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Section A :
Physique théorique.

Finite groups generated by symmetries

by

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SUMMARY. — In this paper, we study the finite groups generated by symmetries in a n-dimensional vector-space over the real, and also the rational numbers; we define also a « root pattern », a « simple root system », and new diagrams including the set of Dynkin diagrams as a subset; the allowed diagrams are shown; if $n > 2$, two new diagrams are found, when we choose the field of real numbers; over the field of rational numbers, the solutions are precisely the Weyl groups of simple *Lie Algebras*. These groups can be used as an essential tool to introduce certain *Lie Algebras*, and for classifying the irreducible modules [*1*] [*3*].

I — SYMMETRIES

Let us denote by V a finite-dimensional vector-space over the field R of real numbers, or over the field Q of rational numbers. If S is a linear involutive mapping in V, i. e. satisfying $S^2 = I$ (identity), we define the following operators:

$$E_+ = \frac{I + S}{2} \qquad E_- = \frac{I - S}{2}.$$

One can easily verify the relations:

$$(1) \quad E_+ + E_- = I \; ; \quad E_+E_- = E_-E_+ = 0 \; ; \quad S = E_+ - E_- \; ; \quad E_\pm^2 = E_\pm.$$

It follows that E_+ and E_- are projection operators associated with a decomposition of V into a direct sum of subspaces V_+ and V_-; we have then:

(2) $\forall x \in V$ [1] $S(x) = S(x_+ + x_-) = x_+ - x_-$ with $x_\pm \in V_\pm$.

The elements belonging to $V_\pm$ are specified by the condition

$$S(x_\pm) = \pm x_\pm.$$

Reciprocally, if V is a direct sum of subspaces V_+ and V_-, the formula (2) defines an involutive operator. S is a symmetry when V_- is one-dimensional; thus:

DEFINITION. — A symmetry S is a linear involutive operator acting in a finite-dimensional vector-space V [over the field of real numbers or over the field of rational numbers], the subset of its fixed points being an hyperplane [i. e. a subspace with one dimension less than V].

Let ε be a linear mapping from V into the field, such that for a given element $a \in V_-$, the following relations hold:

$$\varepsilon(a) = 1; \quad \text{if} \quad x \in V_+, \quad \varepsilon(x) = 0; \quad \text{if} \quad x \in V_- \ (x = \lambda a), \quad \varepsilon(x) = \lambda.$$

Since if $x \in V_+$, $S(x) = x$ and if $x \in V_-$, $S(x) = S(\lambda a) = -x = -\lambda a$ thus, $\forall x \in V$, $S(x) = x - 2\varepsilon(x)a$, the symmetry verifying $S(a) = -a$.

From now on, we shall write such a symmetry as S_a.

II. — ROOT-SYSTEM

Let G be a finite group generated by symmetries acting in V. We suppose that G is an irreducible set of mappings. If we know a finite set of symmetries generating G, we can choose for any symmetry S_i among these, and also among those obtained through products of such generating symmetries, a vector a_i such that $S_i = S_{a_i}$; we call Δ the set of vectors $\{\pm a_i\}$, or « root-system », satisfying:

a) $\forall a \in \Delta$, if $\lambda a \in \Delta$, then $\lambda = \pm 1$ (λ a scalar)

b) $\forall a, b \in \Delta$, $S_a(b) \in \Delta$

c) V is spanned by Δ.

a) Derives from the definition of the set Δ; we can take $S_a(b) \in \Delta$ since the following mapping is a symmetry belonging to G, as a product of gene-

[1] i. e. : « for every x belonging to V ».

rating symmetries: $S_aS_bS_a = S_{S_a(b)}$; so, (b) results from a peculiar choice. Clearly, if $S_a(b) = \lambda b$, then $\lambda = \pm 1$, from $S_a^2 = I$; thus (b) is in agreement with (a). Now, if Δ only spans a proper subspace of V, it would be, at least, a vector $x \neq 0$, belonging to every hyperplane H_{a_i} ($H_{a_i} = V \ominus \{ \lambda a_i \}$; i is fixed but λ runs over the field); so, it would be a vector $x \in \bigcap_i H_{a_i}$, and x would be an invariant vector by each S_i, and also by the whole group G, but G is irreducible, and only the null vector is invariant by G.

REMARK. — From (2), we see:

$$S_a\Delta \subset \Delta, \qquad (\forall a \in \Delta) \quad \text{and} \quad S_a^2\Delta = \Delta \subset S_a\Delta\,;$$

it follows :

$$S_a\Delta = \Delta. \tag{3}$$

Each element $g \in G$ can be written as a finite product of symmetries (since G is finite); from (3), we derive:

$$S_bS_a\Delta = S_b\Delta = \Delta \qquad (\forall a, b \in \Delta).$$

By a recurrent process it immediately follows: $S_nS_m \ldots S_bS_a\Delta = \Delta$ and:

$$g\Delta = \Delta \qquad (\forall g \in G). \tag{4}$$

Further, if a is a root with $ga = \lambda a$, since G is finite, an integer p can be found with: $g^pa = \lambda^pa = a$, and necessary $\lambda = \pm 1$. Over the dual-space V* of V, a positive definite scalar product is defined through the formula:

$$(f \mid g) = \sum_{a \in \Delta} f(a)g(a),$$

if f and g are two linear mappings from V into the field; by duality, a positive definite scalar product $(x \mid y)$ is defined onto V, which is an invariant product by every linear operator conserving the set Δ, thus [from (4)], by every $g \in G$.

From now on, consequently, V is an euclidean space and every mapping of G is orthogonal with respect to this product. G is a subgroup of the orthogonal group. It will be easier to write down a symmetry acting as:

$$S_a \mid x) = \mid x) - \frac{2 \mid a)(a \mid x)}{(a \mid a)}.$$

Or, more briefly:

$$S_a = I - \frac{2 \mid a)(a \mid}{(a \mid a)}.$$

III. — SIMPLE ROOTS

The set Δ being a finite set generating V, we may conclude: $\exists x_0 \in V$ (i. e. x_0 can be found, belonging to V) such that $(x_0 \mid a) \neq 0$ ($\forall a \in \Delta$). We shall always denote by Σ the set of roots « a » such that $(x_0 \mid a) > 0$, $(-\Sigma)$ the set of roots such that $(x_0 \mid a) < 0$. Σ and $(-\Sigma)$ give a partition for Δ:

$$\Delta = \Sigma \cup (-\Sigma) \qquad \text{with} \qquad \Sigma \cap (-\Sigma) = \Phi.$$

V will be equipped with a partial ordering compatible with its structure of vector-space over the real (or the rational) numbers; let us write:

$$x \gg y \qquad \text{when} \qquad x - y \in K(\Sigma)$$

[$K(\Sigma)$ is the set of linear combinations with coefficients $\geqslant 0$, of elements belonging to Σ]; thus the positive roots are the roots which belong to Σ.

Let us consider the subsets $\Omega \subset \Sigma$ such that:

$$K(\Omega) = K(\Sigma)$$

(the inclusion clearly suffices) and define the system of simple roots as

$$\Pi = \cap \Omega.$$

This is a rather direct (but difficult to handle), definition of Π; let us give two remarks which characterize the elements of Π.

REMARK. — If $x_i \in \Sigma$ and $x_i \notin K(\Sigma - x_i)$, (($\Sigma - x_i$) is the set Σ with x_i missing) then $x_i \in \Pi$.

It suffices to prove that if it exists an Ω such that $x_i \notin \Omega$, $K(\Omega) \neq K(\Sigma)$; actually, if $x_i \notin \Omega$

$$\Omega \subset \Sigma - x_i$$

and

$$K(\Omega) \subset K(\Sigma - x_i)$$

which is not equal to $K(\Sigma)$ since $x_i \notin K(\Sigma - x_i)$. Remark $x_i \in \Pi$ implies $x_i \notin K(\Sigma - x_i)$.

If not, the set $\Sigma - x_i = \Omega$ generates $K(\Sigma)$ and does not contain x_i so as $\Pi = \Pi \cap \Omega$, there is a contradiction.

It is necessary to prove that Π is not empty, or equivalently that there exist roots such that $x_i \notin K(\Sigma - x_i)$; this is clear, according to the following remark:

if

$$\Omega \subset \Sigma, \quad K(\Omega) = K(\Sigma), \quad x \in \Omega \quad \text{and} \quad x \in K(\Sigma - x)$$

then

$$K(\Omega - x) = K(\Sigma)$$

for if $y \in K(\Sigma)$:

$$y = \Sigma\lambda_i x_i + \mu x \qquad x_i \in \Omega - x$$

$$x = \Sigma\mu_i x_i + \lambda x \qquad \lambda_i,\ \mu_i,\ \mu,\ \lambda \geqslant 0.$$

According to the fact that $(x \mid x_0) > 0$ for every $x \in K(\Sigma)$

$$\lambda < 1$$

so

$$x = \sum \frac{\mu_i}{1-\lambda} x_i$$

and

$$y = \Sigma\left(\lambda_i + \frac{\mu\mu_i}{1-\lambda}\right) x_i \in K(\Omega - x).$$

So if every $x_i \in \Sigma$ was such that $x \in K(\Sigma - x_i)$ one could construct a sequence of Ω_i, $\Omega_1 = \Sigma - x_1$, $\Omega_2 = \Omega_1 - x_2$, $\ldots$, $\Omega_p = \Phi$, each of them generating $K(\Sigma)$, which is absurd.

— It is clear then that from the previous remarks

$$K(\Pi) = K(\Sigma)$$

and we shall derive with the help of a lemma that Π is in fact a basis for V.

LEMMA I. — $a \in \Delta$ cannot be written as $\lambda_i\alpha_i - \lambda_j\alpha_j$ with

$$\alpha_i,\ \alpha_j \in \Pi,\ \alpha_i \neq \alpha_j,\ \lambda_i,\ \lambda_j > 0.$$

Indeed, we can suppose $a \gg 0$; then, one would get

$$\lambda_i\alpha_i - \lambda_j\alpha_j = \Sigma\mu_k\alpha_k \qquad (\alpha_k \in \Pi,\ \mu_k \geqslant 0)$$

and then the following relation would be deduced:

$$\lambda_j\alpha_j + (\mu_i - \lambda_i)\alpha_i + \sum_{k \neq i} \mu_k\alpha_k = 0.$$

Such a system is not possible when $\lambda_i \leqslant \mu_i$ (since $\lambda_j > 0$ and $\mu_k \geqslant 0$). If $\lambda_i > \mu_i$, it would follow:

$$\alpha_i = \frac{1}{\lambda_i - \mu_i}\left[\lambda_j \alpha_j + \sum_{k \neq i} \mu_k \alpha_k\right].$$

But this relation, written as:

$$\alpha_i = \sum_{k \neq i} \nu_k \alpha_k \quad \text{(with } \nu_k \geqslant 0\text{, let us recall that } i \neq j\text{)},$$

shows that α_i would not belong to Π.

Now if $\alpha_i, \alpha_j \in \Pi$, $\alpha_i \neq \alpha_j$:

$$S_{\alpha_i}(\alpha_j) = \alpha_j - \frac{2(\alpha_i \mid \alpha_j)}{(\alpha_i \mid \alpha_i)} \alpha_i$$

and from condition (*b*) for root-systems, $S_{\alpha_i}(\alpha_j) \in \Delta$; from the lemma I we conclude that

$$\frac{-2(\alpha_i \mid \alpha_j)}{(\alpha_i \mid \alpha_i)} \geqslant 0$$

and:

$$(\alpha_i \mid \alpha_j) \leqslant 0.$$

Now we are able to prove the linear independence of the α_i's (over the corresponding field R or Q according to our primitive choice of the field). If the α_i's were linearly dependent, one could write $\sum_k \nu_k \alpha_k = 0$ ($\nu_k \neq 0$, necessarily some ν_k's would be > 0, others < 0); one could deduce, considering separately these terms:

$$\sum_{i \in I} \lambda_i \alpha_i - \sum_{j \in J} \mu_j \alpha_j = 0.$$

Or:

$$\sum_{i \in I} \lambda_i \alpha_i = \sum_{j \in J} \mu_j \alpha_j = u$$

(with $I \cap J = \Phi$; $\lambda_i, \mu_j > 0$ and the families I, J of indices *i*, *j* verifying I, $J \neq \Phi$).

Using $(\alpha_i \mid \alpha_j) \leqslant 0$, one could obtain from:

$$\sum_{I,J} \lambda_i \mu_j (\alpha_i \mid \alpha_j) = (u \mid u) \geqslant 0, \; \lambda_i \mu_j = 0,$$

contrary to $\lambda_i, \mu_j > 0$ (it makes no difference between R and Q).

LEMMA II. — $\forall(\alpha, \alpha_i)$, $\alpha \in \Sigma$, $\alpha_i \in \Pi$ (with $\alpha \neq \alpha$), then $S_{\alpha_i}(\alpha) \in \Sigma$. We can write:

$$\alpha = \Sigma\lambda_j\alpha_j, \quad \text{and} \quad S_{\alpha_i}(\alpha) = \alpha - \mu\alpha_i = (\lambda_i - \mu)\alpha_i + \sum_{j \neq i}\lambda_j\alpha_j\,;$$

α is not proportional to α_i, the numbers λ_j $(j \neq i)$ are $\geqslant 0$ and at least one among them $\lambda_{j_0} \neq 0$. $S_{\alpha_i}(\alpha) \in \Delta$ and its coefficients are all together $\geqslant 0$, or all together $\leqslant 0$; $\lambda_{j_0} > 0$, thus all are $\geqslant 0$ and $S_{\alpha_i}(\alpha) \in \Sigma$.

$S_{\alpha_i}(\alpha_i) = -\alpha_i \in (-\Sigma)$, there is only one positive root, α_i, such that its image through S_{α_i} belongs to $(-\Sigma)$.

IV. — TOTAL ORDERING

Σ is always considered as a fixed set; with respect to the basis $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ we consider the lexicographic ordering, noticed as $x > y$ (we decide $\alpha_1 > \alpha_2 \ldots > \alpha_n$); it is a total ordering over V compatible with its structure of vector-space over R or over Q if the elements $x > 0$ are those written as

$$x = \lambda_j\alpha_j + \sum_{k>j}\lambda_k\alpha_k \quad \text{with} \quad \lambda_j > 0.$$

If $x \gg y$ then $x > y$, and if $a \in \Delta$ with $a > 0$, then $a \in \Sigma$.

LEMMA III. — $\forall x \in K(\Sigma)$ $(= K(\Pi))$, $\exists\alpha_{i_0} \in \Pi$ such that $x > S_{\alpha_{i_0}}(x)$ $(x \neq 0)$. If, $\forall\alpha_i \in \Pi$, $(x \mid x_i) \leqslant 0$, one could deduce

$$(x \mid x) = \sum_{i,j}\lambda_i\lambda_j(\alpha_i \mid \alpha_j) \leqslant 0 \qquad (\text{since } \lambda_i, \lambda_j \geqslant 0),$$

it would follow $x = 0$; thus x_{i_0} can be found, such that $(x \mid \alpha_{i_0}) > 0$; from

$$S_{\alpha_{i_0}}(x) = x - \frac{2(x \mid \alpha_{i_0})}{(\alpha_{i_0} \mid \alpha_{i_0})}\alpha_{i_0}, \quad \text{it follows} \quad x - S_{\alpha_{i_0}}(x) > 0.$$

Fundamental theorem. — Every root can be written as $S_iS_j \ldots S_k\alpha_l$, where $\alpha_i, \alpha_j, \ldots, \alpha_k, \alpha_l$ are simple roots [$S_i = S_{\alpha_i}$, $S_j = S_{\alpha_j}$, and so on ...]:

Let us denote by W the set of roots which have the form $S_iS_j \ldots S_k\alpha_l$ (these are roots from property (*b*) of root-systems); since $S_iS_i\alpha_l = \alpha_l$, $\alpha_l \in W$ and every simple root belongs to W; further, $S_iW \subset W(\forall i)$. If α is a positive root, let us suppose that all the positive roots β satisfying $\alpha > \beta$, belong to the set W; α_i can be found (from lemma III) such that $\alpha > S_i\alpha$; if $\alpha \neq \alpha l$, from lemma II, we deduce $S_i\alpha \gg 0$, thus $S_i\alpha \in W$, and $\alpha = S_i(S_i\alpha) \in W$ too; if $\alpha = \alpha_i$, then $\alpha \in W$ (since $\Pi \subset W$): in both cases $\alpha \in W$ and α_n is a simple root belonging to W and satisfying $\alpha > \alpha_n$ ($\forall \alpha \in \Sigma$), so that we see by a recurrent process that $W \supset \Sigma$; further, if $\alpha \in W$, $(-a) \in W$ because if

$$\alpha = S_iS_j \ldots S_k\alpha_l, \qquad -\alpha = S_iS_j \ldots S_kS_l\alpha_l.$$

Thus $W \supset (-\Sigma)$ and we conclude $W = \Delta$.

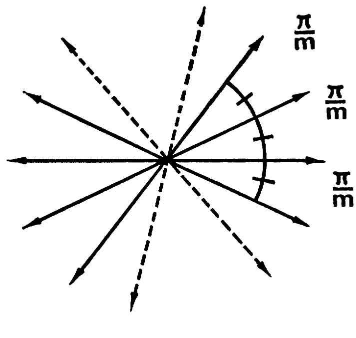
From the formula $gS_\alpha g^{-1} = S_{g.\alpha}$ ($\forall g \in G$), which is an immediate generalization of $S_aS_bS_a = S_{S_a(b)}$, we see that for every root $\alpha = S_iS_j \ldots S_k\alpha_l$, the corresponding symmetry is

$$S_\alpha = (S_iS_j \ldots S_k)S_l(S_iS_j \ldots S_k)^{-1} = S_iS_j \ldots S_kS_lS_k \ldots S_jS_i.$$

And every symmetry S_α is obtained through products of symmetries S_α corresponding to simple roots only.

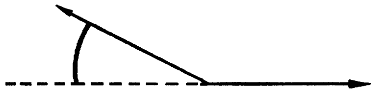
V. — CLASSIFICATION

Now, in order to obtain G, we must consider all possible sets of roots, or all possible corresponding sets of simple roots; if S_i and S_j are two symmetries, H_i and H_j the corresponding fixed hyperplanes, $H_i \cap H_j$ is a subspace of two dimensions less than V, which is left fixed by the mapping S_iS_j: V is the direct sum $V = (H_i \cap H_j) + \Gamma_{ij}$, where Γ_{ij} is a plane, over which S_iS_j induces a rotation; the rotation angle is necessary commensurable with π, since the order of S_iS_j is finite. Further, there are at least two roots in Γ_{ij}, a, b, the angle between them being precisely the rotation angle, which can be written as $\frac{k\pi}{m}$, $\frac{k}{m}$ being an irreducible fraction; we can now, in the Γ_{ij}-plane, construct a subsystem of roots, departing from both the roots a, b, constructing the roots $S_a(b)$ and $S_b(a)$, then the new ones obtained by symmetries of these roots, through each other, and so on. We obtain a subsystem as the following one:



Looking at such a system, since every positive root must be a linear combination of simple roots with positive coefficients, we see that the two simple roots which generate the system, are in the following position:

$\frac{\pi}{m}$



the angle between them is $\frac{\pi}{m}$. Thus, between any system of two simple roots, α_i and α_j, the angle is $\pi - \frac{\pi}{n_{ij}} = \theta_{ij}$ where n_{ij} is an integer. The simple roots can be replaced with unit length vectors a_i, a_j, and we have to search for allowable configurations defined by $\cos^2 \theta_{ij} = \cos^2 \frac{\pi}{n_{ij}}$ where n_{ij} is an integer $\geqslant 2$ $(\cos \theta_{ij} < 0)$.

It is useful to consider $4(a_i \mid a_j)^2 = 4 \cos^2 \frac{\pi}{n_{ij}}$, and to define a diagram for any one allowable configuration, as a collection of points u_i, $i = 1, 2, \ldots, n$, and lines connecting these according to the rule: u_i and u_j are not connected if $(a_i \mid a_j) = 0$ and u_i and u_j are connected by $4(a_i \mid a_j)^2 = 1, 2$ or 3 lines when this equality holds [i. e. respectively when $n_{ij} = 3,4$ or 6]; the case $n_{ij} = 2$ corresponds to $(a_i \mid a_j) = 0$. If $4(a_i \mid a_j)^2$ is not an integer but verifies:

$$p < 4(a_i \mid a_j)^2 < p + 1$$

p an integer, u_i and u_j will be connected with p lines, together with another dashed line. For example, if $n_{ij} = 5$, $2 < 4(a_i \mid a_j)^2 < 3$, the diagram is

Thus, when $n_{ij} = 3,4$ or 6, we recognize the Dynkin diagrams, and the corresponding groups are the well-known Weyl groups of simple Lie Algebras. But, otherwise, if $n = 2$, all values $n_{1,2} = 2, 3, 4 \ldots$ are solutions, and if $n > 2$, there are *only two solutions*, because, looking at connected diagrams, one sees:

1° *If n is the number of vertices (points) of a diagram, then the number of pairs of connected points is less than n.*

PROOF. — Let

$$a = \sum_1^n a_i, \quad \text{then} \quad 0 < (a \mid a) = n + 2 \sum_{i<j} (a_i \mid a_j).$$

If $(a_i \mid a_j) \neq 0$, then $2(a_i \mid a_j) \leqslant -1$.
Hence the inequality shows that the number of pairs a_i, a_j with $(a \mid a_j) \neq 0$ is less than n.

2° *A diagram contains no cycles* (a cycle is a sequence of points $u_1, \ldots, u_k$ such that u_i is connected to u_{i+1}, $i \leqslant k - 1$ and u_k connected to u_1).

PROOF. — The subset forming a cycle violates the former condition.

3° *The number of non-dashed lines* (counting multiplicities) *issuing from a vertex is less than four.*

PROOF. — Let u be a vertex, $v_1, v_2, \ldots, v_k$ the vertices connected to u. No two v_i are connected since there are no cycles. Hence $(v \mid v_j) = 0$, $i \neq j$ (now for simplicity, we denote in the same way simple roots and vertices). In the space spanned by u and the v_i we can choose a vector v_0 such that $(v_0 \mid v_0) = 1$ and $v_0, v_1, \ldots, v_k$ are mutually orthogonal. Since u and the v_i, $i \geqslant 1$, are linearly independent, u is not orthogonal to v_0 and so $(u \mid v_0) \neq 0$. Since

$$u = \sum_0^k (u \mid v_j) v_j, \quad (u \mid u) = (u \mid v_0)^2 + (u \mid v_1)^2 + \ldots + (u \mid v_k)^2 = 1,$$

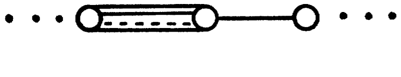
Hence

$$\sum_1^k (u \mid v_i)^2 < 1 \quad \text{and} \quad 4 \sum_1^k (u \mid v_i)^2 < 4.$$

Since $4(u \mid v_i)^2$ is the number of non-dashed lines connecting u and v_i whenever there is no dashed line between them, or otherwise is greater than this number, the result follows.

4° With any dashed line, there are at least two non-dashed lines issuing from a vertex $\left(\text{the first case, } n_{ij} \text{ increasing, is } n_{ij} = 5, \text{ when } \theta_{ij} = \pi - \dfrac{\pi}{5}\right)$.

It readily follows that when the dimension of V is $n \geqslant 3$, and $n_{ij} \geqslant 7$, there is no solution because the corresponding diagrams (we speak about connected diagrams) are such that there are at least four non-dashed lines issuing from one vertex:



5° *Let* Π *be an allowable configuration and let* $v_1, v_2, \ldots, v_k$ *be vectors of* Π *such that the corresponding points of the diagram form a simple chain in the sense that each one is connected to the next by a single line. Let* Π' *be the collection of vectors of* Π *which are not in the simple chain* $v_1, \ldots, v_k$ *together with the vector* $v = \sum_{1}^{k} v_i$; *then* Π' *is an allowable configuration*:

PROOF. — We have $2(v_i \mid v_{i+1}) = -1$, for $i = 1, 2, \ldots, k-1$. Hence

$$(v \mid v) = k + 2\sum_{i<j}(v_i \mid v_j).$$

Since there are no cycles $(v_i \mid v_j) = 0$ if $i < j$, unless $j = i + 1$. Hence

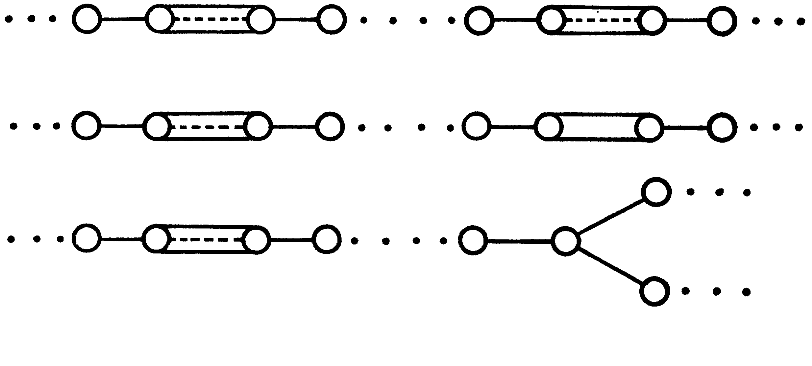
$$(v \mid v) = k - (k-1) = 1$$

and v is a unit vector.

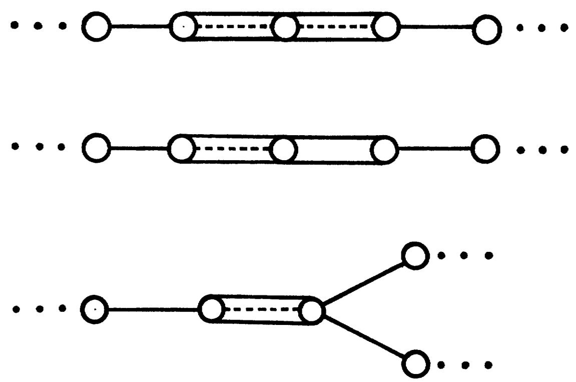
Now let $u \in \Pi$, $u \neq v_i$. Then u is connected with at most one of the v_i, say v_j, since there are no cycles. Then

$$(u \mid v) = \left(u \,\middle|\, \sum_{1}^{k} v_i\right) = (u \mid v_j) \quad \text{and} \quad 4(u \mid v)^2 = 4(u \mid v_j)^2 = 4\cos^2\frac{\pi}{n_{ij}},$$

as required; the diagram of Π' is obtained from that of Π by shrinking the simple chain to a point; thus we replace all the vertices by the single vertex v and we join this to any $u \in \Pi$, $u \neq v_i$ by the total number of non-dashed lines connecting u to any one of the v_j in the original diagram; we get the same result for dashed lines, but always one dashed line will connect two vertices. Application of this to the following graphs:



reduces these respectively to:

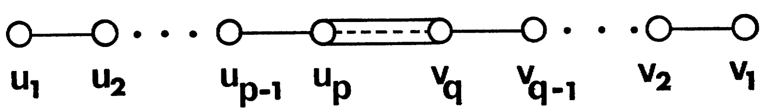


But these last ones are not allowable (four non-dashed lines issuing from a vertex).

The possibilities are among the following type of diagrams:

(i. e. there is, on each side, a finite chain).

Consider the peculiar one:



And let us write

$$u = \sum_{1}^{p} i u_i, \qquad v = \sum_{1}^{q} j v_j.$$

Since

$$2(u_i \mid u_{i+1}) = -1 \quad \text{and} \quad 2(v_j \mid v_{j+1}) = -1$$

we have:

$$(u \mid u) = \sum_{1}^{p} i^2 - \sum_{1}^{p-1} i(i+1) = p^2 - \frac{p(p-1)}{2} = \frac{p(p+1)}{2}$$

$$(v \mid v) = \frac{q(q+1)}{2}.$$

And

$$(u \mid v) = pq(u_p \mid u_q) = pq \cos\left(\pi - \frac{\pi}{5}\right), \quad (u \mid v)^2 = p^2q^2 \cos^2 \frac{\pi}{5}.$$

By Schwarz inequality:

$$pq \cos^2 \frac{\pi}{5} < \frac{(p+1)(q+1)}{4}$$

p and q are integers $\geqslant 1$; if $n \geqslant 3$, q, for instance, is > 1. The solutions are:

$$p = 2, \qquad q = 1$$

$$p = 3, \qquad q = 1$$

corresponding to the following diagrams:

When the field is R, the field of real numbers, it is easy to see that these two diagrams give actual solutions since the corresponding euclidean systems can be constructed.

When the field is Q, the field of rational numbers, one can multiply the non-simple roots by integer multipliers in such a way that the new ones can be written as:

$$\alpha' = \pm \Sigma m_i \alpha_i$$

with m_i integers $\geqslant 0$, $\alpha_i \in \Pi$; now, these are considered as new non-simple roots, and the condition:

$$S_{\alpha_i}(\alpha') = \alpha' - \frac{2(\alpha' \mid \alpha_i)}{(\alpha_i \mid \alpha_i)} \alpha_i \in \Sigma \quad \text{when} \quad \alpha \in \Sigma \quad \text{and} \quad \alpha_i \in \Pi$$

shows that necessarily, the scalars $-\frac{2(\alpha \mid \alpha_i)}{(\alpha_i \mid \alpha_i)}$ (from now on, we drop the « prime » for brevity) are integers. More generally, the scalars $-\frac{2(\alpha \mid \beta)}{(\beta \mid \beta)}$

($\forall \alpha, \beta \in \Sigma$) are necessarily integers, since every root can be considered as simple, according to the choice of x_0, and Σ. From:

$$\frac{4(\alpha \mid \beta)^2}{(\alpha \mid \alpha)(\beta \mid \beta)} = \left(-\frac{2(\alpha \mid \beta)}{(\alpha \mid \alpha)}\right)\left(-\frac{2(\alpha \mid \beta)}{(\beta \mid \beta)}\right) < 4 \quad (\text{if } \alpha \neq \pm \beta).$$

We deduce that the scalars $-\frac{2(\alpha \mid \beta)}{(\alpha \mid \alpha)}$ are integers verifying:

$$-3 \leqslant \frac{-2(\alpha \mid \beta)}{(\alpha \mid \alpha)} \leqslant 3.$$

So we find precisely as solutions all Weyl groups of simple *Lie Algebras*, whatever the dimension of V may be.

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