

# ANNALES DE L'I. H. P., SECTION A

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*Annales de l'I. H. P., section A*, tome 3, n° 2 (1965), p. 161-174

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## Finite groups generated by symmetries

by

M. SIRUGUE and J. C. TROTIN

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SUMMARY. — In this paper, we study the finite groups generated by symmetries in a  $n$ -dimensional vector-space over the real, and also the rational numbers; we define also a « root pattern », a « simple root system », and new diagrams including the set of Dynkin diagrams as a subset; the allowed diagrams are shown; if  $n > 2$ , two new diagrams are found, when we choose the field of real numbers; over the field of rational numbers, the solutions are precisely the Weyl groups of simple *Lie Algebras*. These groups can be used as an essential tool to introduce certain *Lie Algebras*, and for classifying the irreducible modules [1] [3].

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### I — SYMMETRIES

Let us denote by  $V$  a finite-dimensional vector-space over the field  $R$  of real numbers, or over the field  $Q$  of rational numbers. If  $S$  is a linear involutive mapping in  $V$ , i. e. satisfying  $S^2 = I$  (identity), we define the following operators:

$$E_+ = \frac{I + S}{2} \quad E_- = \frac{I - S}{2}.$$

One can easily verify the relations:

$$(1) \quad E_+ + E_- = I; \quad E_+ E_- = E_- E_+ = 0; \quad S = E_+ - E_-; \quad E_{\pm}^2 = E_{\pm}.$$

It follows that  $E_+$  and  $E_-$  are projection operators associated with a

decomposition of  $V$  into a direct sum of subspaces  $V_+$  and  $V_-$ ; we have then:

$$(2) \quad \forall x \in V \quad (1) S(x) = S(x_+ + x_-) = x_+ - x_- \quad \text{with} \quad x_{\pm} \in V_{\pm}.$$

The elements belonging to  $V_{\pm}$  are specified by the condition

$$S(x_{\pm}) = \pm x_{\pm}.$$

Reciprocally, if  $V$  is a direct sum of subspaces  $V_+$  and  $V_-$ , the formula (2) defines an involutive operator.  $S$  is a symmetry when  $V_-$  is one-dimensional; thus:

DEFINITION. — A symmetry  $S$  is a linear involutive operator acting in a finite-dimensional vector-space  $V$  [over the field of real numbers or over the field of rational numbers], the subset of its fixed points being an hyperplane [i. e. a subspace with one dimension less than  $V$ ].

Let  $\varepsilon$  be a linear mapping from  $V$  into the field, such that for a given element  $a \in V_-$ , the following relations hold:

$$\varepsilon(a) = 1; \quad \text{if } x \in V_+, \quad \varepsilon(x) = 0; \quad \text{if } x \in V_- (x = \lambda a), \quad \varepsilon(x) = \lambda.$$

Since if  $x \in V_+$ ,  $S(x) = x$  and if  $x \in V_-$ ,  $S(x) = S(\lambda a) = -x = -\lambda a$  thus,  $\forall x \in V$ ,  $S(x) = x - 2\varepsilon(x)a$ , the symmetry verifying  $S(a) = -a$ .

From now on, we shall write such a symmetry as  $S_a$ .

## II. — ROOT-SYSTEM

Let  $G$  be a finite group generated by symmetries acting in  $V$ . We suppose that  $G$  is an irreducible set of mappings. If we know a finite set of symmetries generating  $G$ , we can choose for any symmetry  $S_i$  among these, and also among those obtained through products of such generating symmetries, a vector  $a_i$  such that  $S_i = S_{a_i}$ ; we call  $\Delta$  the set of vectors  $\{\pm a_i\}$ , or « root-system », satisfying:

$$a) \quad \forall a \in \Delta, \quad \text{if } \lambda a \in \Delta, \quad \text{then } \lambda = \pm 1 \quad (\lambda \text{ a scalar})$$

$$b) \quad \forall a, b \in \Delta, \quad S_a(b) \in \Delta$$

$$c) \quad V \text{ is spanned by } \Delta.$$

a) Derives from the definition of the set  $\Delta$ ; we can take  $S_a(b) \in \Delta$  since the following mapping is a symmetry belonging to  $G$ , as a product of gene-

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(1) i. e. : « for every  $x$  belonging to  $V$  ».

rating symmetries:  $S_a S_b S_a = S_{S_a(b)}$ ; so, (b) results from a peculiar choice. Clearly, if  $S_a(b) = \lambda b$ , then  $\lambda = \pm 1$ , from  $S_a^2 = I$ ; thus (b) is in agreement with (a). Now, if  $\Delta$  only spans a proper subspace of  $V$ , it would be, at least, a vector  $x \neq 0$ , belonging to every hyperplane  $H_{a_i}$  ( $H_{a_i} = V \ominus \{\lambda a_i\}$ ;  $i$  is fixed but  $\lambda$  runs over the field); so, it would be a vector  $x \in \bigcap_i H_{a_i}$ , and  $x$  would be an invariant vector by each  $S_i$ , and also by the whole group  $G$ , but  $G$  is irreducible, and only the null vector is invariant by  $G$ .

REMARK. — From (2), we see:

$$S_a \Delta \subset \Delta, \quad (\forall a \in \Delta) \quad \text{and} \quad S_a^2 \Delta = \Delta \subset S_a \Delta;$$

it follows :

$$(3) \quad S_a \Delta = \Delta.$$

Each element  $g \in G$  can be written as a finite product of symmetries (since  $G$  is finite); from (3), we derive:

$$S_b S_a \Delta = S_b \Delta = \Delta \quad (\forall a, b \in \Delta).$$

By a recurrent process it immediately follows:  $S_n S_m \dots S_b S_a \Delta = \Delta$  and :

$$(4) \quad g \Delta = \Delta \quad (\forall g \in G).$$

Further, if  $a$  is a root with  $ga = \lambda a$ , since  $G$  is finite, an integer  $p$  can be found with:  $g^p a = \lambda^p a = a$ , and necessary  $\lambda = \pm 1$ . Over the dual-space  $V^*$  of  $V$ , a positive definite scalar product is defined through the formula:

$$(f | g) = \sum_{a \in \Delta} f(a)g(a),$$

if  $f$  and  $g$  are two linear mappings from  $V$  into the field; by duality, a positive definite scalar product  $(x | y)$  is defined onto  $V$ , which is an invariant product by every linear operator conserving the set  $\Delta$ , thus [from (4)], by every  $g \in G$ .

From now on, consequently,  $V$  is an euclidean space and every mapping of  $G$  is orthogonal with respect to this product.  $G$  is a subgroup of the orthogonal group. It will be easier to write down a symmetry acting as:

$$S_a | x) = | x) - \frac{2 | a) (a | x)}{(a | a)}.$$

Or, more briefly:

$$S_a = I - \frac{2 | a) (a |}{(a | a)}.$$

## III. — SIMPLE ROOTS

The set  $\Delta$  being a finite set generating  $V$ , we may conclude:  $\exists x_0 \in V$  (i. e.  $x_0$  can be found, belonging to  $V$ ) such that  $(x_0 | a) \neq 0$  ( $\forall a \in \Delta$ ). We shall always denote by  $\Sigma$  the set of roots «  $a$  » such that  $(x_0 | a) > 0$ , ( $-\Sigma$ ) the set of roots such that  $(x_0 | a) < 0$ .  $\Sigma$  and ( $-\Sigma$ ) give a partition for  $\Delta$ :

$$\Delta = \Sigma \cup (-\Sigma) \quad \text{with} \quad \Sigma \cap (-\Sigma) = \Phi.$$

$V$  will be equipped with a partial ordering compatible with its structure of vector-space over the real (or the rational) numbers; let us write:

$$x \gg y \quad \text{when} \quad x - y \in K(\Sigma)$$

[ $K(\Sigma)$  is the set of linear combinations with coefficients  $\geq 0$ , of elements belonging to  $\Sigma$ ]; thus the positive roots are the roots which belong to  $\Sigma$ .

Let us consider the subsets  $\Omega \subset \Sigma$  such that:

$$K(\Omega) = K(\Sigma)$$

(the inclusion clearly suffices) and define the system of simple roots as

$$\Pi = \cap \Omega.$$

This is a rather direct (but difficult to handle), definition of  $\Pi$ ; let us give two remarks which characterize the elements of  $\Pi$ .

REMARK. — If  $x_i \in \Sigma$  and  $x_i \notin K(\Sigma - x_i)$ , ( $(\Sigma - x_i)$  is the set  $\Sigma$  with  $x_i$  missing) then  $x_i \in \Pi$ .

It suffices to prove that if it exists an  $\Omega$  such that  $x_i \notin \Omega$ ,  $K(\Omega) \neq K(\Sigma)$ ; actually, if  $x_i \notin \Omega$

$$\Omega \subset \Sigma - x_i$$

and

$$K(\Omega) \subset K(\Sigma - x_i)$$

which is not equal to  $K(\Sigma)$  since  $x_i \notin K(\Sigma - x_i)$ . Remark  $x_i \in \Pi$  implies  $x_i \notin K(\Sigma - x_i)$ .

If not, the set  $\Sigma - x_i = \Omega$  generates  $K(\Sigma)$  and does not contain  $x_i$  so as  $\Pi = \Pi \cap \Omega$ , there is a contradiction.

It is necessary to prove that  $\Pi$  is not empty, or equivalently that there

exist roots such that  $x_i \notin K(\Sigma - x_i)$ ; this is clear, according to the following remark:

if

$$\Omega \subset \Sigma, \quad K(\Omega) = K(\Sigma), \quad x \in \Omega \quad \text{and} \quad x \in K(\Sigma - x)$$

then

$$K(\Omega - x) = K(\Sigma)$$

for if  $y \in K(\Sigma)$ :

$$\begin{aligned} y &= \sum \lambda_i x_i + \mu x & x_i &\in \Omega - x \\ x &= \sum \mu_i x_i + \lambda x & \lambda_i, \mu_i, \mu, \lambda &\geq 0. \end{aligned}$$

According to the fact that  $(x | x_0) > 0$  for every  $x \in K(\Sigma)$

$$\lambda < 1$$

so

$$x = \sum \frac{\mu_i}{1 - \lambda} x_i$$

and

$$y = \sum \left( \lambda_i + \frac{\mu \mu_i}{1 - \lambda} \right) x_i \in K(\Omega - x).$$

So if every  $x_i \in \Sigma$  was such that  $x \in K(\Sigma - x_i)$  one could construct a sequence of  $\Omega_i$ ,  $\Omega_1 = \Sigma - x_1$ ,  $\Omega_2 = \Omega_1 - x_2$ ,  $\dots$ ,  $\Omega_p = \Phi$ , each of them generating  $K(\Sigma)$ , which is absurd.

— It is clear then that from the previous remarks

$$K(\Pi) = K(\Sigma)$$

and we shall derive with the help of a lemma that  $\Pi$  is in fact a basis for  $V$ .

LEMMA I. —  $a \in \Delta$  cannot be written as  $\lambda_i \alpha_i - \lambda_j \alpha_j$  with

$$\alpha_i, \alpha_j \in \Pi, \quad \alpha_i \neq \alpha_j, \quad \lambda_i, \lambda_j > 0.$$

Indeed, we can suppose  $a \gg 0$ ; then, one would get

$$\lambda_i \alpha_i - \lambda_j \alpha_j = \sum \mu_k \alpha_k \quad (\alpha_k \in \Pi, \mu_k \geq 0)$$

and then the following relation would be deduced:

$$\lambda_j \alpha_j + (\mu_i - \lambda_i) \alpha_i + \sum_{k \neq i} \mu_k \alpha_k = 0.$$

Such a system is not possible when  $\lambda_i \leq \mu_i$  (since  $\lambda_j > 0$  and  $\mu_k \geq 0$ ). If  $\lambda_i > \mu_i$ , it would follow:

$$\alpha_i = \frac{1}{\lambda_i - \mu_i} \left[ \lambda_j \alpha_j + \sum_{k \neq i} \mu_k \alpha_k \right].$$

But this relation, written as:

$$\alpha_i = \sum_{k \neq i} \nu_k \alpha_k \quad (\text{with } \nu_k \geq 0, \text{ let us recall that } i \neq j),$$

shows that  $\alpha_i$  would not belong to  $\Pi$ .

Now if  $\alpha_i, \alpha_j \in \Pi$ ,  $\alpha_i \neq \alpha_j$ :

$$S_{\alpha_i}(\alpha_j) = \alpha_j - \frac{2(\alpha_i | \alpha_j)}{(\alpha_i | \alpha_i)} \alpha_i$$

and from condition (b) for root-systems,  $S_{\alpha_i}(\alpha_j) \in \Delta$ ; from the lemma I we conclude that

$$\frac{-2(\alpha_i | \alpha_j)}{(\alpha_i | \alpha_i)} \geq 0$$

and:

$$(\alpha_i | \alpha_j) \leq 0.$$

Now we are able to prove the linear independence of the  $\alpha'_i$ 's (over the corresponding field  $\mathbb{R}$  or  $\mathbb{Q}$  according to our primitive choice of the field). If the  $\alpha'_i$ 's were linearly dependent, one could write  $\sum_k \nu_k \alpha_k = 0$  ( $\nu_k \neq 0$ , necessarily some  $\nu_k$ 's would be  $> 0$ , others  $< 0$ ); one could deduce, considering separately these terms:

$$\sum_{i \in I} \lambda_i \alpha_i - \sum_{j \in J} \mu_j \alpha_j = 0.$$

Or:

$$\sum_{i \in I} \lambda_i \alpha_i = \sum_{j \in J} \mu_j \alpha_j = u$$

(with  $I \cap J = \Phi$ ;  $\lambda_i, \mu_j > 0$  and the families  $I, J$  of indices  $i, j$  verifying  $I, J \neq \Phi$ ).

Using  $(\alpha_i | \alpha_j) \leq 0$ , one could obtain from:

$$\sum_{I, J} \lambda_i \mu_j (\alpha_i | \alpha_j) = (u | u) \geq 0, \quad \lambda_i \mu_j = 0,$$

contrary to  $\lambda_i, \mu_j > 0$  (it makes no difference between  $\mathbb{R}$  and  $\mathbb{Q}$ ).

LEMMA II. —  $\forall(\alpha, \alpha_i), \alpha \in \Sigma, \alpha_i \in \Pi$  (with  $\alpha \neq \alpha_i$ ), then  $S_{\alpha_i}(\alpha) \in \Sigma$ . We can write:

$$\alpha = \sum \lambda_j \alpha_j, \quad \text{and} \quad S_{\alpha_i}(\alpha) = \alpha - \mu \alpha_i = (\lambda_i - \mu) \alpha_i + \sum_{j \neq i} \lambda_j \alpha_j;$$

$\alpha$  is not proportional to  $\alpha_i$ , the numbers  $\lambda_j$  ( $j \neq i$ ) are  $\geq 0$  and at least one among them  $\lambda_j \neq 0$ .  $S_{\alpha_i}(\alpha) \in \Delta$  and its coefficients are all together  $\geq 0$ , or all together  $\leq 0$ ;  $\lambda_{j_0} > 0$ , thus all are  $\geq 0$  and  $S_{\alpha_i}(\alpha) \in \Sigma$ .

$S_{\alpha_i}(\alpha) = -\alpha_i \in (-\Sigma)$ , there is only one positive root,  $\alpha_i$ , such that its image through  $S_{\alpha_i}$  belongs to  $(-\Sigma)$ .

#### IV. — TOTAL ORDERING

$\Sigma$  is always considered as a fixed set; with respect to the basis  $(\alpha_1, \alpha_2, \dots, \alpha_n)$  we consider the lexicographic ordering, noticed as  $x > y$  (we decide  $\alpha_1 > \alpha_2 > \dots > \alpha_n$ ); it is a total ordering over  $V$  compatible with its structure of vector-space over  $R$  or over  $Q$  if the elements  $x > 0$  are those written as

$$x = \lambda_j \alpha_j + \sum_{k > j} \lambda_k \alpha_k \quad \text{with} \quad \lambda_j > 0.$$

If  $x \gg y$  then  $x > y$ , and if  $a \in \Delta$  with  $a > 0$ , then  $a \in \Sigma$ .

LEMMA III. —  $\forall x \in K(\Sigma)$  ( $= K(\Pi)$ ),  $\exists \alpha_{i_0} \in \Pi$  such that  $x > S_{\alpha_{i_0}}(x)$  ( $x \neq 0$ ). If,  $\forall \alpha_i \in \Pi, (x | \alpha_i) \leq 0$ , one could deduce

$$(x | x) = \sum_{i,j} \lambda_i \lambda_j (\alpha_i | \alpha_j) \leq 0 \quad (\text{since } \lambda_i, \lambda_j \geq 0),$$

it would follow  $x = 0$ ; thus  $\alpha_{i_0}$  can be found, such that  $(x | \alpha_{i_0}) > 0$ ; from

$$S_{\alpha_{i_0}}(x) = x - \frac{2(x | \alpha_{i_0})}{(\alpha_{i_0} | \alpha_{i_0})} \alpha_{i_0}, \quad \text{it follows} \quad x - S_{\alpha_{i_0}}(x) > 0.$$

*Fundamental theorem.* — Every root can be written as  $S_i S_j \dots S_k \alpha_l$ , where  $\alpha_i, \alpha_j, \dots, \alpha_k, \alpha_l$  are simple roots [ $S_i = S_{\alpha_i}, S_j = S_{\alpha_j}$ , and so on ...]:

Let us denote by  $W$  the set of roots which have the form  $S_i S_j \dots S_k \alpha_l$  (these are roots from property (b) of root-systems); since  $S_i S_i \alpha_l = \alpha_l, \alpha_l \in W$  and every simple root belongs to  $W$ ; further,  $S_i W \subset W(\forall i)$ . If  $\alpha$  is a positive root, let us suppose that all the positive roots  $\beta$  satisfying  $\alpha > \beta$ , belong to the set  $W$ ;  $\alpha_i$  can be found (from lemma III) such that  $\alpha > S_i \alpha$ ; if  $\alpha \neq \alpha_i$ ,

from lemma II, we deduce  $S_i\alpha \gg 0$ , thus  $S_i\alpha \in W$ , and  $\alpha = S_i(S_i\alpha) \in W$  too; if  $\alpha = \alpha_i$ , then  $\alpha \in W$  (since  $\Pi \subset W$ ): in both cases  $\alpha \in W$  and  $\alpha_n$  is a simple root belonging to  $W$  and satisfying  $\alpha > \alpha_n$  ( $\forall \alpha \in \Sigma$ ), so that we see by a recurrent process that  $W \supset \Sigma$ ; further, if  $\alpha \in W$ ,  $(-a) \in W$  because if

$$\alpha = S_i S_j \dots S_k \alpha_l, \quad -\alpha = S_i S_j \dots S_k S_l \alpha_l.$$

Thus  $W \supset (-\Sigma)$  and we conclude  $W = \Delta$ .

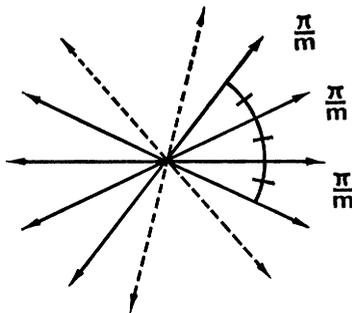
From the formula  $gS_\alpha g^{-1} = S_{g.\alpha}$  ( $\forall g \in G$ ), which is an immediate generalization of  $S_a S_b S_a = S_{S_a(b)}$ , we see that for every root  $\alpha = S_i S_j \dots S_k \alpha_l$ , the corresponding symmetry is

$$S_\alpha = (S_i S_j \dots S_k) S_l (S_i S_j \dots S_k)^{-1} = S_i S_j \dots S_k S_l S_k \dots S_j S_i.$$

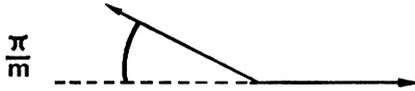
*And every symmetry  $S_\alpha$  is obtained through products of symmetries  $S_\alpha$  corresponding to simple roots only.*

### V. — CLASSIFICATION

Now, in order to obtain  $G$ , we must consider all possible sets of roots, or all possible corresponding sets of simple roots; if  $S_i$  and  $S_j$  are two symmetries,  $H_i$  and  $H_j$  the corresponding fixed hyperplanes,  $H_i \cap H_j$  is a subspace of two dimensions less than  $V$ , which is left fixed by the mapping  $S_i S_j$ :  $V$  is the direct sum  $V = (H_i \cap H_j) + \Gamma_{ij}$ , where  $\Gamma_{ij}$  is a plane, over which  $S_i S_j$  induces a rotation; the rotation angle is necessary commensurable with  $\pi$ , since the order of  $S_i S_j$  is finite. Further, there are at least two roots in  $\Gamma_{ij}$ ,  $a, b$ , the angle between them being precisely the rotation angle, which can be written as  $\frac{k\pi}{m}, \frac{k}{m}$  being an irreducible fraction; we can now, in the  $\Gamma_{ij}$ -plane, construct a subsystem of roots, departing from both the roots  $a, b$ , constructing the roots  $S_a(b)$  and  $S_b(a)$ , then the new ones obtained by symmetries of these roots, through each other, and so on. We obtain a subsystem as the following one:



Looking at such a system, since every positive root must be a linear combination of simple roots with positive coefficients, we see that the two simple roots which generate the system, are in the following position:



the angle between them is  $\frac{\pi}{m}$ . Thus, between any system of two simple roots,  $\alpha_i$  and  $\alpha_j$ , the angle is  $\pi - \frac{\pi}{n_{ij}} = \theta_{ij}$  where  $n_{ij}$  is an integer. The simple roots can be replaced with unit length vectors  $a_i, a_j$ , and we have to search for allowable configurations defined by  $\cos^2 \theta_{ij} = \cos^2 \frac{\pi}{n_{ij}}$  where  $n_{ij}$  is an integer  $\geq 2$  ( $\cos \theta_{ij} < 0$ ).

It is useful to consider  $4(a_i | a_j)^2 = 4 \cos^2 \frac{\pi}{n_{ij}}$ , and to define a diagram for any one allowable configuration, as a collection of points  $u_i, i = 1, 2, \dots, n$ , and lines connecting these according to the rule:  $u_i$  and  $u_j$  are not connected if  $(a_i | a_j) = 0$  and  $u_i$  and  $u_j$  are connected by  $4(a_i | a_j)^2 = 1, 2$  or  $3$  lines when this equality holds [i. e. respectively when  $n_{ij} = 3, 4$  or  $6$ ]; the case  $n_{ij} = 2$  corresponds to  $(a_i | a_j) = 0$ . If  $4(a_i | a_j)^2$  is not an integer but verifies:

$$p < 4(a_i | a_j)^2 < p + 1$$

$p$  an integer,  $u_i$  and  $u_j$  will be connected with  $p$  lines, together with another dashed line. For example, if  $n_{ij} = 5, 2 < 4(a_i | a_j)^2 < 3$ , the diagram is



Thus, when  $n_{ij} = 3, 4$  or  $6$ , we recognize the Dynkin diagrams, and the corresponding groups are the well-known Weyl groups of simple Lie Algebras. But, otherwise, if  $n = 2$ , all values  $n_{1,2} = 2, 3, 4 \dots$  are solutions, and if  $n > 2$ , there are *only two solutions*, because, looking at connected diagrams, one sees:

<sup>10</sup> If  $n$  is the number of vertices (points) of a diagram, then the number of pairs of connected points is less than  $n$ .

PROOF. — Let

$$a = \sum_1^n a_i, \quad \text{then} \quad 0 < (a | a) = n + 2 \sum_{i < j} (a_i | a_j).$$

If  $(a_i | a_j) \neq 0$ , then  $2(a_i | a_j) \leq -1$ .

Hence the inequality shows that the number of pairs  $a_i, a_j$  with  $(a_i | a_j) \neq 0$  is less than  $n$ .

2° *A diagram contains no cycles* (a cycle is a sequence of points  $u_1, \dots, u_k$  such that  $u_i$  is connected to  $u_{i+1}$ ,  $i \leq k - 1$  and  $u_k$  connected to  $u_1$ ).

PROOF. — The subset forming a cycle violates the former condition.

3° *The number of non-dashed lines* (counting multiplicities) *issuing from a vertex is less than four.*

PROOF. — Let  $u$  be a vertex,  $v_1, v_2, \dots, v_k$  the vertices connected to  $u$ . No two  $v_i$  are connected since there are no cycles. Hence  $(v_i | v_j) = 0$ ,  $i \neq j$  (now for simplicity, we denote in the same way simple roots and vertices). In the space spanned by  $u$  and the  $v_i$  we can choose a vector  $v_0$  such that  $(v_0 | v_0) = 1$  and  $v_0, v_1, \dots, v_k$  are mutually orthogonal. Since  $u$  and the  $v_i, i \geq 1$ , are linearly independent,  $u$  is not orthogonal to  $v_0$  and so  $(u | v_0) \neq 0$ . Since

$$u = \sum_0^k (u | v_j) v_j, \quad (u | u) = (u | v_0)^2 + (u | v_1)^2 + \dots + (u | v_k)^2 = 1,$$

Hence

$$\sum_1^k (u | v_i)^2 < 1 \quad \text{and} \quad 4 \sum_1^k (u | v_i)^2 < 4.$$

Since  $4(u | v_i)^2$  is the number of non-dashed lines connecting  $u$  and  $v_i$  whenever there is no dashed line between them, or otherwise is greater than this number, the result follows.

4° With any dashed line, there are at least two non-dashed lines issuing from a vertex (the first case,  $n_{ij}$  increasing, is  $n_{ij} = 5$ , when  $\theta_{ij} = \pi - \frac{\pi}{5}$ ).

It readily follows that when the dimension of  $V$  is  $n \geq 3$ , and  $n_{ij} \geq 7$ , there is no solution because the corresponding diagrams (we speak about connected diagrams) are such that there are at least four non-dashed lines issuing from one vertex:



5° Let  $\Pi$  be an allowable configuration and let  $v_1, v_2, \dots, v_k$  be vectors of  $\Pi$  such that the corresponding points of the diagram form a simple chain in the sense that each one is connected to the next by a single line. Let  $\Pi'$  be the collection of vectors of  $\Pi$  which are not in the simple chain  $v_1, \dots, v_k$  together with the vector  $v = \sum_1^k v_i$ ; then  $\Pi'$  is an allowable configuration:

PROOF. — We have  $2(v_i | v_{i+1}) = -1$ , for  $i = 1, 2, \dots, k - 1$ . Hence

$$(v | v) = k + 2 \sum_{i < j} (v_i | v_j).$$

Since there are no cycles  $(v_i | v_j) = 0$  if  $i < j$ , unless  $j = i + 1$ . Hence

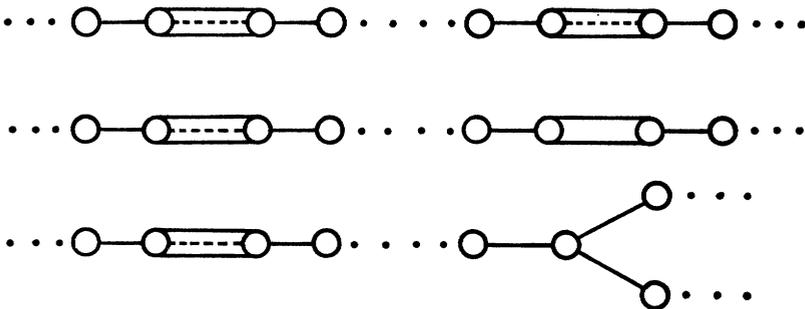
$$(v | v) = k - (k - 1) = 1$$

and  $v$  is a unit vector.

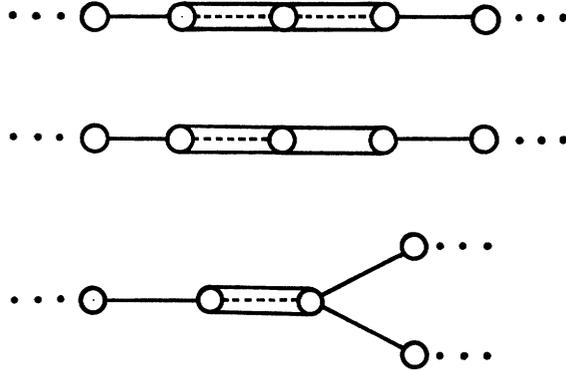
Now let  $u \in \Pi$ ,  $u \neq v_i$ . Then  $u$  is connected with at most one of the  $v_i$ , say  $v_j$ , since there are no cycles. Then

$$(u | v) = \left( u \left| \sum_1^k v_i \right. \right) = (u | v_j) \quad \text{and} \quad 4(u | v)^2 = 4(u | v_j)^2 = 4 \cos^2 \frac{\pi}{n_{ij}},$$

as required; the diagram of  $\Pi'$  is obtained from that of  $\Pi$  by shrinking the simple chain to a point; thus we replace all the vertices by the single vertex  $v$  and we join this to any  $u \in \Pi$ ,  $u \neq v_i$  by the total number of non-dashed lines connecting  $u$  to any one of the  $v_j$  in the original diagram; we get the same result for dashed lines, but always one dashed line will connect two vertices. Application of this to the following graphs:



reduces these respectively to:



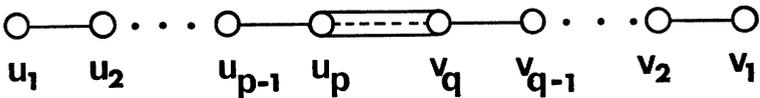
But these last ones are not allowable (four non-dashed lines issuing from a vertex).

The possibilities are among the following type of diagrams:



(i. e. there is, on each side, a finite chain).

Consider the peculiar one:



And let us write

$$u = \sum_1^p i u_i, \quad v = \sum_1^q j v_j.$$

Since

$$2(u_i | u_{i+1}) = -1 \quad \text{and} \quad 2(v_j | v_{j+1}) = -1$$

we have:

$$(u | u) = \sum_1^p i^2 - \sum_1^{p-1} i(i+1) = p^2 - \frac{p(p-1)}{2} = \frac{p(p+1)}{2}$$

$$(v | v) = \frac{q(q + 1)}{2}.$$

And

$$(u | v) = pq(u_p | u_q) = pq \cos \left( \pi - \frac{\pi}{5} \right), \quad (u | v)^2 = p^2 q^2 \cos^2 \frac{\pi}{5}.$$

By Schwarz inequality:

$$pq \cos^2 \frac{\pi}{5} < \frac{(p + 1)(q + 1)}{4}$$

$p$  and  $q$  are integers  $\geq 1$ ; if  $n \geq 3$ ,  $q$ , for instance, is  $> 1$ . The solutions are:

$$p = 2, \quad q = 1$$

$$p = 3, \quad q = 1$$

corresponding to the following diagrams:



When the field is  $\mathbb{R}$ , the field of real numbers, it is easy to see that these two diagrams give actual solutions since the corresponding euclidean systems can be constructed.

When the field is  $\mathbb{Q}$ , the field of rational numbers, one can multiply the non-simple roots by integer multipliers in such a way that the new ones can be written as:

$$\alpha' = \pm \sum m_i \alpha_i$$

with  $m_i$  integers  $\geq 0$ ,  $\alpha_i \in \Pi$ ; now, these are considered as new non-simple roots, and the condition:

$$S_{\alpha_i}(\alpha') = \alpha' - \frac{2(\alpha' | \alpha_i)}{(\alpha_i | \alpha_i)} \alpha_i \in \Sigma \quad \text{when } \alpha \in \Sigma \quad \text{and } \alpha_i \in \Pi$$

shows that necessarily, the scalars  $-\frac{2(\alpha | \alpha_i)}{(\alpha_i | \alpha_i)}$  (from now on, we drop the « prime » for brevity) are integers. More generally, the scalars  $-\frac{2(\alpha | \beta)}{(\beta | \beta)}$

$(\forall \alpha, \beta \in \Sigma)$  are necessarily integers, since every root can be considered as simple, according to the choice of  $x_0$ , and  $\Sigma$ . From:

$$\frac{4(\alpha | \beta)^2}{(\alpha | \alpha)(\beta | \beta)} = \left(-\frac{2(\alpha | \beta)}{(\alpha | \alpha)}\right) \left(-\frac{2(\alpha | \beta)}{(\beta | \beta)}\right) < 4 \quad (\text{if } \alpha \neq \pm \beta).$$

We deduce that the scalars  $-\frac{2(\alpha | \beta)}{(\alpha | \alpha)}$  are integers verifying:

$$-3 \leq \frac{-2(\alpha | \beta)}{(\alpha | \alpha)} \leq 3.$$

So we find precisely as solutions all Weyl groups of simple *Lie Algebras*, whatever the dimension of  $V$  may be.

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#### ACKNOWLEDGEMENTS

We are indebted to Dr. H. Bacry and Dr. N. Straumann for reading the manuscript and for their critical comments.

(Manuscrit reçu le 19 juillet 1965).

