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Canonical theory of the two-body problem in the classical relativistic electrodynamics

by

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SUMMARY. — We consider two charged particles of finite mass and assume that the first particle moves in the retarded field of the second one, while the second particle moves in the advanced field of the first one. Equations of motion for this system are not differential-difference equations, as is the case in other conceivable two-body problems, but can be reduced to ordinary differential equations. The main result of this paper is derivation of the Euler homogeneity relation expressed by means of momenta.

INTRODUCTION

The two-body problem in the classical relativistic electrodynamics has a long history [*1*]. Darwin [*2*] considered it in the first post-newtonian approximation. Fokker [*3*] realized that the problem, when considered within the framework of the Maxwell field theory, is not of mechanical nature; he introduced an action principle which eliminates radiation and allows to consider the two-body problem as a mechanical one. But Fokker's theory leads to differential-difference equations which, in general, have solutions depending on arbitrary functions [*4*] [*5*], not arbitrary parameters, as in the case of a really mechanical theory. Similar differential-difference equations arise in Synge's theory [*6*]. Consequently, the theories of Fokker and Synge cannot be considered complete, unless an additional and independent principle of selection of admissible solutions is added. The author [*7*] [*8*] formulated this principle for the case of two particles

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moving along a nearly circular orbit; in the general case such a principle is not known.

It might seem that differential-difference equations will unavoidably appear in any kind of the electromagnetic two-body problem, because the finite velocity of interaction is the most important physical fact to be taken into account. The author realized, however, some time ago [*9*] [*10*] that it is not so; there is a case when equations of motion are reducible to ordinary differential equations. This case arises when the first particle moves in the retarded (advanced) Lienard-Wiechert potential of the second particle, while the second particle moves in the advanced (retarded) potential of the first particle. Since solutions of the two cases do not differ considerably [*9*], the best idea is to consider both of them as a time-symmetric description of motion.

In this paper we develope the canonical theory of the two-body problem for the two reducible cases.

THE LAGRANGIAN FORM OF THE THEORY

The assumption that one particle acts as an emitter and the other as an absorber of radiation may be formulated in the best way by means of an action principle. We assume that equations of motion of the system of two particles are the Euler-Lagrange equations for the variational principle $\delta S = 0$, where

$$\text{(1)}\quad S = -m_1 \int \sqrt{dx_{(1)\mu} dx_{(1)}{}^{\mu}} - m_2 \int \sqrt{dx_{(2)\mu} dx_{(2)}{}^{\mu}}$$
$$- e_1 e_2 \int\int 2\theta[\varepsilon(x^0_{(1)} - x^0_{(2)})]\delta[(x_{(1)\lambda} - x_{(2)\lambda})(x^{\lambda}_{(1)} - x^{\lambda}_{(2)})] dx_{(1)\mu} dx_{(2)}{}^{\mu}.$$

m_a, $a = 1, 2$, denotes mass, e_a charge and $x_{(a)\mu}$, $\mu = 0, 1, 2, 3$, a coordinate. θ is the Heaviside step function and δ is the Dirac delta-function; $2\theta(\varepsilon x^0)\delta(x_\mu x^\mu)$ is, for $\varepsilon = +1$, the retarded Green function of the d'Alembert equation and, for $\varepsilon = -1$, the advanced Green function. The integral (1) is Lorentz invariant and, moreover, is invariant with respect to an arbitrary change of parametrization: parameter τ_1 on the first world-line may be replaced by a new parameter $\tau_1' = F(\tau_1)$, where F is an arbitrary monotonically increasing function. Similarly, parameter τ_2 on the second world-line may be replaced by a new parameter $\tau_2' = G(\tau_2)$; both parameters τ_1 and τ_2 are entirely independent. However, in order to obtain equations of motion in the form of differential equations, not of differen-

We choose the first parameter arbitrarily and the second one in such a way that interacting events on the two world-lines have the same parameter. One can say that we introduce on the two world-lines a common parameter which is the same for events which can be connected by a light signal. Such a parameter is the closest relativistic counterpart of the newtonian universal time.

Denoting a common parameter by τ and performing in the interaction term of (1) integration over the second world-line, we obtain

$$(2) \quad S = -\int m_1\sqrt{\dot{x}_{(1)\mu}\dot{x}_{(1)}{}^{\mu}} + m_2\sqrt{\dot{x}_{(2)\mu}\dot{x}_{(2)}{}^{\mu}} + e_1e_2\frac{\dot{x}_{(1)\mu}\dot{x}_{(2)}{}^{\mu}}{|(x_{(1)\mu} - x_{(2)\mu})\dot{x}_{(2)}{}^{\mu}|}\,d\tau.$$

The integral (2) is numerically equal to (1) but in (1) the condition

$$[x_{(1)\mu} - x_{(2)\mu}][x_{(1)}{}^{\mu} - x_{(2)}{}^{\mu}] = 0$$

is « written in » while in (2) it has to be remembered. One may take this constraint into account by means of a Lagrange multiplier; we prefer here an alternative method. We shall introduce the Lagrange generalized coordinates which take constraints automatically into account.

Let us put

$$(3) \quad \begin{aligned} x_{(1)\mu} - x_{(2)\mu} &= x_\mu, \\ \alpha x_{(1)\mu} + \beta x_{(2)\mu} &= \zeta_\mu, \quad \alpha + \beta = 1. \end{aligned}$$

The Lagrangian takes on the form

$$(4) \quad \begin{aligned} -L = m_1\sqrt{\dot{\zeta}^\mu\dot{\zeta}_\mu + 2\beta\dot{\zeta}^\mu\dot{x}_\mu + \beta^2\dot{x}^\mu\dot{x}_\mu} + m_2\sqrt{\dot{\zeta}^\mu\dot{\zeta}_\mu - 2\alpha\dot{\zeta}^\mu\dot{x}_\mu + \alpha^2\dot{x}^\mu\dot{x}_\mu} \\ + e_1e_2\frac{\dot{\zeta}^\mu\dot{\zeta}_\mu + (\beta-\alpha)\dot{\zeta}^\mu\dot{x}_\mu - \alpha\beta\dot{x}^\mu\dot{x}_\mu}{|\dot{\zeta}^\mu x_\mu|}, \end{aligned}$$

where we understand that

$$(5) \quad x^0 \equiv \varepsilon\sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$$

and consequently

$$(6) \quad \dot{x}^0 \equiv -\frac{x_i}{x_0}\dot{x}^i \ , \quad i = 1, 2, 3.$$

Equations of motion take on the form

$$(7) \quad \frac{d}{d\tau}\frac{\partial L}{\partial\dot{\zeta}^\mu} = 0 \ , \quad \frac{d}{d\tau}\left(\frac{\partial L}{\partial\dot{x}^i}\right)_{\text{tot}} - \left(\frac{\partial L}{\partial x^i}\right)_{\text{tot}} = 0,$$

where

$$(8) \quad \left(\frac{\partial L}{\partial\dot{x}^i}\right)_{\text{tot}} = \frac{\partial L}{\partial\dot{x}^i} + \frac{\partial L}{\partial\dot{x}^0}\frac{\partial\dot{x}^0}{\partial\dot{x}^i} = \frac{\partial L}{\partial\dot{x}^i} - \frac{x_i}{x_0}\frac{\partial L}{\partial\dot{x}_0}.$$

THE CANONICAL FORM OF THE THEORY

Let us introduce the canonically conjugated momenta

$$(9) \qquad -\frac{\partial \mathrm{L}}{\partial \dot{\zeta}^\mu} = \mathrm{P}_\mu \quad , \quad -\left(\frac{\partial \mathrm{L}}{\partial \dot{x}^i}\right)_{\mathrm{tot}} = p_i.$$

The momentum $\vec{p} = (p^1, p^2, p^3)$ has no simple transformation properties with respect to the Lorentz group. It will be therefore convenient to introduce the four-vector

$$(10) \qquad -\frac{\partial \mathrm{L}}{\partial \dot{x}^\mu} = \pi_\mu.$$

π_μ has no mechanical meaning; such a meaning has only

$$(11) \qquad p_i = \pi_i - \frac{\pi_0}{x_0} x_i = \frac{1}{x_0}(x_0 \pi_i - x_i \pi_0).$$

Since $x_i \pi_k - x_k \pi_i = x_i p_k - x_k p_i$, we may say that altough π_μ contains an unphysical component, the bivector

$$(12) \qquad \Omega_{\mu\nu} = x_\mu \pi_\nu - x_\nu \pi_\mu$$

contains only physical components of π_μ (i. e. p_i). In the subsequent calculation we shall use the four-vector π_μ; the result of our calculations will be correct if it is possible to express it by means of the bivector $\Omega_{\mu\nu}$.

We may formulate now the main goal of this paper. The Lagrangian (4) is a homogeneous function of the velocities $\dot{\zeta}^\mu$ and $\dot{x}^i$. It is therefore impossible to calculate the Hamiltonian; in fact, because of the Euler homogeneity relation

$$(13) \qquad \dot{\zeta}^\mu \frac{\partial \mathrm{L}}{\partial \dot{\zeta}^\mu} + \dot{x}^i \left(\frac{\partial \mathrm{L}}{\partial \dot{x}^i}\right)_{\mathrm{tot}} = \mathrm{L},$$

the Hamiltonian is identically equal to zero. But it is well known in the variational calculus [*11*] that, if it is possible to express the Euler homogeneity relation by means of momenta, one obtains a parameter-invariant relation which for all practical purposes plays the role of the Hamiltonian. In particular, transition to geometrical optics may be based on the homogeneity relation expressed by means of momenta.

Our aim is to write down the homogeneity relation (13) by means of momenta. Using the definitions (9) and dividing by $-$ L we may write the homogeneity relation in a parameter-invariant form:

$$\frac{\dot{\zeta}^\mu}{-\mathrm{L}}\mathrm{P}_\mu + \frac{\dot{x}^i}{-\mathrm{L}}p_i = 1. \tag{14}$$

It follows from (11) and (6) that

$$\dot{x}^i p_i = \dot{x}^\mu \pi_\mu. \tag{15}$$

Consequently, the homogeneity relation takes on the Lorentz invariant form

$$\frac{\dot{\zeta}^\mu}{-\mathrm{L}}\mathrm{P}_\mu + \frac{\dot{x}^\mu}{-\mathrm{L}}\pi_\mu = 1. \tag{16}$$

We see that only parameter-invariant quotients $\dot{\zeta}^\mu/-\mathrm{L}$ and $\dot{x}^\mu/-\mathrm{L}$ enter the homogeneity relation.

Let us solve the definitions (9) and (10) of P_μ and π_μ with respect to $\dot{\zeta}_\mu$ and $\dot{x}_\mu$, considering in the process all scalar coefficients as given. We obtain

$$\dot{\zeta}_\mu = \frac{1}{1-(e_1e_2)^2\dfrac{xy}{\mathrm{D}^2}}\left\{\left(\beta^2 y + \alpha^2 x - 2e_1e_2\alpha\beta\frac{xy}{\mathrm{D}}\right)\left(\mathrm{P}_\mu + e_1e_2\frac{\mathrm{N}}{\mathrm{D}^2}\varepsilon x_\mu\right) + \left(\alpha x - \beta y - e_1e_2\frac{\beta-\alpha}{\mathrm{D}}xy\right)\pi_\mu\right\}, \tag{17}$$

$$\dot{x}_\mu = \frac{1}{1-(e_1e_2)^2\dfrac{xy}{\mathrm{D}^2}}\left\{\left(\alpha x - \beta y - e_1e_2 xy\frac{\beta-\alpha}{\mathrm{D}}\right)\left(\mathrm{P}_\mu + e_1e_2\frac{\mathrm{N}}{\mathrm{D}^2}\varepsilon x_\mu\right) + \left(x + y + 2e_1e_2\frac{xy}{\mathrm{D}}\right)\pi_\mu\right\}, \tag{18}$$

where

$$x = \frac{1}{m_1}\sqrt{\dot{x}_{(1)\mu}\dot{x}_{(1)}{}^\mu}, \quad y = \frac{1}{m_2}\sqrt{\dot{x}_{(2)\mu}\dot{x}_{(2)}{}^\mu}, \qquad \mathrm{N} = \dot{x}_{(1)\mu}\dot{x}_{(2)}{}^\mu, \quad \mathrm{D} = |\dot{\zeta}^\mu x_\mu|. \tag{19}$$

We shall assume that our parameter τ is a monotonically increasing function of time in some inertial system of reference; hence

$$|\dot{\zeta}^\mu x_\mu| = \varepsilon\dot{\zeta}^\mu x_\mu. \tag{20}$$

On taking into account that

$$-\mathrm{L} = m_1^2 x + m_2^2 y + e_1e_2\frac{\mathrm{N}}{\mathrm{D}}, \tag{21}$$

we may write the homogeneity relation (16) in the form

$$\frac{\dot{\xi}^\mu}{D}P_\mu + \frac{\dot{x}^\mu}{D}\pi_\mu = m_1^2\frac{x}{D} + m_2^2\frac{y}{D} + e_1e_2\frac{N}{D^2}. \tag{22}$$

We see that to express the homogeneity relation entirely by means of momenta, we need three invariants x/D, y/D and N/D^2. It turns out, however, that the last invariant disappears from the homogeneity relation; consequently, we have to find only x/D and y/D.

Let us multiply equation (17) and (18) by x^μ. In the first case we obtain, because of (20), εD; in the second case we obtain zero because of the constraint equation. In this way we obtain two equations for the two unknown parameter-invariant quantities x/D and y/D:

$$\begin{aligned}\varepsilon\left[1 - (e_1e_2)^2\frac{xy}{D^2}\right] &= \left(\beta^2\frac{y}{D} + \alpha^2\frac{x}{D} - 2e_1e_2\alpha\beta\frac{xy}{D^2}\right)P_\mu x^\mu \\ &\quad + \left[\alpha\frac{x}{D} - \beta\frac{y}{D} - e_1e_2(\beta-\alpha)\frac{xy}{D^2}\right]\pi_\mu x^\mu, \\ 0 &= \left[\alpha\frac{x}{D} - \beta\frac{y}{D} - e_1e_2(\beta-\alpha)\frac{xy}{D^2}\right]P_\mu x^\mu + \left(\frac{x}{D} + \frac{y}{D} + 2e_1e_2\frac{xy}{D^2}\right)\pi_\mu x^\mu.\end{aligned} \tag{23}$$

Equations (23) are quadratic and have two solutions but one of them is singular for $e_1e_2 \to 0$ and therefore has to be rejected. The other solution, regular for $e_1e_2 \to 0$, has the form

$$\begin{aligned}\frac{D}{x} &= \varepsilon(\alpha P_\mu x^\mu + \pi_\mu x^\mu) - e_1e_2, \\ \frac{D}{y} &= \varepsilon(\beta P_\mu x^\mu - \pi_\mu x^\mu) - e_1e_2.\end{aligned} \tag{24}$$

Introducing (24), (17) and (18) into the homogeneity relation (22) we get after some rearrangement

$$\begin{aligned}&\alpha\beta P_\mu P^\mu + \frac{1}{(x^\mu P_\mu)^2}\Omega_{\mu\nu}P^\mu\Omega^{\alpha\nu}P_\alpha - \frac{\pi_\mu x^\mu}{P_\mu x^\mu}[m_2^2 - m_1^2 - (\beta-\alpha)P_\mu P^\mu] \\ &+ (m_1^2 + m_2^2 - P_\mu P^\mu)\frac{e_1e_2}{\varepsilon x^\mu P_\mu} + \frac{(e_1e_2)^2}{\varepsilon x^\mu P_\mu}\left\{\frac{m_1^2}{\varepsilon(\alpha P_\mu x^\mu + \pi_\mu x^\mu) - e_1e_2}\right. \\ &\left. + \frac{m_2^2}{\varepsilon(\beta P_\mu x^\mu - \pi_\mu x^\mu) - e_1e_2}\right\} = m_1^2\beta + m_2^2\alpha,\end{aligned} \tag{25}$$

which is just what we sought for, namely the homogeneity relation expressed by means of momenta. The left-hand side of (25) does not depend on π_μ but only on p_i, because

$$\pi_\mu x^\mu = p_i x^i = -\vec{p}\cdot\vec{x}. \tag{26}$$

Consequently, equation (25) has the form

$$\mathrm{F}(\mathrm{P}_\mu, \vec{p}, \vec{x}) = m_1^2\beta + m_2^2\alpha. \tag{27}$$

α and β have been assumed constant during the whole calculation which led to equation (25); they are supposed to satisfy the condition $\alpha + \beta = 1$ but are otherwise arbitrary. It turns out [*10*], however, that just as in the newtonian two-body problem, there exists a preferred choice of α and β which is the most natural and which substantially simplifies theory of the internal motion of two bodies. To show this we shall introduce the notion of a reduced homogeneity relation.

We have the following theorem [*11*]: there exists parameter τ such that the equations of motion take on the canonical form

$$\dot{\zeta}^\mu = \frac{\partial \mathrm{F}(\mathrm{P}_\mu, \vec{p}, \vec{x})}{\partial \mathrm{P}_\mu}, \quad \dot{\mathrm{P}}_\mu = -\frac{\partial \mathrm{F}(\mathrm{P}_\mu, \vec{p}, \vec{x})}{\partial \zeta^\mu}, \tag{28 a}$$

$$\dot{x}^i = \frac{\partial \mathrm{F}(\mathrm{P}_\mu, \vec{p}, \vec{x})}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial \mathrm{F}(\mathrm{P}_\mu, \vec{p}, \vec{x})}{\partial x^i}. \tag{28 b}$$

(One has to remember, however, that altough $\mathrm{F}(\mathrm{P}_\mu, \vec{p}, \vec{x}) = \text{const}$ is always a first integral of the system (28), this constant is not to be determined from the initial conditions, but has to be always equal to the constant determined by the parameter-invariant equation (25)).

Since $\mathrm{P}_\mu = \text{const}$ is obviously an integral of the system (28) and since F does not depend on ζ^μ, we can insert in (28 *b*) instead of momenta P_μ their (constant) numerical values. In this way we obtain a closed system of equations for $\vec{p}$ and $\vec{x}$:

$$\dot{x}^i = \frac{\partial \mathrm{F_R}(\vec{p}, \vec{x})}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial \mathrm{F_R}(\vec{p}, \vec{x})}{\partial x^i}, \tag{29}$$

where $\mathrm{F_R}(\vec{p}, \vec{x})$ arises from $\mathrm{F}(\mathrm{P}_\mu, \vec{p}, \vec{x})$ when all the momenta P_μ are replaced by their numerical values.

We shall call the reduced homogeneity relation, the homogeneity relation (25) in which the momenta P_μ are replaced by their numerical values. The homogeneity relation and the reduced homogeneity relation are given formally by the same function but are entirely different notions. In particular, the reduced homogeneity relation can be substantially simplified by an appropriate choice of α and β; such a choice is not admissible in the homogeneity relation.

If $m_1 = m_2$, the only natural choice is $\alpha = \beta$ and we see that the term proportional to $\pi_\mu x^\mu$ disappears from the reduced homogeneity relation. In the general case this term will also disappear if

$$\beta - \alpha = \frac{m_2^2 - m_1^2}{M^2} \quad , \quad M^2 = P_\mu P^\mu. \tag{30}$$

This equation and the condition $\alpha + \beta = 1$ determine α and β uniquely:

$$\begin{aligned} \alpha &= \frac{1}{2}\left(1 + \frac{m_1^2 - m_2^2}{M^2}\right), \\ \beta &= \frac{1}{2}\left(1 + \frac{m_2^2 - m_1^2}{M^2}\right). \end{aligned} \tag{31}$$

For small velocities $M = m_1 + m_2$ and (31) goes over into the usual classical expression for α and β.

It will be convenient to write down the term $\alpha\beta P_\mu P^\mu$ which, in the reduced homogeneity relation, is simply a constant, on the right-hand side. In this way we obtain the reduced homogeneity relation in the form

$$\begin{aligned} &-\frac{1}{(x^\mu P_\mu)^2}\Omega_{\mu\nu}P^\mu\Omega^{\alpha\nu}P_\alpha + \left(M^2 - m_1^2 - m_2^2\right)\frac{e_1e_2}{\varepsilon x^\mu P_\mu} \\ &- \frac{(e_1e_2)^2}{\varepsilon x^\mu P_\mu}\left\{\frac{m_1^2}{\varepsilon(\alpha P_\mu x^\mu + \pi_\mu x^\mu) - e_1e_2} + \frac{m_2^2}{\varepsilon(\beta P_\mu x^\mu - \pi_\mu x^\mu) - e_1e_2}\right\} \\ &= \frac{1}{4M^2}[M^2 - (m_1 + m_2)^2][M^2 - (m_1 - m_2)^2]. \end{aligned} \tag{32}$$

In the centre of mass system, $P_0 = M$, $P_1 = P_2 = P_3 = 0$ and therefore

$$-\frac{1}{(x^\mu P_\mu)^2}\Omega_{\mu\nu}P^\mu\Omega^{\alpha\nu}P_\alpha \underset{\text{C.M.}}{=\!=} (\vec{p})^2 \quad , \quad \varepsilon x^\mu P_\mu \underset{\text{C.M.}}{=\!=} M\,|\,x\,|. \tag{33}$$

Hence, the reduced homogeneity relation takes on the form

$$\begin{aligned} &(\vec{p})^2 + \frac{M^2 - m_1^2 - m_2^2}{M}\frac{e_1e_2}{|\,\vec{x}\,|} \\ &- \frac{(e_1e_2)^2}{M\,|\,\vec{x}\,|}\left\{\frac{m_1^2}{\alpha M\,|\,\vec{x}\,| - \varepsilon\vec{p}\cdot\vec{x} - e_1e_2} + \frac{m_2^2}{\beta M\,|\,\vec{x}\,| + \varepsilon\vec{p}\cdot\vec{x} - e_1e_2}\right\} \\ &= \frac{1}{4M^2}[M^2 - (m_1 + m_2)^2][M^2 - (m_1 - m_2)^2], \end{aligned} \tag{34}$$

where α and β are given by (31).

Comparing the exact formula (34) with approximate Darwin's Hamiltonian [*2*], to say nothing about Primakoff and Holstein's [*12*] or Kerner's [*13*] Hamiltonians, we see how substantial a simplification has been achieved in the result of our procedure.

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