators C_i of G.

III. DESCRIPTION OF A MODIFIED SYMMETRY SCHEME

One infers by inspection of correspondences (7) and (7′) that our symmetry scheme requires an appropriate choice of some group G_2 such that

$$H^2(P_+^\uparrow, C(G)) \cong G_2 \tag{22}$$

where the order of G_2 must be equal to $(n + 1)$, where n denotes the number of nonvanishing mass-differences of some given particle-family. To summarize: nontrivial mixings between the relativistic invariance and some internal symmetry must be related to a modified symmetry scheme which is associated with the following mathematical objects:

a) The internal symmetry group G (noncompact, non-semisimple).
b) The space-time symmetry $P_+^\uparrow$.
(23) *c*) Some « classification-group » G_2 (which is in general a discrete group, e. g. $\mathbb{Z}_n$).
d) The mass-breaking operator $C(f) \in \mathscr{L}(\mathscr{F}, \mathscr{F})$.

DISCUSSION OF THE CLASSIFICATION SCHEME (23)

According to (23 *a*) one is confronted with a theory where an infinite number of elementary particles occurs in any supermultiplet thus introducing unobserved particles. Finite multiplets may be obtained however by using a Hilbert space with an indefinite metric.

Consider now some given J^P-particle-family with mass-differences $\Delta m_{ik} \neq 0$. According to (7) and (7′) these mass-differences belong to some well-determined extensions $E_\alpha = P_+^\uparrow \perp_\alpha G$ the irreducible representations of which transform the corresponding spinors (or tensors). This is seen by taking for instance the familiar baryon-octet $J^P = 1/2+$:

$$(24)\quad \left\{ \begin{array}{llllll} H^2(P_+^\uparrow, C(G)) = \{ & \{0\}, & \{f_1\}, & \{f_2\}, & \{f_3\} & \} \cong \mathbb{Z}_4 \\ \quad\Big\updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ & \Delta m = 0 & m_\Lambda - m_N & m_\Sigma - m_N & m_\Xi - m_N & \\ \mathrm{Ext}\,(P_+^\uparrow, G) = \{ & E_0, & E_1, & E_2, & E_3 & \} \\ & E_0 = P_+^\uparrow \times G & & & & \end{array} \right.$$

Then, according to diagramm (24) the spinor $\begin{pmatrix} N \\ \Xi \end{pmatrix}$ for instance must transform under an irreducible unitary representation $U(E_3)$. The abstract group $G_2 \cong \mathbb{Z}_4$ accounts for all mass-differences of the supermultiplet which are due, say, to the medium strong interaction. Switching on the eletromagnetic interaction the principle, described by (24) remains unchanged. This has been worked out in [8].

IV. CONSTRUCTION OF A MASS-FORMULA

By virtue of the correspondences (4) and (7) as well as by the postulate, nontrivial couplings of space-time and internal symmetry are related to the commutators (2) in the following way:

$$E_\alpha = P_+^\uparrow \perp_\alpha G \longleftrightarrow [P^\mu, X_k] \neq 0 \tag{25}$$

(where $\perp_\alpha$ is not the direct product).

The correspondence (25) holds for at least one $X_k \in \mathcal{G}$. Otherwise stated: the correspondence

$$\begin{aligned} \{0\} &\longleftrightarrow [P^\mu, X_k] = 0 \qquad \forall X_k \\ \{f\} &\longleftrightarrow [P^\mu, X_k] \neq 0 \qquad \text{for at least one } X_k \in \mathcal{G} \end{aligned} \tag{26}$$

guarantees the existence of the mappings φ and ψ such that

$$\begin{aligned} \Delta m &= \varphi(X_1, \dots, X_n) \qquad & X_k \in \mathcal{G} \\ C(f_\alpha) &= \psi(X_1, \dots, X_n); \qquad & f_\alpha \in \{f\} \in H^2(P_+^\uparrow, C(G)) \end{aligned} \tag{27}$$

holds.

This means that there exists a mass formula of the type (19):

$$m = m_0 + \Delta m(X_1, \dots, X_n)$$

which is associated with the symmetry scheme (23). Referring to the mass-differences Δm due to the medium strong interaction there always exists some real numbers α and α' such that (28) is true:

$$\Sigma \Delta m_{ij} = \Sigma \alpha_i (\Psi, C(f_i)\psi) + \Sigma \alpha'_m (\Psi, C(f_m)\psi) \tag{28}$$

where $\lambda \Sigma \Delta m_{ij} = \Delta m$.

This can be shown to be in agreement with the mass spectrum (see the figure).

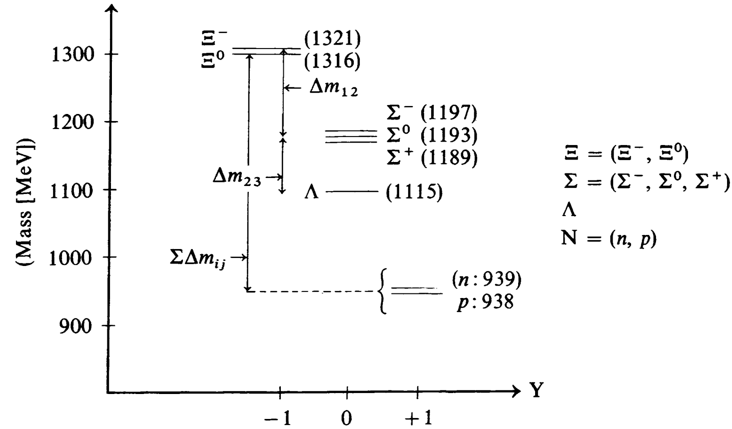
Indeed, if one restricts oneself to the baryon family (N, Λ, Σ, Ξ) for instance, one obtains, by fitting suitably these real numbers, the Gell-Mann-Okubo mass-formula. In fact:

$$\begin{aligned} \Sigma \Delta m_{ij} = m_\Xi - m_N &= \alpha_1(\psi, C(f_1)\psi) + \alpha_2(\psi, C(f_2)\psi) \\ &= \alpha_1(m_\Xi - m_\Sigma) + \alpha_2(m_\Sigma - m_\Lambda) \\ \alpha_1 \neq 0 \quad &, \quad \alpha_2 \neq 0 \end{aligned}$$

This yields for $\alpha_1 = 2$, $\alpha_2 = \frac{3}{2}$:

$$m_\Xi + m_N = \frac{1}{2}(3m_\Lambda + m_\Sigma)$$

(Gell-Mann-Okubo-mass-formula).



$A = 1$: $J^P = \frac{1^+}{2}$ – Baryons

$$\Sigma \Delta m_{ij} = \Delta m_{12} + \Delta m_{23} + \Delta m_{34}$$
$$\Delta m_{23} = m_\Sigma - m_\Lambda$$
$$\Delta m_{12} = m_\Xi - m_\Sigma$$
$$\Delta m_{34} = m_\Lambda - m_N$$

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