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An exponentiation theorem for unbounded derivations

by

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ABSTRACT — We give a sufficient (and necessary) condition to define the exponential of unbounded derivations in C^* -algebras.

1. DEFINITIONS

Let $\mathcal{A}$ be a Banach algebra, a derivation is a linear function D from a dense sub-algebra $\mathcal{A}^{(1)}$ of $\mathcal{A}$, into $\mathcal{A}$, such that

$$(1.1) \quad \begin{aligned} &\forall x \in \mathcal{A}^{(1)} \\ &\forall y \in \mathcal{A}^{(1)} \end{aligned} \quad D(xy) = D(x)y + xD(y)$$

For a $*$ -Banach algebra $\mathcal{A}$, the derivation D is said to be hermitian if:

$$\forall x \in \mathcal{A}^{(1)} \quad x^* \in \mathcal{A}^{(1)} \quad \text{and} \quad D(x^*) = (D(x))^*.$$

The set of the elements x in $\mathcal{A}$ such that the function

$$\zeta \rightarrow \sum_{n=0}^{\infty} \frac{\zeta^n}{n!} D^n(x)$$

exists and is analytic in some neighbourhood of 0, is called « the set of the analytic elements » with respect to this derivation and is written $\mathcal{A}^{(a)}$.

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2. THEOREM

Let $\mathcal{A}$ be a C^* -algebra, D an hermitian closed derivation of $\mathcal{A}$, such as $\mathcal{A}^{(a)}$ is dense in $\mathcal{A}$, then D induces a strongly continuous group $\{\alpha_t | t \in \mathbb{R}\}$ of automorphisms of $\mathcal{A}$.

Proof. — If $x \in \mathcal{A}^{(a)}$, $\exists t_x > 0$ such that $t \in \mathbb{R}$, $|t| \leq t_x$ we can define:

$$\alpha_t(x) = \sum_{n=0}^{\infty} \frac{t^n}{n!} D^n(x)$$

which is absolutely convergent in $\mathcal{A}$.

$\alpha_t(x) \in \mathcal{A}^{(a)}$, since for $|t'| < t_x - |t|$ we shall show that:

$$(2.1) \quad \alpha_{t'}(\alpha_t(x)) = \alpha_{t+t'}(x).$$

We write

$$y = \alpha_t(x); \quad y_j = \sum_{n=0}^j \frac{D^n(x)}{n!} t^n; \quad y_j \in \mathcal{A}^{(1)}.$$

$$D(y_j - y_k) = \sum_{n=k+1}^j \frac{D^{n+1}(x)}{n!} t^n$$

now $\sum_{n=0}^{\infty} \frac{D^n(x)}{n!} t^n$ is analytic on $] -t_x, t_x[$, therefore ([I], 9.3.5) $(\alpha_t(x))'_t$ is absolutely and uniformly converging on the same interval

$$(\alpha_t(x))'_t = \sum_{n=1}^{\infty} \frac{D^n(x)}{(n-1)!} t^{n-1} = \sum_{n=0}^{\infty} \frac{D^{n+1}(x)}{n!} t^n$$

We write $z_j = \sum_{n=0}^j \frac{D^{n+1}(x)}{n!} t^n$, then $(z_j)_j$ is a Cauchy sequence for $\|\cdot\|$ and:

$$z_j - z_k = \sum_{n=k+1}^j \frac{D^{n+1}(x)}{n!} t^n = D(y_j - y_k).$$

So $D(y_j - y_k)$ converges to 0 as j and k go to infinity. Let

$$z = \lim_{j, \infty} D(y_j).$$

Now, $y = \lim_{j, \infty} y_j$. As D is closed, $z = D(y)$

$$D(y_j) = \sum_{n=1}^j \frac{D^n(D(x))}{n!} t^n, \quad \lim_{j, \infty} D(y_j) = \alpha_t(D(x)).$$

hence

$$(2.2) \quad D(\alpha_t(x)) = \alpha_t(D(x))$$

and consequently

$$\begin{aligned} \alpha_{t'}(\alpha_t(x)) &= \lim_{l, \infty} \sum_{k=0}^l \frac{D^k}{k!} \left(\sum_{n=0}^{\infty} \frac{D^n(x)}{n!} t^n \right) t'^k \\ &= \lim_{l, \infty} \sum_{n=0}^{\infty} \sum_{k=0}^l \frac{D^{k+n}(x)}{k!n!} t^n t'^k \end{aligned}$$

which is absolutely converging as l goes to infinity, so we can rearrange the terms:

$$\sum_{m=0}^{\infty} \frac{D^m(x)}{m!} (t + t')^m = \alpha_{t+t'}(x).$$

Through elementary calculations, taking advantage of the absolutely convergence of the series and of the continuity of $*$ one gets:

$$(2.3) \quad \alpha_t(\lambda x + \lambda y) = \lambda \alpha_t(x) + \mu \alpha_t(y)$$

$$(2.4) \quad \alpha_t(xy) = \alpha_t(x) \alpha_t(y)$$

$$(2.5) \quad \alpha_t(x^*) = (\alpha_t(x))^*$$

for $t \in \mathbb{R}$ sufficiently small.

Moreover $\forall t \in \mathbb{R}, \exists m \in \mathbb{N}, |t| < mt_x$; we write

$$\alpha_t(x) = \left[\frac{\alpha_t}{m} \right]^m(x)$$

α_t is now well defined for all $t \in \mathbb{R}$ on $\mathcal{A}^{(a)}$ and fulfils (2.1) and (2.2) for every x in $\mathcal{A}^{(a)}$,

α_t is a $*$ -algebra isomorphism applying $\mathcal{A}^{(a)}$ into $\mathcal{A}^{(a)}$ and $\forall x \in \mathcal{A}^{(a)}, t \rightarrow \alpha_t(x)$ is an analytic function. We shall extend α_t to $\mathcal{A}$. We can assume that $\mathcal{A}$ has a unit element, for, if not, we can define D on $\tilde{\mathcal{A}} = \mathbb{C} \times \mathcal{A}$, the algebra obtained from $\mathcal{A}$ by adjunction of a unit element,

$$D(\lambda, x) = (0, D(x)).$$

Moreover, we can assume that $e \in \mathcal{A}^{(1)}$; because if not one settles: $D(e) = 0$.

Note that $\alpha_t(e) = e$ because $D(e) = 0$. If $y = \alpha_0(y)$ is invertible, there exists a neighbourhood of 0 such that $\alpha_t(y)$ is invertible. Now if $t \rightarrow \alpha_t(y)$ is analytic, then $t \rightarrow (\alpha_t(y))^{-1}$ is also analytic. We can put $\alpha_t(y^{-1}) = (\alpha_t(y))^{-1}$ so $y \in \mathcal{A}^{(a)} \Rightarrow y^{-1} \in \mathcal{A}^{(a)}$ for $x \in \mathcal{A}^{(a)}$; $\lambda \in \mathbb{C}$.

$$\begin{aligned} x - \lambda e \text{ invertible} &\Rightarrow \exists y \text{ and } (x - \lambda e)y = e \\ &\Rightarrow \alpha_t \text{ is well defined on } y \text{ and } [\alpha_t(x) - \lambda e]\alpha_t(y) = e \end{aligned}$$

therefore $(\alpha_t(x) - \lambda e)$ is invertible; hence $\text{Spec}' \alpha_t(x) \subset \text{Spec}' x$.

On the other hand, for an hermitian element y of $\mathcal{A}$:

$$\|y\| = \sup_{\zeta \in \text{Spec}' y} |\zeta|$$

([I], 15.4.14.1); hence:

$$\|\alpha_t(x)\|^2 = \|\alpha_t(x^*x)\| = \sup_{\zeta \in \text{Spec}' \alpha_t(x^*x)} |\zeta| \leq \sup_{\zeta \in \text{Spec}' x^*x} |\zeta| = \|x^*x\| = \|x\|^2$$

and finally $\|\alpha_t(x^*x)\| = \|x\|$ on $\mathcal{A}^{(a)}$. We extend α_t to $\mathcal{A}$ (2.1) to (2.5) still hold $\forall x \in \mathcal{A}$, $\exists (y_n)n$, $y_n \in \mathcal{A}^{(a)}$ and $x = \lim_n y_n$. Therefore

$$\lim_n \|\alpha_t(x) - \alpha_t(y_n)\| = 0.$$

$t \rightarrow \alpha_t(x)$ is continuous as a uniform limit of continuous functions. So that the one-parameter unitary group $\{\alpha_t \mid t \in \mathbb{R}\}$ is strongly continuous.

Comment. — We get an extension to C^* -algebras of the work of E. Nelson on Hilbert spaces ([5]).

3. CONVERSE PROPOSITION

We give a new proof of the result of Kastler-Pool-Poulsen [4], which improves some one of I. Guelfand [3].

Let $\mathcal{E}$ be a Banach space, $\{\alpha_t\}_{t \in \mathbb{R}}$ a strongly continuous one-parameter group of uniformly bounded linear operators, i. e.

$$\exists M > 0 \forall t \in \mathbb{R} \quad \|\alpha_t\| \leq M$$

$\forall x \in \mathcal{E}$, $\forall \rho \in \mathcal{L}_{\mathbb{C}}^1(\mathbb{R})$; let $\alpha(\rho)x = \int_{-\infty}^{+\infty} \alpha_t(x)\rho(t)dt$, which exists in the Bochner's sense since $\|\alpha_t(x)\rho(t)\| \leq M\|x\|\|\rho(t)\|$ and one has that:

$$t \rightarrow \|\alpha_t(x)\rho(t)\| \in \mathcal{L}_{\mathbb{C}}^1(\mathbb{R}).$$

PROPOSITION. — $\mathcal{E}^{(e)} (= \{x \in \mathcal{E} \mid t \in \mathbb{R} \rightarrow \alpha_t(x) \text{ is entire}\})$ is dense in $\mathcal{E}$.

Proof. — Let ρ be a function in $C^{\mathbb{R}}$ so that $\hat{\rho} \in \mathcal{D}$. Then $\rho \in \mathcal{S}$, and $\rho \in \mathcal{L}_C^1(\mathbb{R})$. Moreover, suppose that $\int_{-\infty}^{+\infty} \rho(t) dt = 1$. We notice that

$\forall \varepsilon > 0, \exists \eta > 0$ so that

$$\int_{-\infty}^{-\eta} |\rho(t)| dt \leq \varepsilon \quad \text{and} \quad \int_{\eta}^{+\infty} |\rho(t)| dt \leq \varepsilon.$$

Now if $\rho_n^{(t)} = n\rho(nt)$, $\int_{-\infty}^{-\frac{\eta}{n}} |\rho_n(t)| dt \leq \varepsilon$ and $\int_{\frac{\eta}{n}}^{+\infty} |\rho_n(t)| dt \leq \varepsilon$

$$\begin{aligned} \|\alpha(\rho_n)x - x\| &= \left\| \int_{-\infty}^{+\infty} \rho_n(t) \alpha_t(x) dt - \int_{-\infty}^{+\infty} \rho_n(t) x dt \right\| \\ &\leq \int_{-\infty}^{-\frac{\eta}{n}} |\rho_n(t)| \|\alpha_t(x) - x\| dt + \int_{-\frac{\eta}{n}}^{+\frac{\eta}{n}} \dots + \int_{\frac{\eta}{n}}^{+\infty} \dots \end{aligned}$$

Now, $\forall \varepsilon > 0, \exists n_0$ such that $\forall n \geq n_0$ and $|t| \leq \frac{\eta}{n} \Rightarrow \|\alpha_t(x) - x\| \leq \varepsilon$.

On the other hand $\|\alpha_t(x)\| \leq M \|x\|$, hence

$$\|\alpha(\rho_n)x - x\| \leq [2(M+1)\|x\| + 1]\varepsilon \quad \text{and} \quad x = \lim_n \alpha(\rho_n)x.$$

We prove that $\alpha(\rho_n)x \in \mathcal{E}^{(e)} \quad \forall x \in \mathcal{E}$. Indeed:

$$\begin{aligned} \alpha_r(\alpha(\rho_n)x) &= \alpha_r \left(\int_{-\infty}^{+\infty} \rho_n(t) \alpha_t(x) dt \right) = \int_{-\infty}^{+\infty} \rho_n(t) \alpha_{r+t}(x) dt \\ &= \int_{-\infty}^{+\infty} \rho_n(t-r) \alpha_t(x) dt \\ &= (\rho_n * h)(r) \end{aligned}$$

where $h(r) = \alpha_r(x)$.

Now, h being continuous and bounded, $\rho_n * h \in \mathcal{S}$ and $\widehat{\rho_n * h} = \widehat{\rho_n} \hat{h}$ is a distribution (cf. [6]) with compact support, hence due to the Paley-Wiener theorem $\rho_n * h$ is an entire function.

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