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Section A :
Physique théorique.

Induced representations of the (1 + 4) de Sitter group in an angular momentum basis and the decomposition of these representations with respect to representations of the Lorentz group

by

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ABSTRACT. — An explicit angular momentum basis is used in the construction of the induced unitary irreducible representations of the (1 + 4) de Sitter group, belonging to the principal continuous series. In the description of the covering group of the (1 + 4) de Sitter group we make use of groups of 2 × 2 quaternion matrices in a formalism similar to the SL(2, C) description of the Lorentz group. The above-mentioned representations of the de Sitter group are decomposed with respect to unitary representations of a non-compact subgroup. The latter is isomorphic to the covering group of the Lorentz group. The general features of the decomposition are established by means of global methods. Possible applications are indicated. As an example the derivation of a decomposition formula for matrix elements of finite transformations is treated in some detail.

I. INTRODUCTION

In the present article we are concerned with some aspects of the unitary irreducible representations (abbreviated UIR:s in the following) of the (1 + 4) de Sitter group. In general we denote by $SO_0(1, n)$ the identity component of the group of real linear homogeneous transformations of the variables $(x_0, x_1, \ldots, x_n)$ which leave the quadratic form

$$x_0^2 - x_1^2 - \ldots - x_n^2$$

invariant. We shall consider both single- and double-valued representations of the (1 + 4) de Sitter group, i. e. $SO_0(1, 4)$, i. e. we shall consider true representations of the universal covering group of $SO_0(1, 4)$. In general $\overline{SO_0(1, n)}$ will denote the universal covering group of $SO_0(1, n)$.

The group $SO_0(1, 4)$ appears in a number of physical problems, both as an invariance group and as a dynamical group. In its role as an invariance group, replacing the Poincaré group, it has been discussed for a long time (see e. g. [*1*]-[*3*] for some recent contributions). Within this field the decomposition of the UIR:s of $SO_0(1, 4)$ with respect to UIR:s of the Lorentz group is analogous to the corresponding de composition of the UIR:s of the Poincaré group within the Poincaré-invariant kinematical formalism [*4*], [*5*]. $SO_0(1, 4)$ is also encountered as a dynamical group of the non-relativistic hydrogen atom. The literature concerning this problem is now very extensive; in [*6*] the reader will find a review and a list of references. As an attempt to understand the various mass formulae for the elementary particles and resonances which have been proposed, a $SO_0(1, 4)$-model of a relativistic rotator has been considered [*7*]. In this model one considers the contraction of $SO_0(1, 4)$ with respect to $SO_0(1, 3)$ and one encounters the problem of decomposing the UIR:s of $SO_0(1, 4)$ with respect to those of $SO_0(1, 3)$ [*7*].

The present article is arranged in the following way. In section II we present the quaternion-matrix description of $\overline{SO_0(1, 4)}$ and we study the various subgroup decompositions of $\overline{SO_0(1, 4)}$ that will be of interest for the construction of the induced representations. In section III we construct the UIR:s of $\overline{SO_0(1, 4)}$ belonging to the principal continuous series as induced representations. An angular momentum basis is introduced and in section IV we give some properties of the matrix elements of finite transformations. The decomposition of the UIR:s of $\overline{SO_0(1, 4)}$ studied in the previous sections, with respect to UIR:s of the Lorentz group is given in section V.

II. QUATERNION-MATRIX DESCRIPTION OF $SO_0(1, 4)$

In [*8*] R. Takahashi has constructed UIR:s of $SO_0(1, n)$ as induced representations. The special case of $SO_0(1, 4)$ is treated in detail and the results are compared with the results obtained earlier with infinitesimal methods by Dixmier [9]. As is amply illustrated in [8] it is very convenient in this context to exploit the isomorphism between $\overline{SO_0(1, 4)}$ and a group of 2×2 quaternion matrices. The formalism resembles closely the SL(2, C)-description of $\overline{SO_0(1, 3)}$. However, we have chosen our conventions in a way which is slightly different from that in [8] and therefore we give in this section a short review of some formulae, in our conventions, which will be of importance for the construction of the UIR:s of $\overline{SO_0(1, 4)}$ in the next section (cf. [8], [10] for more details).

Let R and Q denote the real numbers and the real quaternions respectively. For an element $x \in Q$ we write $x \equiv x_1 + ix_2 + jx_3 + kx_4$ where $x_i \in R$, $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$. Furthermore we introduce the notations $\bar{x} \equiv x_1 - ix_2 - jx_3 - kx_4$, $\tilde{x} \equiv -j\bar{x}j \equiv x_1 + ix_2 - jx_3 + kx_4$, $|x| = (x \cdot \bar{x})^{1/2}$. It follows that

$$\overline{(x \cdot y)} = \bar{y} \cdot \bar{x}, \quad \widetilde{(x \cdot y)} = \tilde{y} \cdot \tilde{x}, \quad |x \cdot y| = |x| \cdot |y|, \quad x, y \in Q.$$

The set of 2×2 quaternion matrices

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where

$$\bar{a}b = \bar{c}d, \quad |a|^2 - |c|^2 = 1, \quad |d|^2 - |b|^2 = 1 \tag{2.1}$$

form a group under the usual matrix multiplication. This group is denoted G. In matrix form (2.1) reads

$$\begin{pmatrix} \bar{a}, & -\bar{c} \\ -\bar{b}, & \bar{d} \end{pmatrix}\begin{pmatrix} a, & b \\ c, & d \end{pmatrix} = \begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix}$$

Since in a group a left inverse element is also a right inverse element the order of the matrices on the left hand side can be reversed and one gets an equivalent form of the condition (2.1) namely

$$a\bar{c} = b\bar{d}, \quad |a|^2 - |b|^2 = 1, \quad |d|^2 - |c|^2 = 1. \tag{2.2}$$

The group G is isomorphic to $\overline{SO_0(1, 4)}$. With

$$X \equiv \begin{pmatrix} x_0, & \bar{x} \\ x, & x_0 \end{pmatrix}, \quad x \in Q, \quad x_0 \in R, \tag{2.3}$$

one finds that under the transformation

$$X \xrightarrow{g} X' = gXg^+ = \begin{pmatrix} x'_0, & \bar{x}' \\ x', & x'_0 \end{pmatrix}$$

where $g \in \mathrm{G}$, $(g^{+})_{rs} = \bar{g}_{sr}$, X′ is again of the form (2.3) and one has

$$x_0' - |x'|^2 = x_0^2 - |x|^2 \tag{2.4}$$

We will find it instructive to consider, besides X another quaternion matrix associated with the five-vector $(x_0, x_1, x_2, x_3, x_4)$. We introduce a similarity transformation by the non-singular quaternion matrix

$$\mathrm{C}(j) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1, & -j \\ -j, & 1 \end{pmatrix}$$

and consider

$$\tilde{\mathrm{X}} \equiv \mathrm{C}(j)\mathrm{X}\mathrm{C}^{-1}(j) \equiv \begin{pmatrix} x_0 + x_3 & , x_1 - ix_2 - kx_4 \\ x_1 + ix_2 + kx_4, & x_0 - x_3 \end{pmatrix}$$

which for $x_4 = 0$ reduces to $x_0\mathrm{I}_2 + x\cdot\sigma$, where stands for the Pauli vector matrix and $x \equiv (x_1, x_2, x_3)$. When applying a similarity transformation by C(j) to G we obtain another group $\mathscr{G}$, isomorphic to G and $\overline{\mathrm{SO}_0(1,4)}$. We write

$$\mathscr{G} = \mathrm{C}(j)\mathrm{G}\mathrm{C}^{-1}(j) \quad g = \begin{pmatrix} \alpha, & \beta \\ \gamma, & \delta \end{pmatrix} \tag{2.5}$$

It follows that

$$\tilde{\alpha}\gamma = \tilde{\gamma}\alpha, \quad \tilde{\beta}\delta = \tilde{\delta}\beta, \quad \tilde{\alpha}\delta - \tilde{\gamma}\beta = 1$$

or equivalently

$$\alpha\tilde{\beta} = \beta\tilde{\alpha}, \quad \gamma\tilde{\delta} = \delta\tilde{\gamma}, \quad \alpha\tilde{\delta} - \beta\tilde{\gamma} = 1$$

Thus under the transformation

$$\tilde{\mathrm{X}} \to \tilde{\mathrm{X}}' = g\tilde{\mathrm{X}}g^{+} = \begin{pmatrix} x_0' + x_3' & , x_1' - ix_2' - kx_4' \\ x_1' + ix_2' + kx_4', & x_0' - x_3' \end{pmatrix}$$

the invariance condition (2.4) holds.

Some properties of $\overline{\mathrm{SO}_0(1,4)}$ and its UIR are most easily discussed in terms of properties of G while others are most conveniently studied as properties of $\mathscr{G}$. In the construction of the induced representations one makes use of the decomposition of the group into products of various subgroups and the corresponding coset spaces are used as carrier spaces for the representation spaces. Therefore we next give a survey of facts, which are of interest in this context.

We define U as the set of all $x \in \mathrm{Q}$ for which $|x| = 1$ and consider the following subgroups of G:

$$\begin{aligned}
&\mathrm{K}: \text{ the elements of G of the form } \begin{pmatrix} u, & 0 \\ 0, & v \end{pmatrix} \quad u, v \in \mathrm{U} \\
&\mathrm{N}: \text{ the elements of G of the form } \begin{pmatrix} 1 + x, & -j\tilde{x} \\ -\tilde{x}j, & 1 + \tilde{x} \end{pmatrix} \quad x = -\tilde{x} \\
&\mathrm{A}: \text{ the elements of G of the form } \begin{pmatrix} \mathrm{Ch}^{t/2}, & -j\,\mathrm{Sh}^{t/2} \\ j\,\mathrm{Sh}^{t/2}, & \mathrm{Ch}^{t/2} \end{pmatrix} \quad t \in \mathrm{R} \\
&\mathrm{M}: \text{ the elements of G of the form } \begin{pmatrix} \tilde{\bar{u}}, & 0 \\ 0, & u \end{pmatrix} \quad u \in \mathrm{U}
\end{aligned} \tag{2.6}$$

M is the maximal compact subgroup of G. M is at the same time centralizer of A in K and normalizer of N in K and N is an invariant subgroup of A·N. The Iwasawa decomposition of G now reads

$$\mathrm{G} = \mathrm{N} \cdot \mathrm{A} \cdot \mathrm{K} \tag{2.7}$$

Since an arbitrary element of K can be uniquely decomposed as follows

$$\begin{pmatrix} u, & 0 \\ 0, & v \end{pmatrix} = \begin{pmatrix} \tilde{\bar{v}}, & 0 \\ 0, & v \end{pmatrix} \begin{pmatrix} \tilde{v}u, & 0 \\ 0, & 1 \end{pmatrix}$$

(2.7) can be carried a little bit further and one has

$$\mathrm{G} = \mathrm{NAMU} \tag{2.8}$$

where U is the subgroup of G formed by the elements of the form

$$\begin{pmatrix} u, & 0 \\ 0, & 1 \end{pmatrix}, \quad u \in \mathrm{U}$$

The images under the isomorphism (2.5) of the subgroups K, N, A, and M are denoted $\mathscr{K}$, $\mathscr{N}$, $\mathscr{A}$ and $\mathscr{M}$ respectively. Their explicit form is

$$\begin{aligned}
&\mathscr{K}: \text{ the elements of } \mathscr{G} \text{ of the form } \begin{pmatrix} \tilde{\bar{n}}, & -\tilde{\bar{r}} \\ r, & n \end{pmatrix}, \quad |n|^2 + |r|^2 = 1, \ n\tilde{r} = r\tilde{n} \\
&\mathscr{N}: \text{ the elements of } \mathscr{G} \text{ of the form } \begin{pmatrix} 1, & \mu \\ 0, & 1 \end{pmatrix} \quad \mu \in \mathrm{Z} \\
&\mathscr{A}: \text{ the elements of } \mathscr{G} \text{ of the form } \begin{pmatrix} e^{t/2}, & 0 \\ 0, & \bar{e}^{t/2} \end{pmatrix} \quad t \in \mathrm{R} \\
&\mathscr{M}: \text{ the elements of } \mathscr{G} \text{ of the form } \begin{pmatrix} \tilde{\bar{u}}, & 0 \\ 0, & u \end{pmatrix} \quad u \in \mathrm{U}
\end{aligned}$$

Here Z denotes the set of all $\mu \in \mathrm{Q}$ for which $\mu = \tilde{\mu}$. Furthermore we shall make use of the following decomposition of elements $g \in \mathscr{G}$ for which $\delta \neq 0$:

$$g = \mu \cdot \mathbf{A}(t) m \cdot z \tag{2.10}$$

Here $\mu \in \mathscr{N}$, $\mathbf{A}(t) \in \mathscr{A}$, $m \in \mathscr{M}$, $z \in \mathscr{Z}$ where $\mathscr{Z}$ denotes the group of matrices of the form

$$\begin{pmatrix} 1, & 0 \\ z, & 1 \end{pmatrix}, \qquad z \in \mathrm{Z}$$

(we let μ and z denote both elements of Z and the corresponding elements of $\mathscr{N}$ and $\mathscr{Z}$).

The representation spaces to be considered have as carrier spaces the coset spaces $\mathrm{T}\backslash\mathrm{G}$ and $\mathscr{T}\backslash\mathscr{G}$ respectively, where $\mathrm{T} = \mathrm{NAM}$ and $\mathscr{T} = \mathscr{N}\mathscr{A}\mathscr{M}$. $\mathrm{T}\backslash\mathrm{G}$ and $\mathscr{T}\backslash\mathscr{G}$ may be described by the sets U and Z since these are in a one-to-one correspondence with the groups U and $\mathscr{Z}$ respectively which appear in the decompositions (2.7) and (2.10) (in this context it is necessary to compactify Z by the addition of a point Z_∞ corresponding to elements of $\mathscr{G}$ with $\delta = 0$). In the theory of induced unitary representations there appears multipliers which reflect the transformation properties of the measures on the carrier spaces. Therefore we need the explicit form of the transformations of the elements of $\mathrm{T}\backslash\mathrm{G}$ and $\mathscr{T}\backslash\mathscr{G}$ under right translations with arbitrary elements in G and $\mathscr{G}$ respectively. With the notation (borrowed form [8])

$$ug = \mathrm{T}(u.g)$$

where $t \in \mathrm{T}$, $u.g \in \mathrm{U}$ one gets explicitly

$$u.g = -j(d - jub)^{-1} j(ua + jc)$$

Similarly with

$$zg = \tau(z.g)$$

where $\tau \in \mathscr{T}$, $z.g \in \mathscr{G}$ it follows that $(\delta \neq 0)$

$$z.g = (z\beta + \delta)^{-1}(z\alpha + \gamma).$$

$\mathrm{T}\backslash\mathrm{G}$ and $\mathscr{T}\backslash\mathscr{G}$ are isomorphic. One easily finds that the relation is given by (note that $z = 0$ corresponds to $u = 1$)

$$z = -j(u + 1)^{-1}(u - 1) \tag{2.11}$$

and conversely

$$u = (1 + jz)(1 - jz)^{-1}$$

We denote by $d\mu(u)$ the normalized invariant measure on the group U (which is isomorphic to SU(2)). From (2.11) it then follows that

$$d\mu(u) = \frac{4}{\pi^2} \cdot \frac{dz}{(1 + |z|^2)^3} \equiv d\mu(z) \tag{2.12}$$

where $dz \equiv dz_1 \cdot dz_2 \cdot dz_4$. Furthermore one finds that $M_G(u, g) = (-jub + d)$ is a multiplier with respect to the transformation $u \to u.g$ i. e.

$$M_G(u, g_1 g_2) = M_G(u, g_1) M_G(u.g_1, g_2).$$

Similarly $z\beta + \delta$ is a multiplier with respect to the transformation $z \to z.g$. The transformations of the measures are found to be

$$\begin{aligned} d\mu(u.g) &= |-jub + d|^{-6} d\mu(u) \\ d\mu(z.g) &= \left[\frac{1 + |z|^2}{|z\alpha + \gamma|^2 + |z\beta + \delta|^2}\right]^3 d\mu(z) \end{aligned} \tag{2.13}$$

According to (2.5) and (2.8) one has

$$\mathscr{G} = \mathscr{T} \cdot \mathscr{U}$$

where $\mathscr{U}$ is the image of U under (2.5). Thus for every $g \in \mathscr{G}$ one has

$$g = \tau k \qquad \tau \in \mathscr{T}, \qquad k \in \mathscr{K}$$

a decomposition which is, however, not unique since

$$\tau \cdot k = \tau m m^{-1} k = \tau' k', \quad \tau, \tau' \in \mathscr{T}, \quad k, k' \in \mathscr{K}, \quad m \in \mathscr{M}.$$

As a consequence functions on $\mathscr{T} \backslash \mathscr{G}$ can also be considered as functions on $\mathscr{K}$. This fact will be used in the next section.

III. THE PRINCIPAL CONTINUOUS SERIES OF UNITARY REPRESENTATIONS OF $\mathscr{G}$

In this section the UIR:s of $\mathscr{G}$ belonging to, in the terminology of [8], the principal continuous series will be considered. They are characterized by two real numbers denoted l and ρ. Here l is the weight of a UIR of $\mathscr{M}$ (isomorphic to SU(2)) and ρ is an arbitrary real number when l is an integer and ρ is a real number $\neq 0$ when l is a half-integer (cf. [8], our (l, ρ) correspond essentially to (n, ν) of [8]). As a result of the choice of the quaternion group $\mathscr{G}$ in section I our presentation again differs from that in [8]. Furthermore we will be more explicit concerning a realization of the representation space as a space of functions defined on the maximal compact subgroup $\mathscr{K}$ and satisfying a « covariance condition ». We prefer to give our construction explicitly in terms of $\mathscr{K}$ rather than in terms of $\mathscr{U}$ or $\mathscr{Z}$. since it is usually the subgroup $\mathscr{K}$ that is of greatest interest in the physical applications. The construction of the UIR:s (l, ρ) in a Hilbert space of functions defined on $\mathscr{Z}$ will primarily be used as a starting point because many formulae are simpler in that case, but eventually all relations will be given in terms of functions defined on $\mathscr{K}$. In order to establish the required relation between the two formalisms one can of course use the isomorphism (2.5) and the connection (2.12) between the measures. However it is instructive to see how it can also be derived in an alternative way, namely from the connection between two elements $k \in \mathscr{K}$ and $z \in \mathscr{Z}$ which belong to the same coset with respect to $\mathscr{T}$. For two such elements one has

$$k = \tau \cdot z$$

or in matrix form

$$\begin{pmatrix} \tilde{\bar{n}}, & -\tilde{\bar{r}} \\ r, & n \end{pmatrix} = \begin{pmatrix} \tilde{\delta}^{-1}, & \mu \\ 0\ , & \delta \end{pmatrix} \begin{pmatrix} 1, & 0 \\ z, & 1 \end{pmatrix} \tag{3.1}$$

With $\delta = \lambda \cdot u$, $\lambda > 0$, $u \in \mathrm{U}$ it follows that

$$\lambda^2 = \frac{1}{1 + |z|^2}$$

We now introduce parameters on $\mathscr{K}$ as follows

$$k = k_{12}(\varphi_1)k_{14}(\theta_1)k_{12}(\varphi_2)k_{34}(\psi)k_{14}(\theta_2)k_{12}(\varphi_3) \tag{3.2}$$

where $k_{ij}(\varphi)$ is a rotation of an angle φ in the $i-j$-plane. The normalized invariant measure on $\mathscr{K}$ is then

$$d\mu(k) = (32\pi^4)^{-1} \sin\theta_1 \sin\theta_2 \sin^2\psi d\varphi_1 d\varphi_2 d\varphi_3 d\theta_1 d\theta_2 d\psi$$

From a direct calculation, using the connection between the parameters implied by (3.1) one now obtains

$$d\mu(k) = \frac{4}{\pi^2} \frac{dz}{(1 + |z|^2)^3} d\mu(m) = d\mu(z) d\mu(m)$$

where $d\mu(m)$ stands for the normalized invariant measure on $\mathscr{M}$. Since (3.1) is just the condition that k and z belong to the same coset with respect to $\mathscr{T}$ we may write

$$\int_{\mathscr{K}} \Phi(g) d\mu(k) = \int_{\mathscr{Z}} \Phi(g) d\mu(z)$$

where $\Phi(g)$ is an arbitrary (integrable) class function with respect to $\mathscr{T}$ i. e. $\Phi(\tau \cdot g) = \Phi(g)$.

In the following we use the notation $\delta(g)$ to indicate that the (22)-element of g is δ and we use the notation $k(g)$ for an arbitrary element in $\mathscr{K}$ which belongs to the same right coset of $\mathscr{G}$ with respect to $\mathscr{T}$ as g. Thus $\delta(g) = \delta(\tau)$ for $g = \tau \cdot z$, $gk^{-1}(g) \in \mathscr{T}$ and

$$| \delta(g \cdot k^{-1}(g)) |^2 = \gamma^2 + \delta^2$$

In terms of these notations (2.13) reads

$$d\mu(z.g) = \left| \frac{\delta(zk^{-1}(z))}{\delta(zgk^{-1}(zg))} \right| d\mu(z). \tag{3.3}$$

We start the construction of the induced representations by the introduction of a « covariance condition ». We shall thus consider $(2l + 1)$-component functions $f^l = \{ f^l_m \}^l_{m=-l}$, defined on $\mathscr{G}$, which satisfy the condition

$$f^l_m(\tau g) = e^{i\rho t} \sum_{n=-l}^{n=l} \mathrm{D}^l_{mn}(m) f^l_n(g) \tag{3.4}$$

where $\tau = \mu \mathbf{A}(t) m$, $\mu \in \mathscr{N}$, $\mathbf{A}(t) \in \mathscr{A}$, $m \in \mathscr{M}$. $\mathrm{D}^l_{mn}(m)$ are the matrices of a unitary irreducible $(2l + 1)$-dimensional representation of $\mathscr{M}$. Thus l is a non-negative integer or half-integer. The scalar product in the $(2l + 1)$-dimensional vector space is as usual

$$(f^l(g), h^l(g))_{[l]} \equiv \sum_{m=-l}^{m=l} \overline{f^l_m(g)} h^l_m(g)$$

The covariance condition (3.4) involves unitary representations of the group $\mathscr{A}$ and $\mathscr{M}$ and consequently the scalar product $(f^l(g), h^l(g))_{[l]}$ is a class function with respect to $\mathscr{T}$. Let $\mathscr{H}^l$ denote the Hilbert space of vector functions $f^l(g)$ which satisfy (3.4) and for which

$$(f^l, f^l) \equiv \int_{\mathscr{Z}} (f^l(g), f^l(g))_{[l]} d\mu(z) < \infty$$

From (3.3) and (3.4) it follows that a unitary representation, denoted (l, ρ), is defined by

$$\mathrm{D}^{l,\rho}(g) f^l_m(z) = \left| \frac{\delta(zk^{-1}(z))}{\delta(zgk^{-1}(zg))} \right|^3 f^l_m(zg) \tag{3.5}$$

In (3.5) we can introduce $zg = \tau\cdot(z.g)$ and use (3.4) to get an explicit realization of the representation (l, ρ) on functions $f^l(z)$ defined on Z.

However, with later applications in mind, we note that an arbitrary factor $\tau_1 \in \mathscr{T}$ may be introduced every-where to the left of z in (3.5). Thus with $\tau_1\cdot z = g_1$ we may write

$$\mathrm{D}^{l,\rho}(g)f^l_m(g_1) = \left|\frac{\delta(g_1 k^{-1}(g_1))}{\delta(g_1 g k^{-1}(g_1 g))}\right|^3 f^l_m(g_1 g) \tag{3.6}$$

Equation (3.6) will be used as a starting point for other choices of realizations, in particular in terms of functions defined on $\mathscr{K}$ but also, in section V, in terms of functions defined on a subgroup $\mathscr{B} \subset \mathscr{G}$, isomorphic to the covering group of the Lorentz group.

With $g_1 = k_1 \in \mathscr{K}$ (3.6) reads

$$\mathrm{D}^{l,\rho}(g)f^l_m(k_1) = |\,\delta(k_1 g k^{-1}(k_1 g))\,|^{-3} f^l_m(k_1 g).$$

In analogy with the notation used in section II we write $k(k_1 g) \equiv k_1\cdot g$ (note that $k_1\cdot g$ is not uniquely determin). If $\tau = \mu\mathbf{A}(t)m$, $\mu \in \mathscr{N}$, $\mathbf{A}(t) \in \mathscr{A}$, $m \in \mathscr{M}$ we write $t = t(\tau)$, $m = m(\tau)$ and also in general $t = t(g)$ if $g = \mu\mathbf{A}(t)k$, $k \in \mathscr{K}$. The covariance condition now gives

$$\begin{aligned}\mathrm{D}^{l,\rho}(g)f^l_m(k_1) = {} & |\,\delta(k_1 g(k_1.g)^{-1})\,|^{-3} \\ & \times e^{i\rho t(k_1 g(k_1.g)^{-1})} \sum_{n=-l}^{n=l} \mathrm{D}^l_{mn}(m(k_1 g(k_1.g)^{-1})) f^l_n(k_1.g)\end{aligned} \tag{3.7}$$

which also can be written as

$$\begin{aligned}\mathrm{D}^{l,\rho}(g)f^l_m(k_1) = {} & |\,\delta(k_1 g(k_1.g)^{-1})\,|^{-3-2i\rho} \\ & \times \sum_{n=-l}^{n=l} \mathrm{D}^l_{mn}(m(k_1 g(k_1.g)^{-1})) f^l_n(k_1.g).\end{aligned} \tag{3.8}$$

The indefiniteness in the choice of the element $k_1.g$ is a left factor $m \in \mathscr{M}$. Using the covariance condition, the representation property of $\mathrm{D}^l(m)$ and the fact that $|\,\delta(m)\,| = 1$ it is easily seen that the r. h. s. of (3.7) and (3.8) are independent of the choice of $k_1.g$. Equation (3.8) gives the desired realization of (l, ρ) in terms of functions defined on $\mathscr{K}$. It is clear from the above that the Hilbert space $\mathscr{H}^l$ can equivalently be characterized as the space of vector functions $f^l(g)$ which satisfy (3.4) and for which

$$(f^l, f^l) \equiv \int_{\mathscr{K}} (f^l(g), f^l(g))_{[l]} d\mu(k). < \infty. \tag{3.9}$$

IV. MATRIX ELEMENTS OF FINITE TRANSFORMATIONS IN A REPRESENTATION (l, q)

The matrix elements of a general finite transformation in a UIR of a group play a fundamental role in the harmonic analysis on the group, a subject which is presently a field of common interest to mathematicians and physicists and which is developing rapidly. The induced representations (l, ρ) described in section III provide a framework within which we can in a straightforward way obtain an integral formula for the matrix elements of finite transformations. Infinitesimal methods can also be used in calculating the matrix elements. In this case one obtains from the representation of the elements of the Lie algebra as differential operators on the group parameters a set of differential relations for the matrix elements. In this section we give a short survey of the basic formulae in both approaches.

We start by introducing an orthonormal basis in which the matrix elements are to be calculated. $\mathscr{H}^l$ is an infinite sum of representation spaces for the UIR:s of $\mathscr{K}$ [9], [11]. These can be characterized by two real numbers p and q which are both integral or both half-integral and where $p \geqslant q|$ (see e. g. [12], [13]). $\mathscr{K}$ is isomorphic to $SU(2) \otimes SU(2)$ but p and q are not the weights of these two SU(2)-groups. They are rather associated with the subgroup chain $\mathscr{K} \supset \mathscr{M} \supset U(1)$. We choose to consider this characterization of the UIR:s of $\mathscr{K}$ mainly for physical reasons. It involves explicitly a physical three-dimensional rotation group. However it is also more in the spirit of the general procedure of Gelfand and Zeitlin [14]. The matrix elements corresponding to a transformation $k \in \mathscr{K}$ in a UIR (p, q) of $\mathscr{K}$ in an angular momentum basis for $\mathscr{M}$ are denoted $R^{jj'}_{mm'}(k; p, q)$. Consider an element $k \in \mathscr{K}$ parametrized according to (3.2). The angular momentum basis is chosen so that we have

$$R^{jj'}_{mm'}(k; p, q) = \sum_{r=-\min(j,j')}^{r=\min(j,j')} D^j_{mr}(\varphi_1, \theta_1, \varphi_2) R^{jj'}_t(\psi; p, q) D^{j'}_{rm'}(0, \theta_2, \varphi_3)$$

where as usual $D^j_{mn}(\varphi, \theta, \varphi') = e^{-im\varphi} d^j_{mn}(\theta) \cdot e^{-in\varphi'}$. The possible values of j and j' are $|q| \leqslant j, j' \leqslant p$. Thus the matrix elements in a fixed row l satisfy the covariance condition (3.4):

$$R^{lj'}_{mm'}(m \cdot k; p, q) = \sum_{n=-l}^{n=l} D^l_{m,n}(m) R^{lj'}_{nm'}(k; p, q)$$

and we can construct an orthonormal basis in $\mathscr{H}^l$ as a certain set of $\mathrm{R}^{lj'}_{mm'}(k; p', q')$-functions. According to [9] the following values occur:

$$\begin{aligned} p'&: \ l,\ l+1,\ l+2,\ \ldots \\ q'&: \ -l,\ -l+1,\ \ldots,\ l-1,\ l. \\ j'&: \ |q|,\ |q|+1,\ \ldots,\ p-1,\ p. \\ m'&: \ -j',\ -j'+1,\ \ldots,\ j'-1,\ j'. \end{aligned} \tag{4.1}$$

The dimension of the representation (p, q) is $(p+1)^2 - q^2$ i. e. one has

$$\int_{\mathscr{K}} d\mu(k)\overline{\mathrm{R}^{j_1 j_2}_{m_1 m_2}(k;\, p,\, q)}\mathrm{R}^{j_3 j_4}_{m_3 m_4}(k;\, p',\, q') = [(p+1)^2 - q^2]^{-1}\delta_{m_1 m_3}\delta_{m_2 m_4}\delta_{j_1 j_3}\delta_{j_2 4}\delta_{pp'}\delta_{qq'}.$$

Thus the functions

$$|\, j'm'\,;\, p'q'\,;\, lm\rho\,\rangle \equiv \mathrm{N}(p',\, q'\,;\, l\rho)[(p'+1)^2 - q'^2]^{1/2}\cdot \mathrm{R}^{lj'}_{mm'}(k\,;\, p',\, q') \tag{4.2}$$

where $\mathrm{N}(p', q'; l, \rho)$ is a phase-factor and where p', q', j', m' take the values given in (4.1), form an orthonormal basis in $\mathscr{H}^l$.

In order to obtain the simplest possible form of the matrix elements we note that every $g \in \mathscr{G}$ can be written as [8]

$$g = k_1\mathbf{A}(t)k_2, \qquad k_1,\, k_2 \in \mathscr{K}, \quad t \geqslant 0.$$

(since $\mathbf{A}(t)$ and $m \in \mathscr{M}$ commute, the factors k_1 and k_2 are not uniquely determined but the only indefiniteness is a factor $m \in \mathscr{M}$ to the right in k_1 and to the left in k_2). The matrix elements of k_1 and k_2 in the basis (4.2) are the functions $\mathrm{R}^{jj'}_{mm'}(k_{1,2};\, p,\, q)$. The properties of these functions and explicit formulae for them are known [14], [10]. Thus in order to obtain the matrix elements of a general transformation $g \in \mathscr{G}$ it is sufficient to consider only one kind of new matrix elements namely those of the particular acceleration $\mathbf{A}(t)$.

In order to obtain the explicit integral formula for the matrix elements of $\mathrm{D}^{l,\rho}(\mathbf{A}(t))$ we use (3.8) and get

$$\mathrm{D}^{l,\rho}(\mathbf{A}(t))f^l_m(m\cdot k_{34}(\psi)k_{14}(\theta)k_{12}(\varphi)) = (\mathrm{Ch}\ t - \cos\psi\ \mathrm{Sh}\ t)^{-\frac{3}{2}-i\rho}f^l_m(m\cdot k_{34}(\psi')k_{14}(\theta)k_{12}(\varphi)), \tag{4.3}$$

where

$$\cos\psi' = \frac{\cos\psi - \mathrm{Tg}\ ht}{1 - \cos\psi\ \mathrm{Tg}\ ht}. \tag{4.4}$$

The matrix elements of $\mathrm{D}^{l,\rho}(\mathbf{A}(t))$ are obviously diagonal in j' and m'

and independent of m' and they may thus be denoted $A_{j'}^{pqp'q'}(t;\, l,\, \rho)$. According to section III, (4.2) and (4.3) we thus get

$$A_{j'}^{pqp'q'}(t;\, l,\, \rho) = N(p,\, q;\, l\rho)N(p'q'l,\, \rho)[(p+1)^2 - q^2]^{1/2}$$
$$\times [(p'+1)^2 - q'^2)^{1/2} \cdot \int_{\mathscr{K}} d\mu(k) \sum_{m} R_{mm'}^{lj'}(k;\, pq)(\mathrm{Ch}\, t - \cos\psi\, \mathrm{Sh}\, t)^{-\frac{3}{2}-i\rho} R_{mm'}^{lj'}(k';\, p',\, q')$$

where the elements k and k' differ only with respect to the (34)-rotation in the canonical factorization (3.2). The relation between the parameters ψ and ψ' of the (34)-rotation is given by (4.4). Integration over the parameters belonging to $\mathscr{M}$ yields

$$A_{j'}^{pqp'q'}(t;\, l,\, \rho) = \overline{N(p,\, q;\, l,\, \rho)}N(p',\, q';\, l,\, \rho)[((p+1)^2 - q^2)((p'+1)^2 - q'^2)]^{1/2}$$
$$\times [(2l+1)(2j'+1)]^{-1} \int_0^{\pi} \frac{2}{\pi} \sin^2 \psi d\psi \cdot (\mathrm{Ch}\, t - \cos\psi\, \mathrm{Sh}\, t)^{-\frac{3}{2}-i\rho}$$
$$\times \sum_{k} \overline{R_k^{lj'}(\psi;\, pq)} R_k^{lj'}(\psi';\, p'q').$$

Using the known expressions for the $R_k^{lj'}$-functions the integral can be evaluated in terms of known functions in various ways. However because of the length of the explicit expressions we do not go into further details here.

For a general transformation we now have, in an obvious notation, the matrix elements ($g = k_1 \mathbf{A}(t) k_2$)

$$D_{jmj'm'}^{pqp'q'}(g;\, l,\, \rho)$$
$$= \sum_{j''=\max(|q|,|q'|)}^{j''=\min(p,p')} \sum_{m''=-j''}^{m''=j''} R_{mm''}^{jj''}(k_1;\, p,\, q) A_{j''}^{pqp'q'}(t;\, l,\, \rho) R_{m'',m'}^{j''j'}(k_2;\, p',\, q') \quad (4.5)$$

In the infinitesimal approach one exploits the fact that the elements in a fixed row of the general matrix element $D_{jmj'm'}^{pqp'q'}$ (i. e. fixed p, q, j and m) transform, by multiplication from the right in the argument, as a basis. One then considers infinitesimal transformations. In the basis (4.2) the generator P_0 of accelerations in the 3-direction acts as follows

$$P_0\, |j'm';\, p',\, q';\, lm\rho\rangle$$
$$= a(p',\, q';\, l\rho)[(p'+j'+2)(p-j'+1)]^{1/2}\, |j',\, m',\, p'+1,\, q';\, lm\rho\rangle$$
$$+\, a(p'-1,\, q';\, l\rho)[(p'+j'+1)(p'-j')]^{1/2}\, |j'm',\, p'-1,\, q';\, lm\rho\rangle$$
$$+\, b(p',\, q';\, l\rho)[(j'-q')(j'+q'+1)]^{1/2}\, |j'm',\, p',\, q'+1;\, lm\rho\rangle$$
$$+\, b(p',\, q'-1;\, l\rho)[(j'+q')(j'-q'+1)]^{1/2}\, |j'm';\, p'q'-1;\, lm\rho\rangle \quad (4.6)$$

where

$$a(p, q; l, \rho) = \left[\frac{(p+1-l)(p+2+l)\left(\left(p+\frac{3}{2}\right)^2+\rho^2\right)}{(p+1-q)(p+2-q)(p+1+q)(p+2+q)}\right]^{1/2}$$

$$b(p, q; l, \rho) = \left[\frac{(l-q)(l+q+1)\left(\left(q+\frac{1}{2}\right)^2+\rho^2\right)}{(p-q)(p-q+1)(p+q+1)(p+q+2)}\right]^{1/2}.$$

Equation (4.6) can be derived from the results of Dixmier for a $SU(2)\otimes SU(2)$ basis, by a change of basis [15] or by a passage to new ranges for the parameters in the Gelfand-Zeitlin patterns [16]. In terms of the following parameters on $\mathscr{G}$:

$$g = k_{12}(\varphi_1)k_{14}(\theta_1)k_{12}(\varphi_2)k_{34}(\psi_1)k_{14}(\theta_2)k_{12}(\varphi_3)\mathbf{A}(t)k_{34}(\psi_2)k_{14}(\theta_3)k_{12}(\varphi_4)$$

the differential operator expression for P_0 is

$$\begin{aligned} P_0 = i\Bigg[&\frac{s\theta_2 s\psi_2}{s\psi_1 \operatorname{Sh} t}\left(\frac{s\varphi_2}{s\theta_1}\frac{\partial}{\partial\varphi_1} + c\varphi_2\frac{\partial}{\partial\theta_1} - \frac{c\theta_1 s\varphi_2}{s\theta_1}\frac{\partial}{\partial\varphi_2}\right) \\ &+ \frac{c\theta_2 s\psi_2}{\operatorname{Sh} t}\frac{\partial}{\partial\psi_1} - \frac{s\theta_2 c\psi_1 s\psi_2}{s\psi_1 \operatorname{Sh} t}\frac{\partial}{\partial\theta_2} + c\psi_2\frac{\partial}{\partial t} - \frac{s\psi_2 \operatorname{Ch} t}{\operatorname{Sh} t}\frac{\partial}{\partial\psi_2}\Bigg] \end{aligned}$$

$$(s\varphi \equiv \sin\varphi,\ c\varphi \equiv \cos\varphi) \quad (4.7)$$

Since (4.6) shall be fulfilled with a row of (4.5) as a basis and with P_0 represented by (4.7) a set of differential relations for $A_j^{pqp'q'}$ are obtained (the calculations essentially consist of an elimination of the known compact parts from $D^{pqp'q'}_{jmj'm'}$, for more details see [10]). Four relations of similar structure are obtained. As an example we quote the following:

$$\begin{aligned} &[(p'+j+2)(p'-j+1)]^{1/2}\left[\frac{d}{dt} - p' \operatorname{Coth} t\right]A_j^{pqp'q'}(t; l, \rho) \\ &\quad + \frac{\begin{array}{c}[(j+q)(j-q)(p+j+1)(p-j+1) \\ (j+q')(j-q')(p'+j+2)(p'+j+1)]^{1/2}\end{array}}{j(2j+1)\operatorname{Sh} t} A_{j-1}^{pqp'q'}(t; l, \rho) \\ &\quad + \frac{(p+1)qq'[(p+j+2)(p-j+1)]^{1/2}}{j(j+1)\operatorname{Sh} t} A_j^{pqp'q'}(t; l, \rho) \\ &\quad + \frac{\begin{array}{c}[(j+1+q)(j+1-q)(p+j+2)(p-j) \\ (j+1+q')(j+1-q')(p'+1-j)(p'-j)]^{1/2}\end{array}}{(j+1)(2j+1)\operatorname{Sh} t} A_{j+1}^{pqp'q'}(t; l, \rho) \\ &\quad = -i\left[\frac{\begin{array}{c}(p'+1-l)(p'+2+l)(p'+q'+1) \\ (p'-q'+1)\left(\left(p'+\frac{3}{2}\right)^2+\rho^2\right)\end{array}}{(p'+2+q')(p'+2-q')}\right]^{1/2} A_j^{pqp'+1,q'}(t; l, \rho) \quad (4.8) \end{aligned}$$

The remaining relations correspond to the combinations $(p' - 1, q')$, $(p', q' + 1)$ and $(p', q' - 1)$ of the primed indices on the function on the r. h. s. These relations can be combined to give higher order equations which then contain a smaller number of different $A_j^{pqp'q'}$-functions. In particular it can be shown [17] that the special matrix elements $A_p^{ppp'q'}(t; l, \rho)$ satisfy a second order differential equation which has solutions in terms of Lamé-functions [18].

It may be remarked that relations of the type (4.8) are fulfilled by the matrix elements of any of the UIR:s of $\mathscr{G}$, only the r. h. s. is changed according to which UIR that is considered (cf. [9]). This may be contrasted with the situation in the global approach where the scalar product in the representation space is of two-point character for some supplementary series of representations. In these cases one gets a more complicated integral formula for the matrix elements. However one may expect that by exploiting the invariant measure it should be possible to simplify this into a single-integration formula as has been shown to be the case in a lower-dimensional example [19].

V. DECOMPOSITION OF THE REPRESENTATIONS (l, q) WITH RESPECT TO REPRESENTATIONS OF A NON-COMPACT SUBGROUP

The decomposition of a UIR of a group with respect to UIR:s of a subgroup has become of interest in the latest developments of the Regge pole theory. In this context it is the decomposition of a UIR of SL(2, C) with respect to SU(1, 1) that is of interest. It may be treated in the framework of induced representations (multiplier representations) [20], [21]. Infinitesimal methods can also be used [22], [24]. An essential tool in performing the decomposition is the completeness relation for the UIR:s, the Plancherel formula, of the non-compact subgroup in question. General methods for deriving the Plancherel formula has become available only recently [25]. Therefore rather few explicit results concerning the decomposition problem in the case of non-compact subgroups are known so far (see however [26]-[28] for some examples).

In this section we shall consider the decomposition of a UIR(l, ρ) with respect to UIR:s of a non-compact subgroup, denoted $\mathscr{B}$, which is isomorphic to $\overline{\mathrm{SO}_0(1, 3)}$. $\mathscr{B}$ is that subgroup of $\mathscr{G}$ which operates in the (0,124) space. Its maximal compact subgroup is $\mathscr{M}$. In performing the decomposition we will associate functions defined on $\mathscr{K}$ with functions defined on $\mathscr{B}$ and then use equation (3.6). To this end we need a description of the right cosets of $\mathscr{G}$ with respect to $\mathscr{T}$ in terms of $\mathscr{B}$. Consider therefore the double cosets of $\mathscr{G}$ with respect to $\mathscr{T}$ and $\mathscr{B}$. It turns out that $\mathscr{G}$ can be divided into three distinct double cosets. According to the previous sections every $g \in \mathscr{G}$ can be written

$$g = \tau k_{34}(\psi) k_{14}(\theta_2) k_{12}(\varphi_3)$$

and this factorization is unique. On the other hand an arbitrary element $b \in \mathscr{B}$ can be written as

$$b = m_1 \mathbf{A}_4(t) m_2, \qquad m_1, m_2 \in \mathscr{M}, \quad t \geqslant 0,$$

where $\mathbf{A}_4(t)$ is an acceleration in the 4-direction. A rotation $k_{34}(\psi)$ can be associated with an acceleration $\mathbf{A}_4(t)$ as follows:

i) $0 \leqslant \psi < \frac{\pi}{2}$: $k_{34}(\psi) = \tau \mathbf{A}_4(t)$, $\tau \in \mathscr{T}$, $\cos \psi = (\mathrm{Ch}\ t)^{-1}$, $\delta(\tau) = (\mathrm{Ch}\ t)^{-1/2}$.

ii) $\frac{\pi}{2} < \psi \leqslant \pi$: $k_{34}(\psi) = \tau k_{34}(\pi) \mathbf{A}_4(t)$, $\tau \in \mathscr{T}$, $\cos \psi = -(\mathrm{Ch}\ t)^{-1}$, $\delta(\tau) = (\mathrm{Ch}\ t)^{-1/2}$.

iii) $\psi = \frac{\pi}{2}$, $t = 0$. (5.1)

Since $\mathbf{A}_4(t) k_{14}(\theta_2) k_{12}(\varphi_3) \in \mathscr{B}$ it follows that all elements of $\mathscr{G}$ can be written in one of the following ways:

$$i)\ g = \tau e^+ b, \quad ii)\ g = \tau e^- b, \quad iii)\ g = \tau e^0 b,$$

where $\tau \in \mathscr{T}$, $b \in \mathscr{B}$ and e^+, e^- and e^0 denote $k_{34}(0)$, $k_{34}(\pi)$ and $k_{34}\left(\frac{\pi}{2}\right)$ respectively. The set *iii)* corresponds to a set of elements g caracterized by only five parameters. This set has zero invariant measure and need not be considered in the following.

With the notation $f_m^l(e^\pm b) \equiv f_m^{l\pm}(b)$ the scalar product in $\mathscr{H}^l$ previously given as an integral over $\mathscr{K}$ (cf. (3.9)) can be expressed as follows-

$$(f^l, f^l) \equiv \| f^l \|^2 = \int_{\mathscr{K}} \sum_{n=-l}^{l} | f_n^l(k) |^2 d\mu(k)$$

$$= (2\pi^2)^{-1} \int_0^\pi \sin^2 \psi d\psi \int_0^\pi \sin \theta_2 d\theta_2 \int_0^{2\pi} d\varphi_3 \sum_{n=-l}^{n=l} | f_n^l(k_{34}(\psi) k_{14}(\theta_2) k_{12}(\varphi_3)) |^2$$

$$= \sum_{\varepsilon:\pm} \sum_{n=-l}^{n=l} (2\pi^2)^{-1} \int_0^\infty \frac{\mathrm{Sh}^2\, t}{\mathrm{Ch}^3\, t} dt \int_0^\pi \sin \theta_2 d\theta_2 \int_0^\pi d\varphi_3 | f_n^{l\varepsilon}(\mathbf{A}_4(t) k_{14}(\theta_2) k_{12}(\varphi_3)) |^2. \tag{5.2}$$

Using the covariance condition (3.4) and the notation

$$h_n^{l\pm}(b) \equiv (\mathrm{Ch}\ t(b))^{-\frac{3}{2}} f_n^{l\pm}(b), \quad \| h^{l\pm}(b) \|_{[l]} \equiv (h^{l\pm}(b), h^{l\pm}(b))_{[l]}$$

(5.2) reads

$$\| f^l \|^2 = \sum_{\varepsilon} \int_{\mathscr{B}} \| h^{l\varepsilon}(b) \|_{[l]}^2 d\mu(b) \tag{5.3}$$

where

$$d\mu(b) = (32\pi^4)^{-1} \sin\theta_1 \sin\theta_2 \mathrm{Sh}^2\, t d\varphi_1 d\varphi_2 d\varphi_3 d\theta_1 d\theta_2 dt$$

is the invariant measure on $\mathscr{B}$. Through (5.3) we associate functions f^l which are square-integrable over $\mathscr{K}$ with functions $h^{l\pm}$ square integrable over $\mathscr{B}$. The corresponding Hilbert spaces are denoted $\mathscr{H}^{l\pm}$ and the content of equation (5.3) is that the representation space $\mathscr{H}^l$ is the direct sum of $\mathscr{H}^{l+}$ and $\mathscr{H}^{l-}$ (note that the elements in $\mathscr{H}^{l+}$ and $\mathscr{H}^{l-}$ satisfy different covariance conditions).

The next problem is to determine the properties of $\mathscr{H}^{l\pm}$ as representation spaces of UIR:s of $\mathscr{B}$. From the general formula (3.5) it follows that

$$\mathrm{D}^{l,\rho}(b) f_m^{l\pm}(b_1) = \left| \frac{\delta(e^{\pm} b_1 (k.e^{\pm} b_1)^{-1}}{\delta(e^{\pm} b_1 b (k.e^{\pm} b_1 b)^{-1}} \right|^3 f^{l\pm}(b_1 b) \tag{5.4}$$

Since $| \delta(e^{\pm} b (k \cdot e^{\pm} b)^{-1} | = [\mathrm{Ch}\ t(b)]^{1/2}$ equation (5.4) can be written

$$\mathrm{D}^{l,\rho}(b) h_m^{l\pm}(b_1) = h_m^{l\pm}(b_1 b) \tag{5.5}$$

i. e. in $\mathscr{H}^{l\pm}$ $\mathrm{D}^{l,\rho}(b)$ acts as the right regular representation. However, because of the fact that the functions $\{ h_m^{l\pm}(b) \}$ satisfy covariance conditions, the spaces $\mathscr{H}^{l+}$ and $\mathscr{H}^{l-}$ are not representation spaces of the whole regular representation. Because of the explicit form of the covariance condition the relevant restriction on the regular representation is easily identified.

An important property of the regular representation of $\mathscr{B}$ is the fact that it can be decomposed into a direct integral and sum of UIR:s belonging to the principal series. These representations are characterized by two real numbers, l_0 and ν where $\nu \geqslant 0$ and l_0 is a non-negative integer or half-integer. The decomposition is expressed by means of the generalized fourier coefficients of $h^{l\pm}(b)$ with respect to the UIR:s (l_0, ν). In this context we need the matrix elements in a UIR (l_0, ν) of an arbitrary element $b \in \mathscr{B}$. From a physical point of view it is most natural to perform all calculations in a standard angular momentum basis. The matrix elements in a UIR (l_0, ν) in this basis are denoted $\mathrm{D}^{l_0,\nu}_{jmj'm'}(b)$. The properties of these functions are well-known [21], [29]-[31]. The generalized fourier coefficients of $h_n^{l\pm}(b)$ are defined by

$$\int_{\mathscr{B}} D^{l_0,\nu}_{jmj'm'}(b^{-1})h^{l\pm}_n(b)d\mu(b) \equiv \delta_{j'l}\delta_{m'n}H^{l_0,\nu,l\pm}_{jmn} \tag{5.6}$$

Note that since the possible values of j' are $|l_0|$, $|l_0|+1, \ldots$ it follows that the fourier coefficients can be non-zero only if $l \geqslant |l_0|$. This is of course a consequence of the covariance condition used in the definition of the representations (l, ρ). From (5.6) and the Plancherel formula for the UIR:s of $\mathscr{B}$ we get

$$\| h^{l\pm} \|^2 = (2\pi^4)^{-2}\sum_{l_0}\int_0^\infty \| H^{l_0,\nu,l,\pm} \|^2(l_0^2+\nu^2)d\nu \tag{5.7}$$

where the sum goes over the values $\pm l$, $\pm(l-1), \ldots, \pm 1/2$ or 0 for l_0 and where

$$\| h^{l\pm} \|^2 \equiv \int_{\mathscr{B}} \| h^{l\pm}(b) \|^2_{[l]} d\mu(b)$$

and

$$\| H^{l_0\ \nu,l,\pm} \|^2 = \sum_{j\geq|l_0|}\ \sum_{m=-j}^{m=j}\ \sum_{n=-l}^{n=l} | H^{l_0\nu l\pm}_{jmn} |^2$$

We refer to [32] and [33] for more details concerning the Plancherel formula for $\mathscr{B}$. (Actually, more restrictive conditions on the functions f^l and $h^{l\pm}$ have to be introduced. However we do not here enter into a discussion of the minimal possible mathematical restriction under which the following formulae are valid, but content ourselves with the observation that there exist non-empty spaces for which they are well-defined). We donote by $\mathscr{H}^{l_0,\nu,l,\pm}$ the Hilbert space of $\{ H^{l_0,\nu,l,\pm}_{jmn} \}^{j=|l_0|,|l_0|+1,\ldots}_{|m|\leqslant j,|n|\leqslant l}$ with the norm $\| H^{l_0,\nu,l,\pm} \|$. Thus the content of (5.7) is that the Hilbert spaces $\mathscr{H}^{l\pm}$ can be decomposed into a direct sum and integral of spaces $\mathscr{H}^{l_0,\nu,l,\pm}$. In analogy with (5.7) we write formally

$$\mathscr{H}^{l\pm} = (2\pi^4)^{-2}\sum_{l_0}\int_0^\infty \mathscr{H}^{l_0,\nu,l,\pm}(l_0^2+\nu^2)d\nu. \tag{5.8}$$

Furthermore

$$\mathscr{H}^l = \mathscr{H}^{l+} \oplus \mathscr{H}^{l-}. \tag{5.9}$$

The equations (5.8) and (5.9) express the result of the decomposition of a UIR (l, ρ) of $\mathscr{G}$, isomorphic to $\overline{SO_0(1,4)}$, with respect of UIR:s of $\mathscr{B}$, isomorphic to $\overline{SO_0(1,3)}$. In the decomposition only the discrete parameter l_0 is restricted: $| l_0 | \leqslant l$, whereas all values of the continuous parameter ν occur. This seems to be the general situation when a decomposition of a representation belonging to a principal series is treated. In the case of decomposition of a representation belonging to a supplementary series the situation is more complicated [34], [35].

The general formulae derived above provide the frame-work within which more detailed explicit results may be derived. In the following we give as an example the derivation of a decomposition formula for the matrix element $D^{pqp'q'}_{mj'm'}(b; l, \rho)$. It involves the fourier coefficients of those functions $h^{l\pm}_m(b)$ which are associated with the orthonormal basis in $\mathcal{H}^l$ which was introduced in section IV. We have

$$h^{l\pm}_m(\mathbf{A}_4(t)) = (\mathrm{Ch}\ t)^{-\frac{3}{2}} f^l_m(\tau^{-1} k_{34}(\psi^\pm))$$

where $\psi = \psi^+$ for $0 \leqslant \psi < \pi/2$, $\psi = \psi^-$ for $\frac{\pi}{2} < \psi \leqslant \pi$. $\psi^\pm$ and t are related according to (5.1) i. e. $\delta(\tau^{-1}) = (\mathrm{Ch}\ t)^{1/2}$. Using the covariance condition one can write more generally

$$h^{l\pm}_m(m_1 \mathbf{A}_4(t) m_2) = (\mathrm{Ch}\ t)^{-\frac{3}{2}-i\rho} f^l_m(m_1^\pm k_{34}(\psi^\pm) m_2)$$

where $m^\pm = e^\pm m (e^\pm)^{-1}$, $m \in \mathcal{M}$. The $h^{l\pm}_m$-functions which correspond to the orthonormal basis (5.2) are thus

$$r^{l,j',\pm}_{mm'}(b; p, q, \rho) \equiv (\mathrm{Ch}\ t)^{-\frac{3}{2}-i\rho}[(p+1)^2 - q^2]^{1/2} \times \mathrm{N}(p, q; l, \rho) \sum_k \mathrm{D}^l_{m,k}(m_1^\pm) \mathrm{R}^{lj'}_k(\psi^\pm; p, q) \mathrm{D}^{j'}_{k,m'}(m_2)$$

$(b = m_1 \mathbf{A}_4(t) m_2)$.

For the fourier coefficients we write

$$\int d\mu(b) \mathrm{D}^{l_0;\nu}_{j''m''j'''m'''}(b^{-1}) r^{l,j',\pm}_{mm'}(b; p, q; \rho) \equiv \delta_{lj'''} \delta_{m,m'''} \mathcal{R}^{j'',l,j',\pm}_{m'',m,m'}(l_0, \nu; p, q; \rho).$$

Integration of the compact variables yields

$$\mathcal{R}^{j'',l,j',\pm}_{m'',m,m'}(l_0, \nu; p, q; \rho) = \delta_{m'm''} \delta_{j'j''} \cdot \mathcal{R}^{l,j'}_m(l_0, \nu; p, q; \rho)$$

where

$$\mathcal{R}^{l,j',\pm}_m(l_0, \nu; p, q; \rho) = \mathrm{N}(p, q; l, \rho) \frac{(-1)^{l-j'} \cdot 2 \cdot [(p+1)^2 - q^2]^{1/2}}{\pi(2l+1)(2j'+1)} \cdot \sum_k (\pm 1)^{m-k} \int_0^\infty (\mathrm{Ch}\ t)^{-\frac{3}{2}-i\rho} \mathrm{A}^{j',l}_{-k}(t; l_0, \nu) \mathrm{R}^{l,j'}_k(\psi^\pm; p, q)\ \mathrm{Sh}^2\ t dt \quad (5.10)$$

For a general scalar product in $\mathscr{H}^l$ one has, in an obvious notation:

$$(f^{1l}, f^{2l}) = \sum_{\varepsilon} \int d\mu(b)(h^{1l}(b), h^{2l}(b))_{[l]}$$
$$= (2\pi^4)^{-2} \sum_{\varepsilon} \sum_{l_0} \int_0^{\infty} (l_0^2 + v^2) \sum_j \sum_m \sum_n \overline{\mathrm{H}_{jmn}^{1 l_0, v, l, \varepsilon}} \mathrm{H}_{jmn}^{2 l_0, v, l, \varepsilon} \cdot dv. \quad (5.11)$$

The decomposition formula for the matrix element $\mathrm{D}_{jmj'm'}^{pqp'q'}(b; l, \rho)$ is obtained by substituting for f_n^{1l} an element $|j, m; p, q; ln\rho\rangle$ of the orthonormal basis (4.2) and for f_n^{2l} the element $\mathrm{D}^{l,\rho}(b)\,|\,j', m'; p', q'; ln\rho\rangle$. In order to obtain the fourier coefficients of the latter we note that the transformation $b \to bb_1$, in the argument of the function $h_n^{l\pm}(b)$ induces a transformation

$$\mathrm{H}_{jmn}^{l_0, v, l, \pm} \to \mathrm{H}'^{l_0, v, l, \pm}_{jmn} \equiv \int_{\mathscr{B}} \mathrm{D}_{jmln}^{l_0, v}(b^{-1}) \cdot h_n^{l\pm}(bb_1) d\mu(b)$$
$$= \sum_{j' = |l_0|, |l_0|+1, \ldots} \sum_{m' = -j'}^{j'} \mathrm{D}_{jmj'm'}^{l_0, v}(b_1) \mathrm{H}_{j',m,n}^{l_0, v, l, \pm}$$

of the fourier coefficients i. e. they transform as a standard basis of the representation (l_0, v). The required fourier coefficients are therefore

$$\sum_{j''} \sum_{m''} \mathrm{D}_{j'm'j''m''}^{l_0, v}(b) \mathscr{R}_{m'm}^{j'',l,j',\pm}(l_0, v; p', q'; \rho)$$

Thus, in this particular case (5.11) reads

$$\mathrm{D}_{jmj'm'}^{pqp'q'}(b; l, \rho) = (2\pi^4)^{-2} \cdot \sum_{\varepsilon} \sum_{l_0} \int_0^{\infty} (l_0^2 + v^2)$$
$$\times \sum_n \overline{\mathscr{R}_n^{l,j,\varepsilon}(l_0, v; p, q; \rho)} \mathscr{R}_n^{l,j',\varepsilon}(l_0, v; pq; \rho) \cdot \mathrm{D}_{jmj'm'}^{l_0, v}(b) \mathrm{d}v$$

which is the decomposition formula for the matrix elements of $b \in \mathscr{B}$ in a « $\mathscr{K}$-basis » with respect to matrix elements in a « $\mathscr{B}$-basis ». The particular fourier coefficients $\mathscr{R}_n^{l,j,\pm}(l_0, v, p, q; \rho)$ here appear as the transformation coefficients between the two bases. They can formally be looked upon as matrix elements in a unitary matrix. This property of the coefficients $\mathscr{R}_n^{l,j,\pm}(l_0, v; p, q; \rho)$ is clearly brought out in another approach to the decomposition problem. The approach we have in mind is one in which one starts with a derivation of the functional form of the eigenfunctions of the generators $\vec{M}^2$, M_3, $\vec{M}^2 - \vec{N}^2$ and $\vec{M}\cdot\vec{N}$ ($\vec{M}$ and $\vec{N}$ generate rotations and accelerations respectively in $\mathscr{B}$) and then considers the transformation between this $\mathscr{B}$-basis and the $\mathscr{K}$-basis (4.2). Using the Plancherel formula for the UIR:s of $\mathscr{B}$ one arrives at the same expression (5.10) for the transformation coefficients. In [*23*] more details on this approach are given for the corresponding problem for the group $\overline{SO_0(1,3)}$.

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